INFINITE WAVEGUIDE TERMINATION BY HYBRID FINITE ELEMENT / SERIES SOLUTION

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1 INTRODUCTION

Finite and boundary element simulation techniques\(^1\) are now commonly used in designing audio systems. There may be many components involved, e.g. a compression driver, a transition section changing cross section from circular to rectangular at the throat of the horn, a cylindrical waveguide terminating in a baffle. A model can be used to analyse the entire system. Ideally the wavefronts should be well behaved throughout, as sound progresses through the system. Many possible complications might occur: the driver might not be producing perfectly plane waves, the transition component or horn may be introducing cross modes, diffraction effects at the mouth may be causing problems. An analyst studying in detail the pressure field for a simulation of the entire system may observe a complicated pressure field at the output of the compression driver and conclude that the driver is the cause, whereas cross modes generated in the transition component, being reflected back might be responsible. Hence it is useful to understand the physics of individual parts of the system, which can be done by analyzing separately, with simplified termination conditions. A compression driver should ideally radiate plane waves, without cross modes, into an infinite plane wave tube. The transition component should ideally generate plane waves into an infinite rectangular duct termination. The effects of edge diffraction are eliminated if a baffled cylindrical horn is replaced by an infinite cylindrical horn. In each of these cases the pressure field within the simplified termination region can be efficiently represented by a series solution. By contrast, finite elements (FE), are good for complicated arbitrary geometries, but ‘infinite termination boundary conditions’ are more difficult to include. The current work shows how a Dirichlet to Neumann (DiN) approach can be used to couple series solutions to a truncated finite element mesh.

2 COUPLING SERIES SOLUTION TO ACOUSTIC FINITE ELEMENTS

For steady state vibration at frequency \( f \) with speed of sound \( c \) and wavenumber \( k = \frac{2\pi f}{c} \) the acoustic pressure satisfies the Helmholtz equation.

\[
\nabla^2 p + k^2 p = 0
\]

This can be solved by several techniques, including finite elements, boundary elements, analytical/closed form series solutions and other methods. Figure 1 illustrates a hybrid finite element + series solution approach. Region 1 is the finite element domain. In region 2 the pressure field is represented by the series solution

\[
p(x, y, z) = \sum_{r=1}^{\infty} a_r \phi_r(x, y, z)
\]
The two regions are coupled at $\Gamma_0$, the interface, which is also referred to as the truncation boundary for the FE mesh. The functions $\phi_r(x, y, z)$ must satisfy the Helmholtz equation, the required boundary conditions on $\Gamma_1$, the appropriate radiation condition on any unterminated boundaries in region 2 and be a basis for the vector space of feasible solutions in region 2. It is advantageous if the $\phi_r$ are orthogonal with respect to integration over $\Gamma_0$, i.e. for $r \neq s$

$$\int_{\Gamma_0} \phi_r \phi_s d\Gamma = 0 \quad (3)$$

The above equation is satisfied for all the examples in this paper.

For a given pressure distribution in region 1, the pressure at any point on $\Gamma_0$ is determined by the FE interpolation functions. Hence by integrating over $\Gamma_0$ and using the orthogonality of the $\phi_r$, the modal contribution factors $a_r$ can be computed. By differentiating the series solution (2), the pressure normal gradient on the interface can be determined by

$$\frac{\partial p}{\partial n} = \sum_{r=1}^{n} a_r \frac{\partial \phi_r}{\partial n} \quad (4)$$

and by multiplying by the element shape functions and integrating over $\Gamma_0$, the loading on the FE mesh from region 2 can be determined. Applying this procedure for an arbitrary pressure distribution on $\Gamma_0$ produces an impedance matrix which can be used to modify the FE equations from region 1, including the effect of region 2. Similar methods have been used previously for underwater acoustic analyses².

3 SEMI-INFINITE TUBE

3.1 Series Solution

Using the separability of the Helmholtz equation in cylindrical polar coordinates, the pressure field for propagation of axisymmetric acoustic waves along an infinite rigid cylinder

$$\{(x, r) : x \geq 0, r \leq R\}$$

can be written in the series solution

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$p(x, r) = \sum_{n=1}^{\infty} a_n J_0 (k_n r) e^{-ik_n z}$ \hspace{1cm} (5)

where the radial wavenumbers are solutions of

$J_0(k_R) = 0$ \hspace{1cm} (6)

and the axial wavenumbers $k_n$ are given by

$k_n' = \begin{cases} \sqrt{k_n^2 - k_R^2} & \text{if } k \geq k_n, \\ -i \sqrt{k_n^2 - k_R^2} & \text{if } k < k_n \end{cases}$ \hspace{1cm} (7)

### 3.2 Simple Test Problem

A simple problem, illustrated in figure 2, was used to test the coupling of the series (5) to a FE mesh. A semi-infinite tube, with rigid surfaces and internal air is excited by a piston in part of the flat end. The tube radius is 0.01835m and the piston radius is 0.009175m. The piston has unit velocity. The properties of air in this, and all further examples in this paper are: speed of sound = 340m/s and density=1.2 kgm$^{-3}$.

This problem can be solved directly by the series solution (5), which provides an accurate benchmark result. Three sets of finite element results were computed. The mesh in figure 3, consisting of 56 quadratic 8-noded quadrilaterals, and modelling the air out to one radius, was used for testing the series solution termination, truncated to 5 terms. The same mesh was used for checking the effect of a simple $\rho c$ impedance condition on the truncation boundary. The mesh in figure 4 was used for a perfectly matched layer (PML) termination. This is an artificial, highly absorptive region, which attempts to absorb all the outgoing waves travelling over the truncation boundary.

Figure 3: mesh for testing series $\rho c$ termination

Figure 4: mesh for PML termination

Figure 5 compares pure series solution and FE + $\rho c$ termination SPL on axis at one radius along the tube, i.e. on the truncation surface. At low frequencies the FE results are accurate; all the evanescent terms have decayed before the boundary and the plane travelling wave is absorbed correctly. As the cut-on frequency is approached the exponential decay tail of the first cross mode gets longer and is reflected by the truncation boundary. Figure 6 shows results when the mesh is extended axially by a factor of 10. The FE results are accurate almost to the first cut-on frequency;

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the cross mode has more room to decay before the truncation boundary. Above the cut-on frequency the cross mode becomes a travelling wave, which is reflected from the truncation boundary. The large path length causes rapid variation of the phase of the reflection with frequency. These results illustrate that when a truncation boundary is not working correctly, and producing spurious reflections, then results computed will vary with the position of the truncation boundary.

Figure 5: FE + ρc termination for axial length R to truncation boundary

Figure 6: FE + ρc termination for axial length 10 R to truncation boundary

Figure 7 shows close agreement between the pure series solution, FE +series and FE + PML. The PML-based results are slightly less accurate at the cut-on frequencies as shown in figure 8.
Figure 7: comparison of pure series, FE + series and FE + PML

Figure 8: comparison of pure series, FE + series and FE + PML at first cut-on frequency
3.3 Compression Driver Test Problem

An untuned compression driver, radiating into a plane wave tube, is illustrated by the mesh in figure 9. This is a much more complicated situation, the mechanical components are modelled (dome + former + voicecoil + surround), the geometry is much more complex, and there are thin gaps where viscothermal effects are important. There is no analytical solution to this. The tube radius is identical to the example above. Figure 10 shows SPL computed with FE + series, on axis at one radius along the tube, with two different lengths up to the truncation boundary. The results are virtually identical, indicating that the series termination is behaving correctly, as there are no spurious reflections.

4 DUCT OF ARBITRARY CROSS SECTION

4.1 Series Solution

Using the separability of the Helmholtz equation in cartesian coordinates, the pressure field for propagation of acoustic waves along a duct of arbitrary cross section, extending infinitely in the x-direction, can be expressed as

$$\sum_{r=1}^{\infty} a_r \psi_r(y, z)e^{-i k_r x}$$

where $\psi_r$ is the $(r)$th eigenmode of the cross section, with associated eigenvalue $k_r$.

$$\nabla^2 \psi_r + k_r^2 \psi_r = 0$$

$k_r$ is determined, by equation (7). $\psi_r$ also satisfies a zero normal gradient condition

$$\frac{\partial \psi_r}{\partial n} = 0$$

at the boundary of the cross section.

In a 3D FE model, extending to a plane of constant $x$, the faces on the truncation surface can be conveniently used to setup an auxiliary 2D FE mesh for computing the cross section modes.
4.2 Simple Test Problem

The method above was tested on a rectangular cross section duct \( \{(x, y, z) : |x| \leq 0.01, |y| \leq 0.013, z \geq 0\} \) with excitation of unit velocity on a piston covering the central portion of the flat end \( \{(x, y, 0) : |x| \leq 0.05, |y| \leq 0.065\} \). There is symmetry about the planes \( x=0 \) and \( y=0 \), and so a quarter model was used comprising of 6 x 6 x 6 20-noded quadratic acoustic hexahedra, as shown in figure 11 (series solution case). The axial length modelled was 0.013m. A PML-based model had a mesh extending by 6 quadratic elements past the truncation plane. For this cross section there is an analytical expression for the modes

\[
\psi_{nm}(x, y) = \cos\left(\frac{n\pi x}{W_1}\right) \cos\left(\frac{m\pi y}{W_2}\right)
\]

and so a pure series solution is also available. Figure 12 compares results at (0,0,0.013). As in the axisymmetric tube case, there is close agreement apart from at cut-on frequencies where FE + series is more accurate than FE + PML. At the 4th cut-on frequency both FE results are less accurate due to the mesh density.

4.3 Transition Component

A simple transition between a circular cross section of radius 0.025m and a rectangular cross section 0.07m x 0.028m over an axial length of 0.08m is illustrated in figure 13. Excitation by a piston with unit velocity was applied at the circular end. The rectangular cross section is assumed to extend infinitely. Two FE meshes were used to analyse this. One terminated at the start of the rectangular section. The other, shown in figure 14, continued the rectangular section for a further 0.01m. Both gave identical results. The SPL at the start of the rectangular cross section and a further 0.05m along are shown in figure 15. The two points agree well below the cut-on frequency but start to diverge as the cut-on frequency is approached as the contribution from the first cross mode increases. At the cut-on frequency both curves have an abrupt change. Above the cut-on frequency both plane wave and cross mode are travelling waves with different wavenumbers. Thus minima, associated with interference of these two waves, occur at different frequencies.
5 INFINITE CYLINDRICAL TERMINATION

5.1 Series Solution

The same methodology can be used for an infinite cylindrical termination, i.e.
\( \{ (r, \theta, z) : r \geq R, 0 \leq \theta \leq \Theta, 0 \leq z \leq Z \} \). A suitable series, obtained from separability of the Helmholtz equation in cylindrical polar coordinates is

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \cos \left( \frac{n \pi \theta}{\Theta} \right) \cos \left( \frac{m \pi z}{Z} \right) f_{nm}(r) = \quad (12)
\]

and the radial variation is given by
5.2 Cylindrical Test Problem

A test problem for this case was a cylindrical horn extending from \( \theta = -45^\circ \) to \( \theta = 45^\circ \), with an axial length of 0.15m and a flat throat 0.012m from the axis. Excitation was by a piston with unit velocity extending halfway across the throat. Results were computed with two FE models, taking the truncation radius at 0.1m and with 0.08m. In each case a half model was used, extending from \( \theta = 0^\circ \) to \( \theta = 45^\circ \). The mesh for R=0.1m is shown in figure 16. Figure 17 compares SPL at r=0.075m, \( \theta = 0^\circ \) and in the centre axially. Agreement is very close, indicating that there are no spurious reflections and the FE/series coupling is working correctly.

![Figure 16: mesh for cylindrical termination](image1)

![Figure 17: SPL comparison for R=0.1 & R=0.08](image2)

6 CONCLUSIONS

An approach for coupling a series solution to a truncation surface on a finite element mesh has been shown to be accurate for modelling infinite waveguides. Tests have shown that it is more accurate than a perfectly matched layer approach in the neighbourhood of cut-on frequencies for cross modes. Suitable series solutions can be constructed when the waveguide is bounded by surfaces of constant coordinate in a coordinate system for which the Helmholtz equation is separable. Hence it would, for example, be possible to analyse an infinite conical horn. The method can be extended to some other situations such as ducts of arbitrary cross section.

This use of this approach could help design engineers understand the physics of different components in an audio device, and hence produce improved products.
7 REFERENCES