APPROXIMATION OF IMPEDANCE IN SERIAL ARRAYS OF COUPLED HELMHOLTZ RESONATORS

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1 INTRODUCTION

The transfer matrix method (TMM) is a simple method to model acoustical systems. Using this it is possible to analyse the sound absorption/transmission properties of one and two port systems[1] and model sound absorbing acoustic metamaterials consisting of waveguide structures side-loaded by Helmholtz resonators[2,3,4]. In this paper a serial array of N identical Helmholtz resonators will be modelled using the TMM. This will be done for up to N=3. A method of simplification will then be applied to the impedance terms obtained using the TMM. This method will utilise the large impedance contrast that exists between the neck and cavity of a Helmholtz resonator. By using a ratio of this impedance contrast it is possible to create a small parameter, ε. By non-dimensionalising the rest of the terms in the impedance expressions, the Taylor series expansion can be taken for ε=0. By using the leading order term of these expansions, you then result in an expression for the impedance of a serial array of Helmholtz resonators composed of a polynomial of the same order as N. These expressions are compared against the TMM to assess validity. A resonant frequency analysis is then undertaken to ascertain whether the solution of the polynomials can be used to determines the resonant frequencies of the systems.

2 BACKGROUND THEORY

2.1 The Transfer Matrix Method (TMM)

The transfer matrix method (TMM) provides the relationship between the initial sound pressure, p, and normal acoustic particle velocity, v, at the start (x=0) of a medium and the end (x=L) of a medium. The transfer matrix, T, is derived under the assumption that only plane waves propagate through the medium in the x direction, meaning it provides the solution for a 1D wave propagation problem. The general formulation of the transfer matrix is as follows.

$$\begin{bmatrix} P \\ v \end{bmatrix}_{x=0} = T \begin{bmatrix} P \\ v \end{bmatrix}_{x=L} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} P \\ v \end{bmatrix}_{x=L} \tag{1}$$

The transfer matrix for a single fluid layer is constructed as

$$\begin{bmatrix} P \\ v \end{bmatrix}_{x=0} = \begin{bmatrix} \cos(kL) & iZ\sin(kL) \\ \frac{i}{Z}\sin(kL) & \cos(kL) \end{bmatrix} \begin{bmatrix} P \\ v \end{bmatrix}_{x=L} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} P \\ v \end{bmatrix}_{x=L} \tag{2}$$

where k is the acoustic wavenumber and Z is the characteristic impedance. For a multilayered structure, the relationship between the input and output pressure and acoustic particle velocity are obtained by the multiplication of the transfer matrix of each layer.

2.2 Modelling a serial array of identical Helmholtz resonators using the TMM

Consider a serial array of identical cylindrical Helmholtz resonators, as depicted in figure 1. Here r_w is the waveguide radius, r_n is the HR neck radius, r_c is the HR cavity radius, l_n is the neck length and l_c is the cavity length.

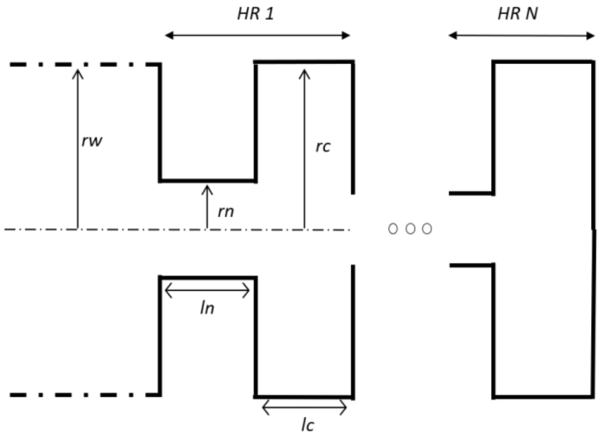

Figure 1: Geometry of a serial array of N identical Helmholtz resonators.

To calculate the impedance of this system, the transfer matrix method is used. The full matrix, T, is derived from the following expression[5]:

$$T = \prod^{N} M_n M_{\Delta l} M_c \tag{3}$$

The transfer matrix for the HR neck and cavity take the following forms, respectively.

$$M_n = \begin{bmatrix} \cos(k_n l_n) & iZ_n \sin(k_n l_n) \\ \frac{i}{Z_n}\sin(k_n l_n) & \cos(k_n l_n) \end{bmatrix} \tag{4}$$

$$M_c = \begin{bmatrix} \cos(k_c l_c) & iZ_c \sin(k_c l_c) \\ \frac{i}{Z_c}\sin(k_c l_c) & \cos(k_c l_c) \end{bmatrix} \tag{5}$$

Here k_i and Z_i denote the acoustic wavenumber and characteristic impedance, where [i=n,c] for either the neck or cavity, respectively. The transfer matrix that accounts for the end corrections of the HR neck is written as:

$$M_{\Delta l} = \begin{bmatrix} 1 & iZ_n k_n \Delta l \\ 0 & 1 \end{bmatrix} \tag{6}$$

Where Δl is arrived at from the addition of two correction lengths, $\Delta l = \Delta l1 + \Delta l2$. $\Delta l1$ is due to pressure radiation at the discontinuity from the neck to the cavity of the HR[6] and $\Delta l2$ comes from the pressure radiation at the discontinuity from the neck to the surrounding medium[7].

(7)

$$\Delta l1 = 0.82\left[1 - 1.35\frac{r_n}{r_c} + 0.31\left(\frac{r_n}{r_c}\right)^3\right]$$

(8)

$$\Delta l2 = 0.6 r_n$$

Where r_W is the hydraulic radius for non-circular ducts. To determine the impedance for the resonator, simply multiply the final T matrix by $[1,0]^T$. This accounts for the velocity termination. From this, the impedance can be found as follows:

(9)

$$z_{HR} = \frac{P_{x=0}}{v_{x=0}} = \frac{T_{11}}{T_{21}}$$

The Reflection coefficient of the one port system described here is determined as

(10)

$$R = \frac{Z_w - Z_{HR}}{Z_w + Z_{HR}}$$

And the absorption coefficient as

(11)

$$\alpha = 1 - |R|^2$$

2.3 Viscothermal Losses

Viscothermal losses within this model are accounted for by evaluating the complex frequency dependant density and bulk modulus for a plane wave propagating through a section of constant cross section[8]. For a circular duct of radius r:

(12)

$$\rho(\omega) = \rho_0\left(1 - \frac{2J_1(rG_r)}{rG_r J_0(rG_r)}\right)$$

(13)

$$K(\omega) = K_0\left(1 + (\gamma - 1)\frac{2J_1(rG_k)}{rG_k J_0(rG_k)}\right)$$

Where $G_r = \sqrt{\frac{-i\omega\rho_0}{\eta}}$ and $G_k = \sqrt{\frac{-i\omega\rho_0 P_r}{\eta}}$ in which ρ_0 is the density of air, $K_0 = \gamma P_0$ is the bulk modulus of air, γ is the ratio of specific heats, P_0 is the atmospheric pressure, P_r is the Prandtl number and η is the dynamic viscosity. The dynamic fluid density and complex compressibility $(1/K(\omega))$ can be used to obtain the characteristic impedance and acoustic wavenumber.

(14)

$$Z(\omega) = \sqrt{\rho(\omega)/C(\omega)}/Sa$$

(15)

$$k(\omega) = \omega\sqrt{\rho(\omega)C(\omega)}$$

3 SIMPLIFICATION OF IMPEDANCE TERMS VIA ASYMPTOTIC APPROXIMATION

To simplify the impedance expressions obtained via the TMM for serial arrays of Helmholtz resonators, a method of asymptotic approximation is used. The premise being the utilisation of the impedance contrast between the neck, Z_n, and cavity, Z_c, which allows for a small order term to be obtained. This small order term is defined as

$$\varepsilon = \sqrt{\frac{Z_c}{Z_n}} \tag{16}$$

The rest of the terms are rendered dimensionless with the following expressions:

$$c = \frac{k_c l_c}{\varepsilon} \tag{17}$$

$$n = \frac{k_n l_n}{\varepsilon} \tag{18}$$

$$\phi = 1 + \frac{\Delta l}{l_n} \tag{19}$$

For simplification purposes a new term, x, is defined as $x = cn\phi$.This explicitly becomes

$$x = \frac{\omega^2 V_c l_n' \rho_n(\omega)}{S_n K_c(\omega)} \tag{20}$$

Here, V_c is the cavity volume, l_n' is the length of the neck plus the correction terms, S_n is the cross sectional area of the neck, $\rho_n(\omega)$ is the dynamic density of air within the neck and $K_c(\omega)$ is the bulk modulus of air within the cavity. From this it is evident that viscous effects dominate in the neck and thermal effects dominate in the cavity.

By solving the Taylor series expansion of the final impedance terms of the serial arrays of HRs, up to N=3, for ε=0, the following impedance expressions are obtained from the leading order terms for the expansions. The subscript number denotes the number of identical HRs within the array.

$$Z_{1HR} = \frac{iK_c(\omega)(x-1)}{S_n V_c} \tag{21}$$

$$Z_{2HR} = \frac{iK_c(\omega)(x^2 - 3x + 1)}{(x-2) S_n V_c} \tag{22}$$

(23)

$$Z_{3HR} = \frac{iK_c(\omega)(x^3 - 5x^2 + 6x - 1)}{(x^2 - 4x + 3)S_n V_c}$$

From these expressions it is evident that by increasing N, you increase the order of the polynomial composed of x terms with the numerator polynomial being of the same order as N, and the denominator being one less. The impedance expression for a single HR can be further simplified and expressed as

$$Z_{1HR} = i\left(\frac{\omega l'_n \rho_n(\omega)}{S_n} - \frac{K_c(\omega)}{\omega V_c}\right) \qquad (24)$$

To assess the validity of these expressions, the absorption coefficients obtained with these approximate expressions are compared with the absorption coefficients of the same systems but modelled with the TMM. The geometry selected can be found in the following table.

Table 1:Geometric properties of the serial array of HRs. All units are [mm].

r_w	r_n	r_c	l_n	l_c
30	2	30	10	30

A plot of the resulting absorption coefficients can be seen in figure

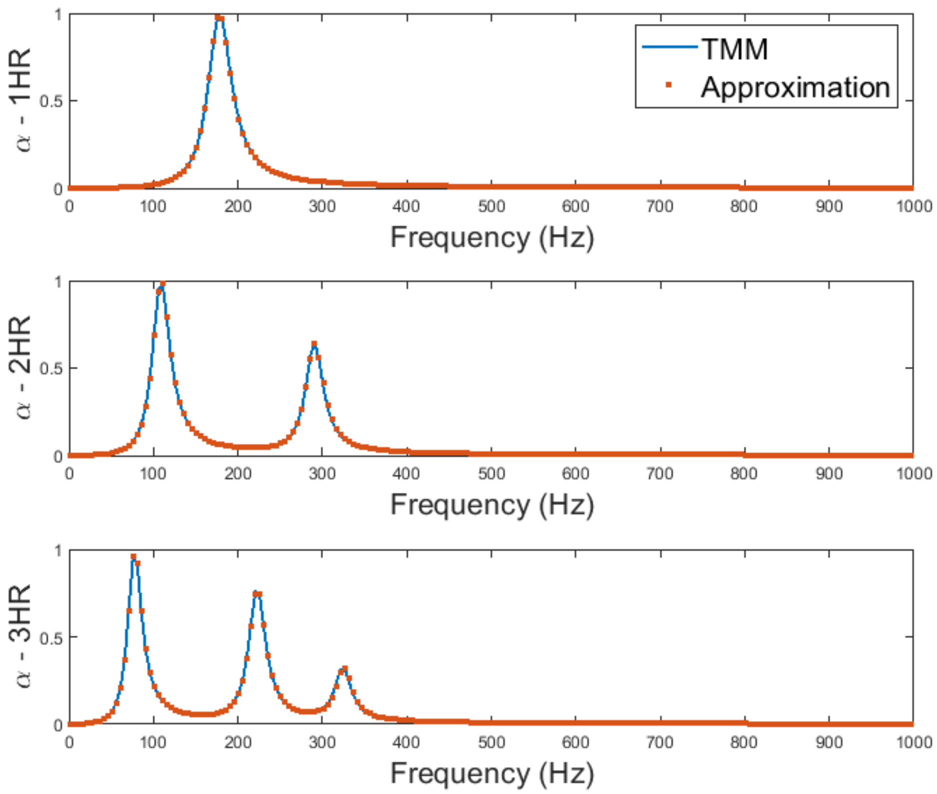

Figure 2:Plot of the absorption coefficient of serial arrays of identical Helmholtz resonators up to N=3 obtained via the impedance approximation method presented here and the TMM.

From this figure it is evident that there is excellent agreement between the two methods for the selected geometry. This provides validation for the proposed methodology of utilising impedance contrasts between sections to simplify expressions obtained using the TMM. One limiting factor worth noting is that the accuracy of the derived expressions are reliant upon there being a large impedance contrast between the neck and cavity. As such, the larger the neck radius becomes in relation to the cavity radius, the more inaccurate these expressions are.

4 RESONANT FREQUENCY ANALYSIS

Due to the simple nature of the approximated impedance expressions and the fact that they are constructed of polynomials, it is possible to derive the resonant frequencies of the 3 systems. To illustrate, the Imaginary component of Z_{HR} is 0 at resonance, therefore it is a simple matter of finding the x value for which the respective polynomial is also equal to 0. Consider an arbitrary solution x=A, by rearranging equation (20) the following relation can be found

$$f_{res}(i) = \frac{1}{2\pi}\left(\frac{S_n}{V_c l'_n}\right)^{0.5} Re\left[\frac{K_c(\omega)}{\rho_n(\omega)}\right]^{0.5} (A(i))^{0.5} \tag{25}$$

Here (i) denotes which resonance is being calculated. Furthermore, as

$$\lim_{\omega\to\infty} Re\left[\frac{K_c(\omega)}{\rho_n(\omega)}\right]^{0.5} = c_0 \tag{26}$$

equation (25) can be simplified to

$$f_{res}(i) = \frac{c_0}{2\pi}\left(\frac{S_n}{V_c l'_n}\right)^{0.5} (A(i))^{0.5} \tag{27}$$

To determine the resonant frequencies of the three systems, it is necessary to find A. This is done by solving the numerator polynomials in the impedance expressions, which can be found in the following table.

Table 2:Values of A for the three HR systems.

No.	A
1	1
2	$(3-\sqrt{5})/2, (3+\sqrt{5})/2$
3	0.19806, 1.5550, 3.2470

From this it is evident that for a single HR, the classical formula for the resonant frequency of a Helmholtz resonator is obtained;

(28)

$$f_{res} = \frac{c_0}{2\pi}\left(\frac{S_n}{V_c l_n'}\right)^{0.5}$$

To assess the accuracy of equation (27) the resonant frequencies obtained using the TMM (frequency of absorption peaks seen in figure (2)) are compared with the resonant frequencies obtained by equation (27). The results can be seen in the following table.

No.	α (TMM)	α (Approx)
1	178	186
2	110, 291	115, 301
3	78, 223, 324	82, 231, 335

From this table it is evident that there is relatively good agreement between the two methods, with only slight discrepancies between the resonant frequencies. It can be seen that the resonant frequency obtained by the TMM is always slightly lower than from the approximation method, this hints as to the reason for the discrepancy. This is the influence of viscothermal losses not being captured within equation (27) as the introduction of losses within the system always reduces the resonant frequencies due to dampening the system.

5 CONCLUSION

A method of simplification has been proposed for impedance terms derived by the TMM. This has been applied to the case of a serial array of N coupled Helmholtz resonators. The simplification method is reliant upon the use of an impedance contrast to create a small order term which can be used in the Taylor expansion of the TMM impedance expressions. By utilising the leading order term from the Taylor series expansions, simple expressions were found composed of polynomials of the same order as N. Excellent agreement was found when comparing the absorption coefficient obtained via the TMM and with this approximation method. It was also found that the resonant frequencies of the systems can be obtained through the solution of the polynomials. However, it is evident that there is a slight discrepancy between the approximate resonant frequency equations and those achieved by the TMM. This is due to viscothermal loss effects not being taken in to account in the approximate expressions.

6 REFERENCES

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