VIBRO-ACOUSTIC ANALYSIS FOR SYSTEM DESIGN IN THE PRESENCE OF UNCERTAINTIES

R S Langley Department of Engineering, University of Cambridge, Cambridge UK

1 INTRODUCTION

This paper provides a review of a family of methods that have been developed for the prediction of the vibro-acoustic performance of complex systems over a broad frequency range. The methods represent various extensions of the Statistical Energy Analysis (SEA) approach to response prediction [1,2], and the main attributes are: (i) the use of a low number of degrees of freedom in the model, and (ii) the prediction of the response statistics without the need to perform Monte Carlo analysis. Both of these attributes contribute towards relatively low computational costs and the allowance of fast design iterations to explore and optimise designs. The paper focusses on work in which the author has been involved over the past 25 years or more, and so there is no intention to provide a comprehensive review of the full range of methods that are available for vibro-acoustic response prediction. In addition, the fact that the underlying equations are readily available in the cited literature means that no equations are presented here, the aim being to provide a descriptive overview of the subject area.

Initially comments are made on the history of the development of mathematical and computational models, and on the growing recognition of the importance of uncertainties. The physics underpinning the present set of prediction methods is then described, and a sequence of methods for problems of increasing complexity is presented. Concluding comments are then made.

2 HISTORICAL CONTEXT

2.1 Design Requirements

Historically the design of engineering systems has passed through a number of distinct phases. In the earliest days design was based on trial and error – the system was built, and then rebuilt if failure occurred. As time progressed, given the experience afforded by trial and error, empirical design guidelines could be established to improve the chance of success. With the growth of scientific knowledge generated by the Scientific Revolution (from the 16th century onwards), the empirical guidelines could be improved and extended to include physics-based calculations. The rapid increase in scientific knowledge, particularly over the past 200 years, has allowed the purely empirical aspects of the design guidelines to be gradually reduced, leading ultimately to design assessment methods based mainly on the use of calculations. This kind of progression is exemplified by ship design, passing from trail and error, to largely empirical design codes, and then to computer based calculations based on scientific and mathematical models. The same progression can be seen, for example, in the design of structures and buildings, and throughout engineering it is now commonplace to employ complex computer models as part of the design process. It should be stressed however that such models do not “design” the system – instead they are used to assess the fitness of a proposed design, with the aim of building only those systems that are known to meet requirements. Design remains a creative process, but the outcome of the design process is much enhanced by the

ability to predict in advance the way in which a particular design will behave. The role of a mathematical or computational model is therefore to provide information that will enable the designer to make informed decisions, and the nature of the required information will depend on the problem at hand. The present work is concerned with the development of computer models to facilitate the design of systems that have good vibro-acoustic performance, and as discussed in what follows, this requires particular consideration of the effects of manufacturing uncertainties.

Before considering vibro-acoustic problems per se, it is helpful to consider a design example from the early aerospace industry. Before his career as a novelist, the author Nevil Shute was the "Chief Calculator" for the R100 airship project (1925-1930), a job which involved overseeing the stressing calculations for the structural frame of the airship. The calculation method used is described in his autobiography *Slide Rule* [3]:

> "the stress calculations for each transverse frame, for instance, required a laborious mathematical computation by a pair of calculators that lasted for two or three months before a satisfactory and true solution to the forces could be guaranteed."

In this context a "calculator" was not an electronic instrument, but rather a person using a slide rule to perform calculations. Clearly there has been a huge advance in computational techniques since Shute's day, in terms of both hardware and software. The problem tackled by Shute's calculators had eight degrees of freedom, and took up to three months to solve. It is now commonplace for problems having millions of degrees of freedom to be solved in a matter of hours using the finite element method implemented on a modern computer. This is not to say that the days of computational difficulties are over: the more we are able to compute, the more challenging and difficult are the problems we set ourselves, and there is a constant feeling that computational methods need to be improved, regardless of the current state of the art. A second quote from Shute is more telling in the present context, and it concerns the completion of the calculation:

> "it produced a satisfaction almost amounting to a religious experience. After literally months of labour, having filled perhaps fifty foolscap sheets with closely pencilled figures, after many disappointments and heartaches, the truth stood revealed, real, and perfect, and unquestionable; the very truth."

This expresses a degree of belief in the accuracy of the calculations that would not be shared today, regardless of the huge advance in computational techniques. At the risk of being facetious, it could be said that there has been a loss of faith in the "religious" truth of the calculations. There is a recognition that uncertainties abound in our calculations: in the material properties assumed for our system, in the detailed manufactured geometry of the system, and in the mathematical models used to approximate the actual system. In some industries these uncertainties have little effect on the performance of the system, but in vibro-acoustics the effect can be immense, as demonstrated by Bernhard [4]. Reference [4] reports measurements made on 57 pick-up trucks manufactured on the same production line: the frequency response function from the wheel/spindle to the sound level at the driver's ear was measured for each truck. At frequencies beyond 100 Hz the noise level in the different trucks differed by more than 20 dB, demonstrating a high sensitivity to small variations introduced in the manufacturing process. A computational model of the truck, developed at the design stage, would of course predict one definite frequency response function, and it is highly unlikely that this would agree exactly with any one of the measurements; it is therefore not possible to share Shute's confidence in the calculations, at least for this type of problem.

The most direct way of predicting the variability in the response of a system would be to repeatedly rerun the computational model with random changes in the input parameters, i.e. Monte Carlo simulation (see for example [5]). This approach faces two difficulties when applied to vibro-acoustic problems: (i) at higher frequencies the model may contain millions of degrees of freedom, and even with modern computer hardware the time required for multiple runs might be unfeasible, and (ii) it is often not clear which of the input parameters are uncertain, or what the appropriate probability distribution might be. These factors have led to the development the alternative methods of analysis

that are the subject of the present paper. The methods have much in common with statistical mechanics, and by way of background this synergy is discussed in the following subsection.

2.2 Complex Systems and Statistical Mechanics

The analysis of a system that has a very large number of degrees of freedom and is subject to uncertainty is by no means a new problem – in fact it can be argued that the whole subject of statistical mechanics is concerned with this issue [6]. For example, a molecular model of a gas in a room has many millions of degrees of freedom, associated with the positions of the individual molecules, and the instantaneous position of each molecule can never be precisely known (even leaving aside quantum effects), and so there is a direct analogy with uncertain engineering systems. In gas dynamics a problem of practical interest would be to consider two adjoining rooms having different initial temperatures. In principle molecular simulations can be performed to compute the changing temperatures in the rooms, perhaps randomised in a Monte Carlo way to allow for uncertainties, but such a calculation would require a vast amount of computational effort. Such a calculation would also be unnecessary; the classical theory of thermodynamics provides a simple heat flow model of this problem in terms of the instantaneous temperatures in each room, yielding two equations for two unknowns. So at first sight there appear to be two competing theories describing the same problem: molecular dynamics and classical thermodynamics. In reality there is only one theory, but with various attributes, and this is illustrated in Figure 1. We can consider: (1) that we have a large complex system for which the equations of motion are known (in this case the equations of molecular dynamics), (2) the system has a degree of uncertainty or randomness, perhaps in the initial conditions and/or in the system properties, (3) given the equations of motion and a description of the uncertainty, we can in principle solve for the statistics of the motion at all times, (4) to better understand the motion we can compute large scale quantities such as temperature and energy from the motion of the individual molecules, (5) most importantly, we might find that these large scale quantities actually obey relatively simple laws – these laws are said to be “emergent” in the sense that they emerge from the very complicated detailed equations. So molecular dynamics lies in boxes (1)-(3) of Figure 1, and classical thermodynamics lies in box (5), but the two theories are not distinct – one theory emerges from another.

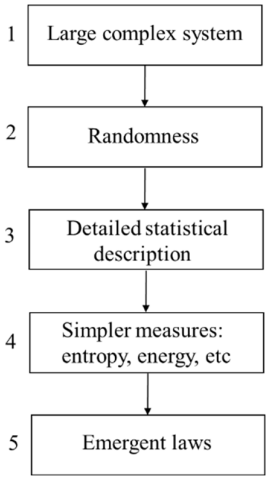

Figure 1 A schematic of statistical mechanics and emergent laws [7]

The standard computational models of structural vibration and acoustics (with Monte Carlo simulations included to allow for the effects of uncertainty) lie within boxes (1)-(3) of Figure 1. The degrees of freedom are generally those associated with a finite element model or a boundary element model of the system, and the computations yield the detailed response (in the deterministic case) or the statistical distribution of the response (in the random case). Given the obvious analogy with statistical mechanics, and motivated in part by high frequency problems in the space industry, much

work was performed in the 1960s to seek emergent laws that govern vibro-acoustic responses so that boxes (1)-(3) might be replaced by box (5). The key breakthroughs were made by Lyon and his co-workers, leading to the development of a method known as Statistical Energy Analysis (SEA) [1]. This approach has since been extended in a number of ways, as described in the present paper, and these extensions are made possible by a remarkable phenomenon known as universality. The dynamic response of a random system is crucially dependent on the statistical distribution of the natural frequencies of the system and it seems obvious that finding this distribution for a complex system should be a very daunting task. However, it turns out that if the system has a sufficient degree of randomness then under certain additional conditions (discussed in what follows) a "universal" distribution is obtained for the natural frequencies, and moreover this distribution depends on a single simple parameter – the modal density, being the average number of resonances in a unit frequency band. This fact enables a strange phenomenon to occur: the statistics of the response of the system can be computed without a detailed knowledge of the statistics of the underlying uncertainties. Universality in the present context is a feature of random matrix theory, and this is discussed in the following section as a background to the subsequent development of the subject.

3 RANDOM MATRIX THEORY AND ITS IMPLICATIONS

3.1 Random Matrix Theory

The natural frequencies of an engineering system are yielded by the solution of a matrix eigenvalue problem. If the system has random properties (for example geometry or material properties) then the matrices will be random, and the resulting statistics of the eigenvalues will have a crucial effect on the statistics of the response of the system. It has long been observed empirically that the eigenvalues of a large random matrix seem to have a universal distribution that depends on only one parameter, the modal density, as mentioned above. This behaviour has also been observed experimentally for systems ranging from aluminium blocks to nuclear energy levels [8,9]. Two questions arise from these observations: (i) why does a universal distribution arise, and (ii) what is the mathematical form of this distribution? Strangely the answer to question (ii) is better established than the answer to question (i). There are various approaches to question (i), but it would be fair to say that there is no universally agreed definitive argument; for example, in reference [10] it is argued that the universal distribution arises from the dominance of the Vandermonde determinant that appears in the transformation from the distribution of the matrix entries to the distribution of the eigenvalues. Regarding question (ii), the answer has been decided by considering a special type of random matrix that is amenable to analytical study. This matrix: (a) is symmetric, (b) has independent zero mean Gaussian entries, (c) has off-diagonal entries of equal variance, and (d) has diagonal entries with variance equal to twice that of the off-diagonal entries. These properties are unchanged under an orthogonal transformation (for example a rotation of the coordinates), and hence this special ensemble of random matrices is known as the Gaussian Orthogonal Ensemble (GOE). The statistical properties of the eigenvalues of the GOE are derived in the text by Mehta [9]. In addition to the joint probability distribution of the eigenvalues, mathematically convenient functions known as distribution functions and cluster functions are also derived. The entries of the eigenvectors of the GOE are approximately statistically independent and Gaussian, although in some cases correlations between the entries can be important, as discussed by Brody et al [11]. The principle of universality implies that these results will apply quite generally to large random matrices, not just the GOE, and in particular the results can apply to the natural frequencies of a random engineering system.

Just to be clear, there is no claim in the foregoing argument that the random system matrices that appear in vibration and acoustics have the form of the GOE. Rather the claim is that the matrices yield a universal eigenvalue distribution, and furthermore this distribution is the same as that obtained for the GOE. It is important to note that there are limitations to the applicability of the principle of universality. The system must be sufficiently random in two senses: (i) there must be strong mixing of the eigenvectors across the random ensemble, i.e. the eigenvectors of one realisation of the matrix must have a non-zero projection onto a reasonable number of the eigenvectors of any other realisation, and (ii) the typical random shift in an eigenvalue must be significant compared to the mean

eigenvalue spacing (a condition referred to as statistical overlap [12]). Condition (i) will generally guarantee condition (ii), but condition (ii) can occur without condition (i), for example in a beam of uniform but random density, in which case the GOE statistics will not occur. In view of these conditions, the GOE distribution is not likely to apply to the lowest natural frequencies of a system, due to the low sensitivity of these frequencies to uncertainties, but is more likely to apply to higher natural frequencies where the sensitivity to uncertainty is much greater. Furthermore, due to condition (i), the GOE is more likely to apply to systems that have global mode shapes, i.e. the mode shapes are non-zero over the whole system. For built-up systems there is a strong tendency for the mode shapes to be localised to particular regions of the system, and in this case the GOE can be applied instead to the uncoupled (or "blocked") modes of the individual subsystems that comprise the total system, as discussed in Sections 4 and 5.

Random matrix theory provides a powerful tool for predicting the response statistics of random engineering systems, even in the absence of detailed knowledge of the statistics of the underlying physical uncertainties. Examples for relatively simple systems are given in the next subsection, and then the case of more complex built-up systems is considered in the following sections.

3.2 Random Matrix Theory and Frequency Response Functions

A frequency response function (FRF) of a system is defined as some measure of the forced response divided by some measure of the applied force, expressed as a function of the forcing frequency. For example, if a harmonic point load is applied to a system then we might consider an FRF consisting of the complex amplitude of the response at some location, divided by the complex amplitude of the point load, expressed as a function of the forcing frequency. Alternatively, we might consider an FRF consisting of the total kinetic energy of the system, divided by the square of the amplitude of the point load, again expressed as a function of the forcing frequency. If the system has random properties then any particular FRF will be different for each possible random configuration of the system. The collection of different curves (as a function of frequency) is known as the ensemble, and an example is shown in Figure 2 [13] for the case of a simply supported plate that has a number of small masses attached at random points.

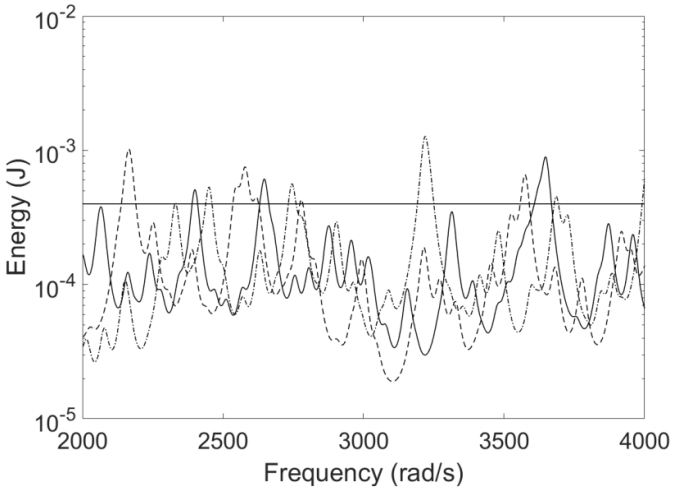

Figure 2 Three members of an ensemble of random FRFs for a plate with masses attached in random locations [13]. The line represents a critical level of response.

The FRF in this case is the total vibrational energy of the plate produced by a unit point force applied at some fixed location, and three members of the ensemble are shown corresponding to different

random arrangements of the masses. To help with design we might seek the following information about the ensemble of curves:

- the mean and variance as a function of frequency
- the statistical distribution at a given frequency
- the probability that the curve will cross a critical value (represented by the line in Figure 2)
- some measure of the fluctuation (with frequency) of the curves

Any FRF of a linear system can be calculated if the following items are known: (i) the mode shapes and natural frequencies, (ii) the damping, (iii) the nature of the excitation. If items (ii) and (iii) are known, and if the natural frequencies and mode shapes have GOE statistics, then providing we know the modal density (the single parameter appearing in the GOE distribution) we have enough information to fully determine the statistical properties of the FRFs. Practical difficulties arise from the fact that the GOE distribution is mathematically complicated, and furthermore the FRF is a nonlinear function of the natural frequencies. These difficulties can be overcome by employing the mathematics of random point process theory [14] together with the cluster (or cumulant) functions associated with the GOE, to yield simple *closed form* results for key statistical quantities [15,16] including those items that are listed above.

As an example, the results obtained using the above approach to predict the mean and variance of the response of a plate were compared to experimental results in reference [17].

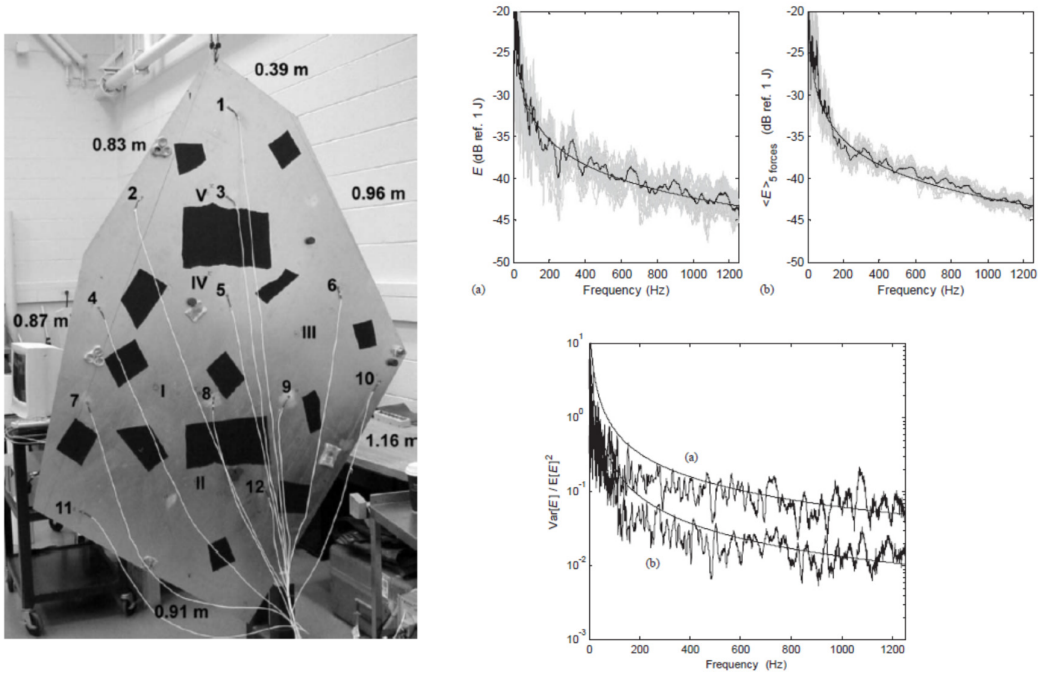

Figure 3 The mean and variance of the energy of a point excited random plate [17]. Left figure – the plate. Upper right - two results for the mean FRF: (a) result for a single point load, (b) an average taken over five different locations of the point load. Lower right – the relative variance of the energy.

The plate was randomised by adding nine masses in random locations, and the kinetic energy of the plate was measured for point force excitation. The masses were employed in 19 different configurations, giving an ensemble of FRFs of this size. The plate itself is shown on the left of Figure 3, and two curves for the mean FRF are shown on the upper right. The first mean curve, labelled (a), corresponds to a single point load, while the second curve, labelled (b), represents a further average taken over five different locations of the point load. The two cases yield practically the same mean

value, and on each plot the analytical mean is compared with the experimental mean and the ensemble of 19 different FRFs. The plot in the lower right of Figure 3 shows the relative variance of the plate energy for the two cases (a) and (b). The variance for case (b) is lower, due to the additional averaging over the load location, and in all cases the theory shows good agreement with the experiment. The divergence of theory and experiment for the variance at low frequencies can be traced to the lack of applicability of the GOE to the lowest modes of the system.

The application of the random matrix (GOE) approach to the other problems listed above (the probability of crossing a critical level, and fluctuation rates) is described in reference [13], where again very simple results requiring minimal computation are obtained. Furthermore, it is shown in reference [18] that the distribution of an FRF at a single frequency is often well approximated by a log-normal distribution, meaning that the mean and variance obtained from the theory can be used to obtain confidence intervals and other statistical measures. A surprising consequence of the GOE that arises for causal FRFs is described in reference [19]: this is the "AE condition", which states that the average value of a function of an FRF is equal to the function evaluated at the average value of the FRF. This property can be of use in experimental work – for example the average impedance can be found by inverting the average of a measured mobility.

It can be concluded that the random matrix approach can be used to predict FRF statistics without computationally expensive finite element or Monte Carlo simulations, providing GOE statistics are applicable. As mentioned in the previous sub-section this is not generally the case for built-up systems, and these systems are considered in the following section.

4 THE RESPONSE OF BUILT-UP SYSTEMS: SEA

As discussed in Section 2.2, in the 1960s there was a research effort to develop a theory of vibro-acoustics that would fit into box (5) in Figure 1, thus avoiding the need for very complex detailed models of the system. To achieve this, the first question to be addressed is the choice of metrics, or degrees of freedom, that are used to describe the response of the system. In statistical mechanics the temperature of a room might be used as a single measure in place of detailed information regarding the state of each molecule of gas, and the temperature is related to the kinetic energy of the gas [6]. In this spirit, Lyon [1] proposed that a complex vibro-acoustic system can be represented as a set of coupled "subsystems" and the appropriate set of degrees of freedom should be the vibrational energy of each subsystem. In this way, instead of employing 10 million degrees of freedom to describe the response of a car using the finite element method, the entire roof panel might be a single subsystem, and the interior acoustic space another, and so on. This approach leads to tens or hundreds of degrees of freedom, rather than millions. Clearly the subsystem vibrational energies do not yield the level of detailed information given by a finite element model, but the information is nonetheless extremely useful for design – for example the vibrational energy of an acoustic volume can be immediately converted into a dB noise level. There is then the question of how the vibrational energies might be found, i.e. is there a set of governing equations for the vibrational energies? Lyon [1] proposed that a set of equations can be derived by considering energy flow for each subsystem: the power input by applied forces must equal the power dissipated by damping plus the net power transferred to connected subsystems. The power dissipated by damping is linearly proportional to the subsystem energy, but it is less obvious how the net power transferred can be expressed in terms of the subsystem energies. Partly by analogy with heat conduction, Lyon proposed that the net power transfer between two coupled subsystems should be proportional to the difference in the subsystem "modal" energies, i.e. the subsystem energy divided by the modal density. The constant of proportionality was termed the "coupling loss factor". The resulting equations are referred to as Statistical Energy Analysis (SEA), the word "statistical" highlighting the fact that the subsystem energies that appear in the equations are ensemble average energies, i.e. corresponding to the average over an ensemble of random systems. To use these equations, the power input to each subsystem from external forcing is computed (often using asymptotic arguments), and then the linear coupled set of SEA equations is solved to yield the vibrational energies.

Since the inception of SEA there have been many attempts to derive the equations from fundamental principles, i.e. to start in box (1) of Figure 1 and then derive the equations that must apply in box (5). Lyon [1] used two distinct approaches, one based on modal analysis and the other based on wave propagation. The main advantage of having some form of fundamental derivation of SEA is that the derivation yields expressions for the coupling loss factors, which are essential components of the theory. Other approaches to the derivation of SEA have included direct analogies to statistical mechanics and the use of the concept of entropy [7,20]. One important point is that the SEA equations are demonstrably "false", meaning that they are not true in general and only emerge in an approximate way from the underlying detailed dynamics under certain conditions. The establishment of these conditions, and the extent to which they apply to any particular engineering system under consideration, has been the source of much debate; much of the tension in this debate can be relaxed at the outset by an explicit recognition of the fact that SEA is always an approximate approach.

As mentioned above, in Lyon's original conception of SEA the subsystem energy was interpreted as the ensemble average of the energy over a collection of random systems. From this point of view there is no reason why the SEA prediction for the energy should agree with energy measurements on a single system tested in a laboratory. One way around this issue is to assume that the response is ergodic, in the sense that ensemble and frequency averages are the same, and to compare measured frequency averages with the SEA predictions. The validity of this approach will depend on the extent to which the ensemble statistics are constant over the specified frequency band, and in some cases the ensemble mean response (for example) can vary fairly rapidly with frequency. An alternative approach is to complement the SEA approach with a prediction of the ensemble variance of the energies, or ultimately with the ensemble probability density function of the energies. In that way confidence intervals can be established for the response, and it would be expected that a laboratory experiment on a single system would fall within these intervals. The SEA equations do not rely on the principle of universality that was discussed in the previous section, but any variance theory, or higher order statistical theory, must make some assumption about the statistical distribution of the properties of the system. If the GOE distribution can be applied to the eigenvalues and eigenvectors of the system then progress can be made, but as discussed in the previous section the GOE does not in general apply to the global modes of a built-up system. This problem was addressed in reference [21], where it was assumed that the modes of each uncoupled (or blocked) subsystem are governed independently by the GOE, and on this basis a relatively simple set of equations was derived for the variance of the energies. The application of SEA then becomes a three step process: (i) the standard SEA equations are used to compute the average energies, (ii) the variance equations [21] are used to predict the ensemble variance of the energies, and (iii) confidence intervals for the energies are established by assuming a suitable two parameter distribution. Regarding step (iii), it is shown in reference [18] that the log-normal distribution usually provides a good approximation.

In reference [17] the SEA approach is applied to a structure composed of a cylinder and three plates, and a comparison is made with experimental measurements; some of the results obtained are reproduced here to provide an example of the methodology. The structure is shown in Figure 4, and the results presented in what follows concern the case in which one of the plates (plate 1 say) is subjected to a harmonic point force. The SEA model consists of four subsystems, one for each of the system components, and so the SEA equations for the mean energies comprise a set of four simultaneous linear equations which are easily solved. The variance of the energies can then be found using the method described in reference [21]. The structure was randomised by the addition of small masses in random locations, and measurements were taken for an ensemble of 25 different mass configurations. Results for the mean and relative variance of the subsystem energies are shown in Figure 5, where good agreement with the experimental results can be seen. It can be seen that the mean energy decreases with distance from the excitation point – the driven subsystem (plate 1) has the highest energy, and the cylinder and other plates have lower energies. In contrast the relative variance of the energy increases with distance from the excitation point; in qualitative terms, more randomness is encountered on longer vibration transmission paths.

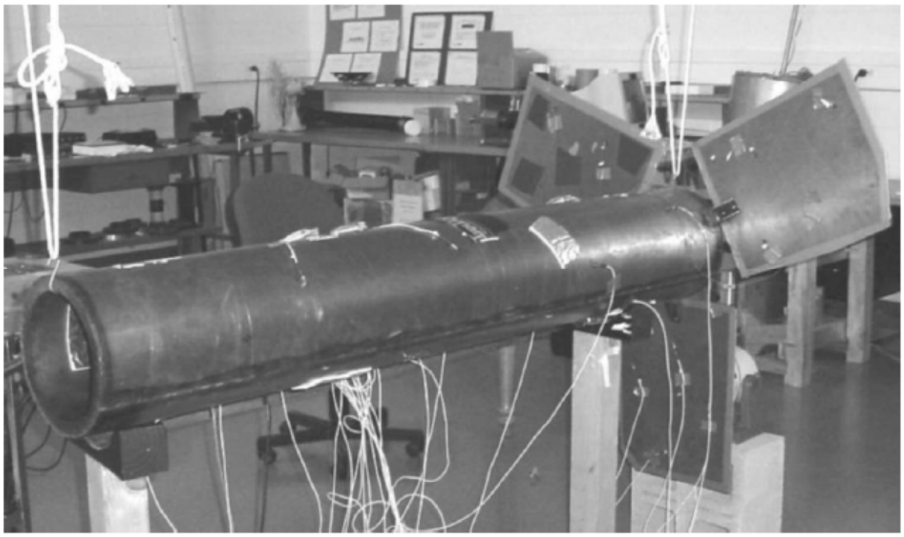

Figure 4 Test structure considered in reference [17], consisting of a cylinder connected to three plates. The plates have equal areas and properties.

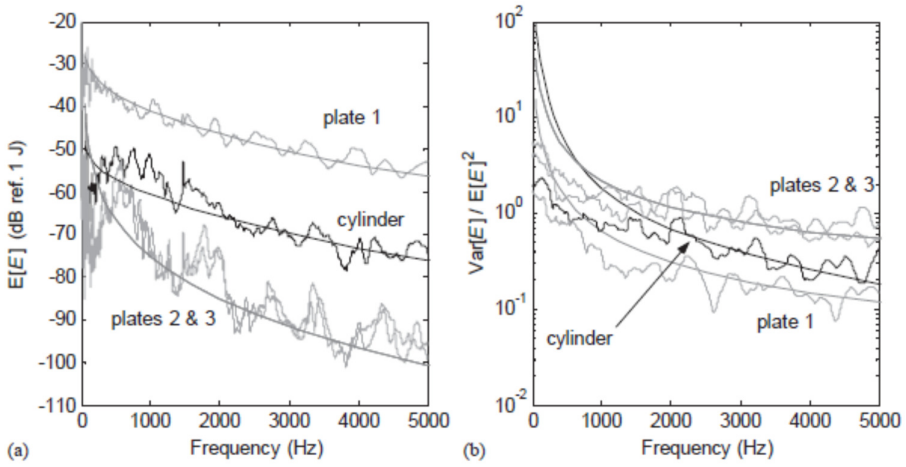

Figure 5 The mean and relative variance of the subsystem energies when plate 1 is driven by a harmonic point force. The smooth curves are the SEA predictions, the irregular curves are experimental measurements on an ensemble of 25 random systems [17].

The worsening prediction of the variance at lower frequencies is due to the fact that the theory relies on the assumption of GOE modal statistics for each subsystem, and this assumption is more valid at higher frequencies. Given the mean and variance, confidence bounds on the subsystem energies can be derived by assuming that the energies have a lognormal distribution. In Figure 6 the results obtained in this way for the 99% confidence levels are compared to the experimental ensemble of results.

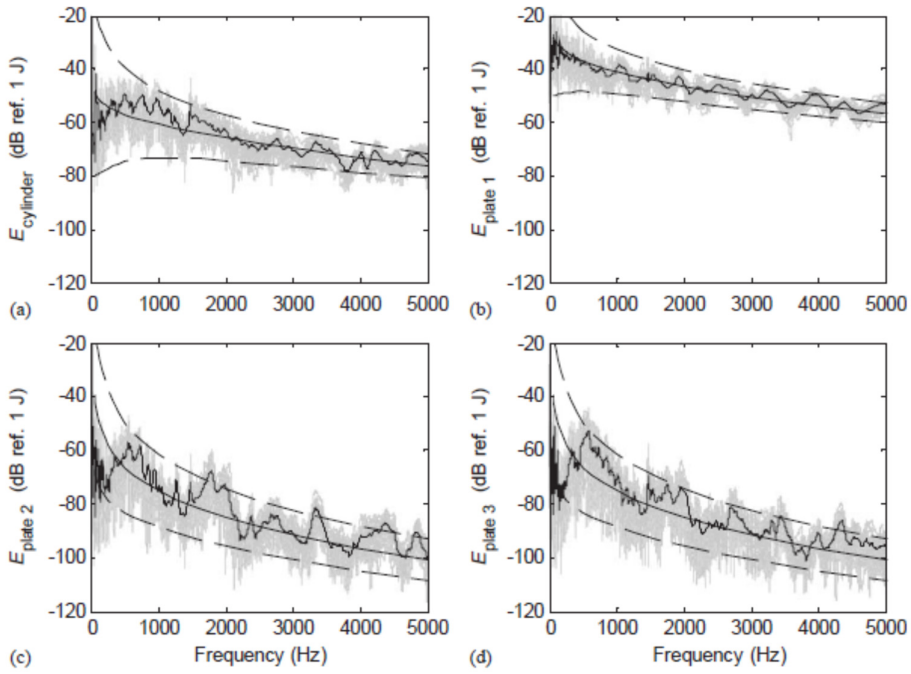

Figure 6 99% confidence bounds (dashed line) compared to the experimental ensemble of results for each of the subsystems The SEA and experimental means are also shown (solid curves) [17].

It should be emphasised that the SEA approach requires only four degrees of freedom for this problem, and so the computational effort needed to produce the results shown in Figures 5 and 6 is minimal. A finite element model of the system over the considered frequency range would require very many degrees of freedom, and moreover the model would need to be randomised and solved multiple times to compute the response statistics.

In reference [22] the SEA mean and variance equations are extended to allow the prediction of a number of higher order statistical properties of the energy FRFs of built-up systems. These include the probability that a critical level will be crossed within a particular frequency band of excitation.

The SEA mean and variance approach described in this section involves the assumption that the modes of each subsystem have GOE statistics. There are systems for which this is not true for all subsystems over any realistic frequency range, and in that case an extension to the approach is required, as discussed in the following section.

4 THE RESPONSE OF BUILT-UP SYSTEMS: HYBRID FE-SEA

There are situations in which a system cannot be represented as an assembly of SEA subsystems with each subsystem meeting, for example, the “GOE requirements” of the theory. For example, the door pillars and side rails of a car have very few modes over the frequency range of interest, and ideally these systems should be modelled using the finite element (FE) method rather than represented as SEA subsystems. If FE is used for such components, and SEA for other parts of the system, then the immediate problem arises as to how to couple the two methods in the same model. This is a challenging problem: the FE equations are based on dynamic equilibrium and compatibility, whereas the SEA equations are based on energy flow. Futhermore, the SEA equations incorporate systems randomness whereas the FE equations do not. A solution to this problem is described in what follows.

The archetypal situation that arises is that of an SEA subsystem surrounded by FE components, and progress can be made by considering the response of the SEA subsystem to have two elements. Firstly we can consider putting prescribed dynamic displacements on the FE nodes that are coupled to the subsystem; this motion will generate waves that propagate into the subsystem and are subsequently reflected from the boundaries. If we ignore the reflections, and consider only the response associated with the initially generated waves, then this part of the response is known as the "direct field response". To view this physically we could imagine there is some device in the interior of the subsystem that absorbs waves, so they never reach the boundary. The direct field imposes forces on the boundary, and there is a linear matrix relation between the forces and the boundary displacements – this relation can be encapsulated in a "direct field dynamic stiffness matrix" that yields the forces given the displacements. Although the direct field dynamic stiffness matrix may sound like a fairly abstract concept, it can actually be readily calculated by a variety of methods [23] and for the purposes of the present argument it can be assumed to be known. So, thus far, we have coupled the FE model to the direct field response of the SEA subsystem, but we have ignored the reflected waves. This part of the response is known as the "reverberant response", and to allow for randomness in the subsystem it can be represented as a diffuse field of random waves. The question then arises as to how these waves will apply forces to the boundary of the subsystem. The answer is surprisingly simple: it turns out that the cross-spectrum of the forces applied to the FE model is proportional to the imaginary part of the direct field dynamic stiffness matrix times the subsystem energy – this result is known as the "diffuse field reciprocity relation" [24]. In summary, the motions of the FE system inject power into the SEA subsystem via the direct field dynamic stiffness matrix, and the SEA reverberant response applies forces to the boundary via the diffuse field reciprocity relation. Given these facts a coupled hybrid FE-SEA analysis method can be developed to allow the use of FE and SEA in the same computational model of a complex structure [25]. The method has two coupled sets of equations: a set for the cross-spectrum of the response of the FE degrees of freedom, and a set of SEA equations for the subsystem energies. The equations are linear in the unknowns, if they are solved in sequence – first the SEA equations, and then the FE equations (to be clear, no iteration is required, a single solve is performed). One additional feature of the approach is that expressions for the coupling loss factors that are required for the SEA equations are found immediately in terms of the direct field dynamic stiffness matrices and the properties of the FE model. The hybrid method was originally developed to yield ensemble average response quantities [25], but was then extended to yield the ensemble variance of the response [26].

An academic example of the application of the hybrid FE-SEA method is given in reference [26]. The structure considered is shown in the upper right of Figure 7, and comprises a stiff framework to which are attached four panels. In the hybrid model, the framework and the in-plane motion of the panels were modelled using FE, and the bending motion of the panels was modelled using SEA. Benchmark results were obtained by performing Monte Carlo simulations for a fully FE model that was randomised by the addition of small masses to the panels to generate an ensemble of 100 systems. A force was applied to a point on the framework (shown by an arrow in the Figure); the response at another point on the framework and the response of a panel are shown in Figure 7, where a comparison is made with benchmark results for both the mean and relative variance of the response quantities. The number of degrees of freedom in the hybrid model is around ten times less than in the full FE model, and the hybrid model yields results for the mean and variance in a single run, obviating the need for repeated Monte Carlo simulations. The reduction in computational effort afforded by the hybrid method is therefore very significant, and the accuracy shown in Figure 7 is good. It can be noted that the underlying modally-sparse deterministic behaviour of the framework is very clear in the response, leading to large scale fluctuations in the mean values (for example). These fluctuations would not be captured by a fully SEA model of the system.

Finally, it can be noted that the hybrid method can be enhanced by the introduction of parametric uncertainty in the FE part of the model [27] and by the incorporation of experimental results [28]. Furthermore, the "FE" part of the model can represent any deterministic method, including the boundary element method (BEM) and computational fluid dynamics (CFD) and so FE, BEM, CFD, and SEA can all be used in the same model.

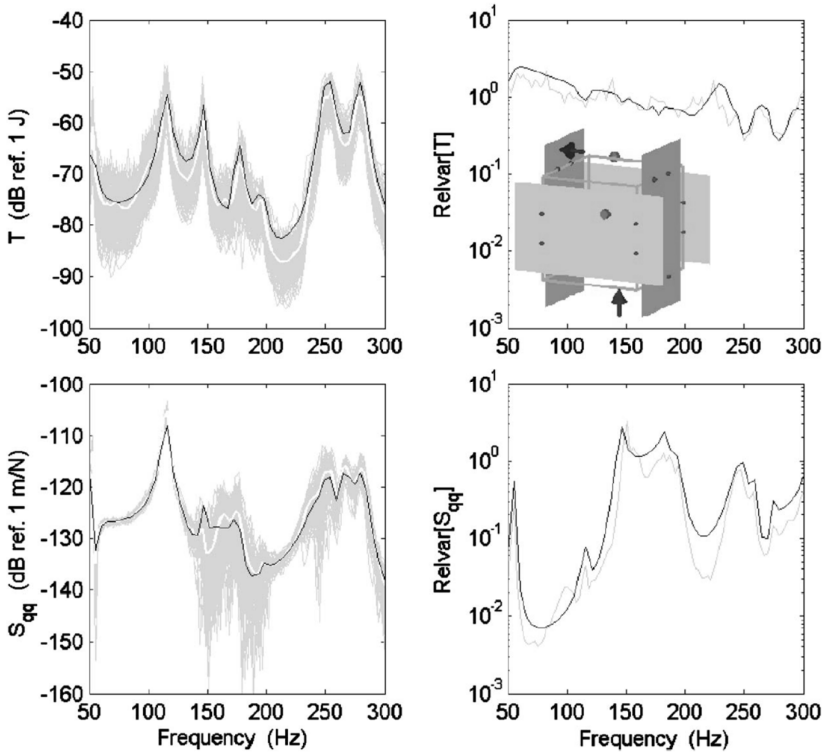

Figure 7 Results yielded by a hybrid FE-SEA model of a plate-beam structure compared to benchmark Monte Carlo results generated using a fully FE model. Top row: the kinetic energy of a panel, bottom row: the cross-spectrum of the response at a point on the framework. Left column: mean values and the Monte Carlo ensemble, right column: relative variance. The darker curve is the hybrid prediction, the lighter curve is the Monte Carlo result [26].

5 CONCLUDING REMARKS

The example problems that have been considered in this paper are academic-type problems drawn from the literature, i.e. the examples concern idealised systems that have been selected to show the main features of the various methods. To apply the methods to a range of real engineering systems requires the theory to be incorporated into a general purpose software package. Aspects of the methods presented here were implemented in the commercial package AutoSEA, developed by Vibro-Acoustic Sciences Inc (Vasci), and then in the package VA One when Vasci was acquired by the ESI Group. The methods are implemented in the Dassault Systemes code Wave6.

As mentioned in the introduction, it should be emphasised that the present paper has focussed on work in which the author has been directly involved over past decades, and this is reflected in the reference list. Of course, a range of other approaches are available in the literature.

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