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FRICTION, DAMPING AND BOWED STRING PITCH AND TIMBRE

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ABSTRACT

Any analysis of the physics of bowed string dymanics must consider in detail the nature of the friction between bow and string. For example, the simplest model of friction in which the opposing force is independent of the sliding speed, will predict steady slip rather than self-excited vibrations. The friction law is thus critical to a realistic model of the bowed string. An empirical friction law is proposed which is a function of both sliding velocity and acceleration. The effect of this law in the generation of self-excited vibrations is demonstrated for a single degree of freedom (spring-mass-damper) system with various levels of damping. This suggests that, for sufficiently energetic excitation, vibration rather than steady slip will occur, even for highly damped systems. The argument is extended to multi degree of freedom systems (bowed string) to analyse the effect of modal damping and friction law on the relative excitation of modes, and thence to decribe the influence of bowing conditions on timbre.

1. INTRODUCTION

To simplify the analysis as much as possible, but retaining the essential features, the string is modelled as a block, attached to a rigid wall by a simple spring and dashpot. The system is driven by the frictional force between the string and bow: a simple one degree of freedom structure with a non-linear excitation term. The configuration is shown in Fig. 1. The governing equation for this system is

$$m\ddot{x} + r\dot{x} + sx = F(\dot{x}, \ddot{x}), \tag{1}$$

where m is the mass of the block, s is the spring constant and r is the damping coefficient. The frictional force is given by $F(\dot{x}, \ddot{x})$.

The simplest model for friction (Coulomb's law) assumes that the frictional force is independent of the sliding speed. However, there are many instances, including bowed string dynamics, where this model does not give predictions which have the correct quality. The observed phenomenon is that a small initial disturbance in the string–bow system can be amplified by energy transfer from the bow to the vibrating string, even under significant levels of damping.

Smith [1] showed that an important factor was the friction induced heating and the effect it has on the rosin. Other investigators have shown that there is a velocity dependence, and Lindop and Jensen [2] demonstrate this computationally based on a qualitative understanding of surface interactions. The experimental results of Wang [3] show "hysteresis" for relative velocities near to zero. As the relative velocity approaches zero, the frictional force rises to local maximum and then begins to fall. It does not reach zero until the relative velocity has actually changed sign and then it finally reaches a lower minumun than the initial maximum. Hunt *et al* [4] also noted this effect and concluded that the frictional force was dependant on another variable besides velocity, and proposed that this be acceleration. The model presented has been constructed to match these results qualitatively.

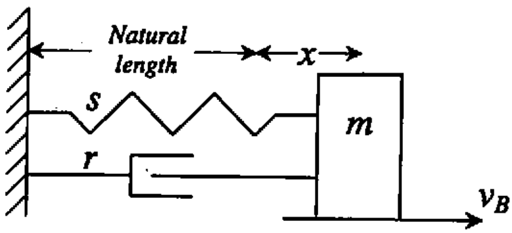

Fig. 1 Friction driven spring–mass–damper system.

2. THE PROPOSED FRICTION MODEL

It is well known that the elastic limit for a microscopic piece of material is far higher than for the bulk material, and the frictional contact area is comprised of many small contacts on the surface irregularities. Thus it would seem reasonable that the two contacting bodies could move a small distance relative to each other, without disrupting the temporary bonds between them, and the opposing frictional force would be predominantly due to the elastic force applied by the stretched irregularities. Such bonds would remain unbroken for a period of time equal to the maximum extension of the bond divided by the velocity. For very low relative velocities, this time period could introduce a hysteretic effect [A J McMillan 5], whilst for moderate velocities the frictional force might be expected to increase in accordance with the increase of the number of bonds being broken in a time period. We refer to this quantity, τ, as the "characteristic period" of the interface.

The mathematical model proposed considers frictional force as a function of slip velocity and acceleration. Given that the bow speed is v_B, the relative velocity of the sliding surfaces v_R is $v_R = \dot{x} - v_B$. If the bow is taken to have constant velocity, the relative acceleration is the same as the instantaneous acceleration of the surface of the string, $\ddot{x}$. Now, for reasonably high slip velocity, the model should predict Coulomb's law, whilst for velocities close to zero stick type behaviour should be manifest. Thse two types of behaviour are introduced by having two parts to the friction function;

$$F_1(v_R, \ddot{x}) = -\mu_k mg \frac{2}{\pi} \arctan\left(\frac{v_R \mu_k}{|\ddot{x}|\tau}\right), \tag{2}$$

which approximates to $F_1 = \mp\mu_k mg$ for large and steady slip velocities, and

$$F_2(v_R, \ddot{x}) = mg \begin{cases} -\dfrac{|\ddot{x}|\tau}{v_R - \ddot{x}\tau/\mu_k}, & \text{if } \operatorname{sgn}(\ddot{x})v_R < 0 \text{ or } > \dfrac{2|\ddot{x}|\tau}{\mu_k}; \\ \operatorname{sgn}(\ddot{x})\mu_s \sin(\Omega v_R + \Phi), & \text{otherwise.} \end{cases} \tag{3}$$

In the above equations μ_s is the coefficient of static friction, μ_k is the coefficient of kinematic friction in the Coulomb friction regime and τ is the "charateristic period" of the interacting surfaces, which determines the "width" of the quasi static friction region. In addition, $\Phi = \arcsin(\mu_k/\mu_s)$ and $\Omega = \mu_k(\pi - \Phi)/\ddot{x}\tau$. The overall frictional force is given by $F(v_R, \ddot{x}) = F_1(v_R, \ddot{x}) + F_2(v_R, \ddot{x})$. Fig. 2 shows this as graph of frictional force versus relative sliding velocity for a mass sliding with sinusoidal motion, $x = 0.5 \times 10^{-6} \sin(200t)$, relative to a surface. The shape of this graph compares well with the experimental results presented by Wang.

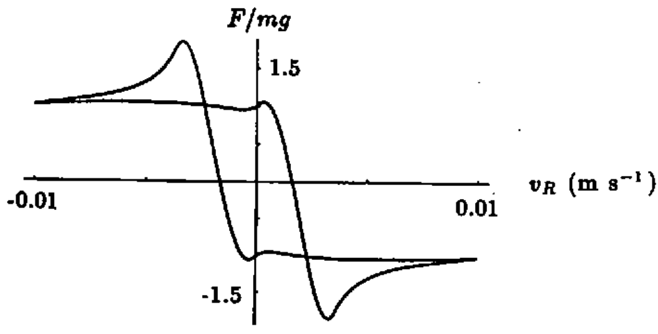

Fig. 2 Friction force *vs* sliding velocity.

3. THE FRICTION DRIVEN SYSTEM

It is convenient to use a non-dimensionalised form of the governing equations;

$$\ddot{\bar{x}} + 2\nu\dot{\bar{x}} + \bar{x} = -\frac{2}{\pi}\arctan\left(\frac{\dot{\bar{x}} - \lambda}{|\ddot{\bar{x}}|\bar{\tau}}\right)$$

$$+\begin{cases} -\dfrac{|\ddot{\bar{x}}|\bar{\tau}}{(\dot{\bar{x}}-\lambda)-\ddot{\bar{x}}\bar{\tau}}, & \text{if } \operatorname{sgn}(\ddot{\bar{x}})(\dot{\bar{x}}-\lambda)<0 \text{ or } >2|\ddot{\bar{x}}|\bar{\tau}; \\ \dfrac{\operatorname{sgn}(\ddot{\bar{x}})}{\sin\Phi}\sin\left(\dfrac{(\pi-\Phi)(\dot{\bar{x}}-\lambda)}{\ddot{\bar{x}}\bar{\tau}}+\Phi\right), & \text{otherwise.} \end{cases} \quad (4)$$

Here $\bar{x} = \dfrac{s}{m}\dfrac{x}{\mu_k g}$ is the non-dimensionalised displacement, $\bar{\tau} = \dfrac{\tau}{\mu_k}\sqrt{\dfrac{s}{m}}$ is the non-dimensionalised characteristic period, and the other two independent variables are $\nu = \dfrac{r}{2\sqrt{ms}}$ and $\lambda = \dfrac{v_B}{\mu_k g}\sqrt{\dfrac{s}{m}}$. From this point the over bar notation is dropped.

We write the right hand side of equation (4) as the functional $F(\dot{x}(t), \ddot{x}(t))$:

$$\ddot{x} + 2\nu\dot{x} + x = F(t). \quad (5)$$

If the system is started from rest at $t = 0$, then

$$x(t) = \int_0^t g(t-\grave{t})F(\grave{t})d\grave{t}, \quad (6)$$

and $g(t) = \frac{1}{\omega}e^{-\nu t}\sin(\omega t)$, where $\omega = \sqrt{1-\nu^2}$. It is convenient to define two functions

$$S^{(1)}(t) = \int_0^t e^{-\nu(t-\grave{t})}\cos\{\omega(t-\grave{t})\}F(\grave{t})d\grave{t} \quad \text{and} \quad S^{(2)}(t) = \int_0^t e^{-\nu(t-\grave{t})}\sin\{\omega(t-\grave{t})\}F(\grave{t})d\grave{t}.$$

Then,

$$\dot{x}(t) = S^{(1)}(t) - \frac{\nu}{\omega}S^{(2)}(t) \quad \text{and} \quad \ddot{x}(t) = \left(\frac{\nu^2}{\omega} - \omega\right)S^{(2)}(t) - 2\nu S^{(1)}(t) + F(t).$$

These coupled integral equations must be solved iteratively. We assume that $F(t)$ may be approximated by $F(t) = F_j$ for $(j-1)\Delta t < t < j\Delta t$, for sufficiently small time steps Δt. Then the integrals $S^{(1)}(t)$ and $S^{(2)}(t)$ may be written in terms of their values at previous time steps;

$$S_j^{(1)} = T_2 S_{j-1}^{(1)} - T_1 S_{j-1}^{(2)} + (T_6 + T_3)F_j \quad (7)$$

and

$$S_j^{(2)} = T_1 S_{j-1}^{(1)} + T_2 S_{j-1}^{(2)} + (T_4 - T_5)F_j, \quad (8)$$

where,

$$\begin{aligned} T_1 &= e^{-\nu\Delta t}\sin(\omega\Delta t) & T_2 &= e^{-\nu\Delta t}\cos(\omega\Delta t) \\ T_3 &= \omega e^{-\nu\Delta t}\sin(\omega\Delta t) & T_4 &= \omega\left[1 - e^{-\nu\Delta t}\cos(\omega\Delta t)\right]. \\ T_5 &= \nu e^{-\nu\Delta t}\sin(\omega\Delta t) & T_6 &= \nu\left[1 - e^{-\nu\Delta t}\sin(\omega\Delta t)\right] \end{aligned} \tag{9}$$

To iterate, start by putting $F_j = F_{j-1}$ and use this to calculate $\dot{x}$ and $\ddot{x}$. Then recalculate F_j based on these values. When the iteration has converged, reset $S^{(1)}$ and $S^{(2)}$; only the most up to date value of these needs to be stored. The values for F_0 and $\ddot{x}$ at $t = 0$ are calculated interatively, based on setting the block velocity $\dot{x}(0) = 0$. In the first iteration, the block acceleration $\ddot{x}(0)$ is taken as zero.

So far we have solved for initial conditions $x(0) = 0$, and $\dot{x}(0) = 0$. For $x(0) = x_0$ and $\dot{x}(0) = v_0$, we may write $\dot{x}^* = \dot{x} - v_0$ and $x^* = x - x_0 - v_0 t$, so that

$$\ddot{x}^* + 2\nu\dot{x}^* + x^* = F^*(t), \tag{10}$$

where $F^*(t) = \left\{F(t) - 2\nu v_0 - (x_0 + v_0 t)\right\}$. The problem can now be solved exactly as before, but in terms of x^* and F^*.

4. RESULTS OF THE NUMERICAL SIMULATIONS

In the results presented in Figs. 3 and 4, the outer loop represents self-excited vibration; the fixed point represents a condition of steady slip at contant velocity. When initial conditions are sufficiently close to the outer loop, the motion of the string will increase so that the trace on the phase plane will spiral out to meet it. This represents an increase in the total energy in the string; the source of this energy is the bow. It is transmitted to the string by frictional force, at a rate great enough to overcome energy losses through damping. If the initial conditions are close to the fixed point representing steady slip, the trace will spiral in to meet that condition. In this case the friction and damping combine to act as a mechanism for removing energy from the string.

The results presented in Fig. 3 are for a system with $\nu = 10^{-3}$, $\lambda = 3.0$, $\tau = 0.1 \times \sqrt{10}$ and $\Phi = 1.0$. The unstable limit cycle is rather small, and in particular, does not enclose the point $(0,0)$, which might be considered as the "usual initial condition". One might consider this system to be prone to self-excited vibration, with a frequency of about 0.96 rad s^{-1}. The results for a similar system having greater damping, $\nu = 10^{-2}$, are given in Fig. 4. In this case the unstable limit cycle is very much larger and encloses the point $(0,0)$, so is prone to steady slip behaviour. However, given sufficiently energetic initial conditions it can be made to self-excite, with frequency 0.98 rad s^{-1}, and this outer loop is rather similar to that in Fig. 3. Since the unforced frequency of both of these systems is 1 rad s^{-1}, the friction excitation mechanism is "pitch flattening", a phenomenon described by McIntyre and Woodhouse [6].

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In both cases the step in the outer loop occurs when $0 < \mathrm{sgn}(\ddot{x})(\dot{x} - \lambda) < 2|\ddot{x}|\tau$, *i.e.* when the relative velocity is passing though zero and the friction becomes "quasi static". The graphs presented show only one cycle; subsequent cycles have similar, but not identical paths and durations, and the position of the step can shift along the top surface. In some cycles there may be more than one step. Although the motion is not strictly periodic, it is clear that the phase plane paths are confined to lie within a finite band, so it may be classified as "quasi-periodic". This is a consequence of hysteresis in the friction law.

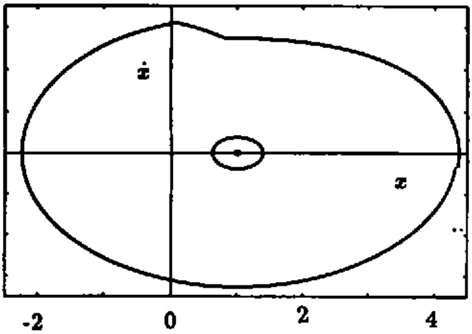

Fig. 3 Phase plane plot for $\nu = 10^{-3}$.

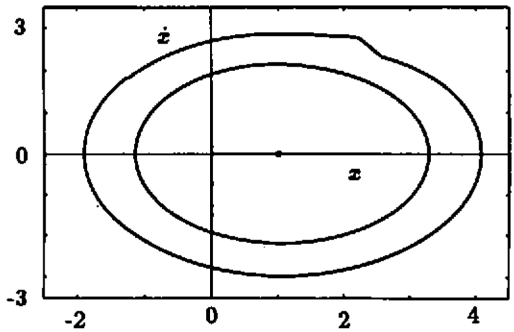

Fig. 4 Phase plane plot for $\nu = 10^{-2}$.

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F Holes and Bass Bar Effects on Plate Tuning

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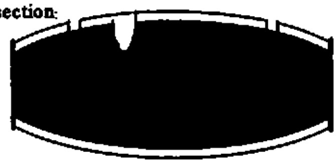
In this study, measuring the frequencies has been used as a tool to guide the working process of the next instrument being built.
While making the back and belly of an instrument of the violin family, can the maker anticipate what tuning is to be obtained on the finished front ?
Traditionnally, one does not retouch from the inside the thicknesses ot the table after the f holes and bass bar are done, in order to obtain the nice smooth and regular internal curve that is usually aimed for. That is why the maker must know in advance the effects the cutting of holes and the addition of bar will have on the frequencies of modes, in order to achieve the final tuning he wishes, between the modes of the front and with those of the back.
In that way, 3 violins, 4 violas and 3 cellos were measured during their making, between 1985 and 1990, with a very simple equipment, thanks to Carleen Hutchins's directions given in CAS NL #39, 1983.
Some conclusions are given on the influence of holes (position, shape and cutting) as well as of the bass bar (gluing and shaping), on the tuning of the instrument's table.

This study has been conducted in the course of making the following 10 instruments: violins n°6, 7 and 8, violas n°2, 3, 4 and 5 and cellos n°1, 3 and 4. Cello n°1, made in 1981 has in fact had a second front made for wich was built after cello 3 & 4. The musician playing it had a bus accident and the front had been severely damaged, needing replacement.
The method used is the one indicated by Carleen Hutchins to measure modes 1, 2 and 5 with the help of a frequency generator, a high speaker, and we used powdered sugar or copper sulphate to visualize the modal lines, thus checking wich mode we were up to. The shapes were very predictable and regular whith traditionnal archings and thicknesses coherent and symetrical as Sacconi describes them.
The goal we were aiming for was the evaluation of the effects of F holes (FF) and bass bar (BB) on those 3 modes, to anticipate the change of frequency they were going to imply, while working the thicknesses on the next instrument.
As a matter of fact, the process of making the instruments was made in the following order:

1. making the ribs
2. cutting the outline of back and belly from the ribs's shape
3. carving the outside archings of both plates and insert the purfling
4. carving the inside of back and belly and thickness them to a certain tuning
5. cutting f holes
6. gluing and shaping bass bar
7. getting the final tuning between back and front.

To obtain the regular inside curve of the instrument, this order schould be kept without touching the table's thicknesses after stage 4, otherwise one cannot control any more the shape of the internal volume of the front.

transverse section:



Moreover, to change frequencies significantly after FF and BB means scraping wood in an irregular manner, wich damages the FF, and leaves dirty ridges of wood along the bass bar and at its end. That does leave irregular stiffness points and the maker knows that empirically.

We have measured the frequences of modes 1, 2, 5 before, after and during stages 5 (FF) and 6 (BB) of the working process:

instrument	belly without ff and bb			belly with ff cut			belly with ff and bb		
	1	2	5	1	2	5	1	2	5
violin n° 6	65	182	389	92	170	333	93	175	372
violin n° 7	90	170	356	82	160	307	90	175	353
violin n° 8	86	164	342	79	153	304	86	162	349
viola n° 2	83	122	297	68	115	271	75	132	301
viola n° 3	63	123	279	58	116	239	64	128	277
viola n° 4	62	107	253	56	102	229	68	115	265
viola n° 5	68	113	254	60	104	218	71	111	256
cello n° 3	29	52	117	27	51	102	32	59	126
cello n° 4	30	53	117	27	52	103	31	63	122
cello n° 1	36	57	141	32	53	117	37	62	143

If one consider Carleen Hutchins's proposal of being within 1 to 4 % to consider to be « in tune » or « equal » in frequency, one sees here that the differences are quite subtle between the instruments, wich are made on the same mould (exept for violin n° 6) and with close archings for each category.

Comparison of the effect of ff and of the cumulated effect of ff + bb tells us more:

instrument	belly without ff and bb			effect of ff			effect of ff + bb		
	1	2	5	1	2	5	1	2	5
violin n° 6	65	182	389	27	-12	-56	28	-7	-17
violin n° 7	90	170	356	-8	-10	-49	0	5	-3
violin n° 8	86	164	342	-7	-11	-38	0	-2	7
viola n° 2	83	122	297	-15	-7	-26	-8	10	4
viola n° 3	63	123	279	-5	-7	-40	1	5	-2
viola n° 4	62	107	253	-6	-5	-24	6	8	12
viola n° 5	68	113	254	-8	-9	-36	3	-2	2
cello n° 3	29	52	117	-2	-1	-15	3	7	9
cello n° 4	30	53	117	-3	-1	-14	1	10	5
cello n° 1	36	57	141	-4	-4	-24	1	5	2

The weight of the plate was recorded at each step:

instrument	weigths of the front in gramms				weight in gramms of:		
	plain	with ff	with bb		ff	bb	ff + bb
violin n° 6	92	88	90		4	2	-2
violin n° 7	76	73	79		3	6	3
violin n° 8	72	71	75		1	4	3
viola n° 2	106	105	112		1	7	6
viola n° 3	103	100	107		3	7	4
viola n° 4	89	85	92		4	7	3
viola n° 5	88	87	94		1	7	6
cello n° 3	513		480				-33
cello n° 4	490	485	517		5	32	27
cello n° 1	565		592				27

For violin n°6, the front was pretty heavy to start with, having been left thicker and heavier. The consequence was that the cutting of holes took 5 g of wood off. The back had a fifth mode tuned at 350 Hz, and to get the table closer, the bar has been thinned down up to 2 g, without going lower that 372 Hz. The bar has been worked a lot sideways and in its cuvature to evoid to affect too much its strength , but the result was absurd with a weight of 2 g and not the right tuning. One could conclude here that FF and BB could'nt compensate for that table wich was too stiff in the fifth mode to have it an octave higher than the second mode.

In this instrument, the ff holes had the effect of lowering the second mode by 12 Hz and the bar had raised it by 5, giving at the end the 175 Hz that were wanted: The tuning of mode 2 with the back was obtained, wich gives a easy response on the finished instrument. Was it possible to get the same result on the next violin n° 7 if we were going to thin down the table to get the fifth mode an octave appart ? What frequency was going to have mode 1 ?

For violin n°7, it was decided to lower the 5th mode to 356 Hz before the ff holes would be cut, knowing that it had been possible to lower it down 56 Hz with ff and go up 39 Hz with a very small bar in violin n° 6.
At the same time, a detailed study of the cutting of f holes of violin n° 7 was then undertaken in order to understand how their shape and position influences the rigidity of the table:

stage of work	weight	mode 1	mode 2	mode 5
before ff and bb	**76 g**	**90**	**170**	**355**
drilled top holes at diameter 5.5mm	id	90	170	362
drilled bottom holes at diameter 5.5 mm	id	90	171	359
fine saw cut from top to bottom hole left	75	**86**	**167**	**330**
fine saw cut from top to bottom hole right	75	**83**	**163**	**314**
opening arms tow. bridge: 74 between ff	74	83	162	315
open. top holes up diagonally tow. centre	74	83	163	316
open. bott. holes down diag. tow. outside	73	83	163	**312**
widening the bottom holes tow. wing	73	83	163	312
opening the bottom holes tow. the CC	73	83	**161**	312
widening the arms towards the CC	73	83	160	**309**
last opening of arm towards bridge 1mm	73	83	160	310
finishing upper curve of top holes	73	82	160	309
finishing bottom hole	**73**	**82**	**160**	**308**

Nearly all the flexibility is given with a very fine sawcut made by a jeweller's saw from top to bottom of the Fs. It is useful to note that the opening of the arms do not affect the tuning, so it gives the chance to tune the helmholtz resonance of the body in widening the f holes, without the fear of giving too much flexibility to the table. It must be understood that the FF were positionned with the outside line of the arms lying along outside arching's level lines, and that line was sawn.
The experience on violin n°7 was worthwile: the balance in mode 5 between ff holes and bb giving a lowering of 49 Hz followed by a highering of 46 HZ. The tuning at 353 Hz can be considered as a success ! Second mode was an octave lower at 175 Hz (note F) and mode one between F and F sharp.

For cello n° 4, one first tried to compensate completely the effects of ff with the bass bar.
The table was thinned down to the following tunins and the work was recorded:

stage of work	**weight**	**mode 1**	**mode 2**	**mode 5**
before ff and bb	**565**	**30**	**53**	**117**
with ff cut		**27**	**52**	**103**
with bb glued, not shaped	627	34	67	139
bb finished	**592**	**31**	**63**	**122**
reworked thicknesses (should'nt be done)	545	29	56	118
bar lowered	542	29	56	116
reworked thicknesses for lowering mode 2	528	29	54	114
lowered bar	*522*	*28*	*53*	*112*
varnished instrument:table taken off		***29***	***58***	***119***
bass bar taken off		**24**	**48**	**96**
new bar not shaped	534	31	59	128
new bar shaped	**529**	**30**	**58**	**125**
3rd bar shaped	**530**	**29**	**58**	**123**

While working the back, one did not succed in having the aimed frequencies: (mode 2: 55 Hz, mode 5: 119Hz). At that point, it was decided at least to tune the second mode, wich gives an easy response, and Carleen Hutchins proposes it as a priority.
One sees the difficulties to get the right tuning when the tuning is not right before ff and bb.
The reworking was meant to lower mode 2, but we went too far in that direction, the mode 5 became too low and we had to take the front off the finished instrument to change the bar. We then could see the effect of the dryed varnish on the modes. With the new bar, we did not try any more to get the fifth mode to the same tuning as the back. At the end of the same year a sinking of the table occured and the bar was changed again, with a bigger tension when it was glued. Mode 2 and 5 are a semi-tone appart.
One can see that whatever effort is made to use the shaping of the bar to affect the tuning, it is always the fifth mode that goes down quite a lot, because of the situation of the bar on nodal line of the belly. It was tried to do first a final height at the center of the bar, measuring the modes, and then shaping progressively the arms: What happened is that mode 1 and 2 were practiquely unmoved (2 Hz during the whole process), while mode 5 was lowered progressively.

For cello n° 1, the work of ff holes 1 was also studied. The tuning of the back, still on the instrument, was recorded to be C (Ideally 65 / 130 Hz but it had been tuned by ear). So one planned to tune the table to that note. But the front was too stiff in mode 5 and was reworked after ff and bb were finished. Mode 2 ended up « in tune » with the back at 62 Hz, but the fifth mode was more than a semi-tone higher than the back's.

stage of work	weight	mode 1	mode 2	mode 5
before ff and bb	**513 g**	**36**	57	**141**
drilled bottom holes at diameter 6.5 mm	id	36	59	141
drilled top holes at diameter 6.5mm	512	36	59	141
fine saw cut from top to bottom hole left	512	35	58	**135**
fine saw cut from top to bottom hole right	512	34	58	**131**
opening arms tow. bridge: 157 between ff	**507**	34	58	131
hollowed the bottom wing from outside	505	34	58	131
open bottom holes	502	34	57	129
joining bottom holes to arms	501	33	57	129
finished top holes and joints with the arms	**500**	**34**	**56**	**128**
thicknesses reworked		**32**	**53**	**117**
after ff and bb		**37**	**62**	**143**

For viola n°5 and violin n°8, an expected effect on the tuning of ff and bb was aimed for, a simple average of what as measured on the preceeding instruments of the same type.

instrument	aims with ff and bb			bb:expected effect			ff: expected effect			plain table tuning		
	1	2	5	1	2	5	1	2	5	1	2	5
violin n° 8	82	165	330	8	18	50	-8	-10	-50	82	167	330
viola n° 5	55	110	220	14	15	55	-5	-6	-35	64	101	200

The tables weres tuned at stage 4 and not retouched. For the viola, the tuning of mode 5 at 220 Hz was impossible with that model (41 cm long, and pretty narrow). It may be worth trying around 65 / 130 / 260 Hz. The plain table tuning started up at 67 / 111 / 256 Hz for a final tuning at 71 / 111 / 256 to match the back on its fifth mode: 84 / 127 / 258.
The violin started with 85 / 164 / 344 before ff and bb. It got closed to what was wanted with modes 2 at 162 Hz for both plates (E), mode 5 at 340 Hz in the front and 326 Hz in the back (between Eb and E#), and mode 1 was at 86 Hz in the belly (below E#).

In conclusion, we succeded to foresee a tuning on violins, but violas and cellos were not as predictable. Their models are more variable and the notes wanted may have been too low for their sizes. The ff holes give flexibility to what the makers call the « pump effect » (mode 5) 4 to 5 times more than the lateral bouncing (mode 2), and that, more in their length and diagonal cutting of the arching than in their width. In violins n° 7 and 8 and in violas n° 3 and 5, ff and bb have nearly been able to compensate each other's changes of tune of modes 2 and 5. In cellos, the ff holes practically have'nt affected the mode 2, while the bb sometimes doubles the frequency the fifth mode had lost with the ffs. A reasonable choice in the height and shaping of the bar can affect around 3 to 4 times more mode 5 than mode 2.