POSITION MEASUREMENT OF VERTICAL LINE ARRAYS USING MULTI-DIMENSIONAL SCALING

A.A. Beard[1] & P.R. Atkins[2]

[1] DERA(Winfrith), [2] University of Birmingham

1. INTRODUCTION

The motivation for this work arose from the need to measure accurately the position of a large number of photomultiplier sensors used in an underwater neutrino telescope. Such a device might take the form of a very large-scale open framework, built using semi-rigid construction techniques and located in a deep oceanic area. The structure would be expected to experience extensive deformations due to gravity and ocean currents. The concepts of using wide-band, spread-spectrum, time-of-flight measurements were investigated and demonstrated using air acoustic scale models. Spread-spectrum techniques were used to provide continuous estimations of the required measurements and to allow other active and passive sonar systems to operate within the same frequency band. Multi-dimensional scaling methods are a key component in the estimation of the geometry of the sensors from the time-of-flight measurements. These methods provide a robust performance in the presence of measurement noise and non-reciprocal sound speed velocities.

It became apparent that these multi-dimensional scaling methods could be effectively applied to other acoustic systems which deform during deployment. Typical examples might include flexible arrays deployed from ships or sonobuoys and bottom mounted vertical line arrays as shown in Figure 1. Normal practice would be to tether the vertical line arrays to a sea bed mounted base units which would often contain additional hydrophones for conventional ping-around calibrations and for tracking purposes. Using the techniques presented here, a minimum of four stable reference points are required, which should ideally be distributed across the three principal co-ordinate axes. In order to determine the position and attitude of the active sections of the line array, a minimum of two sensors capable of transmitting and receiving are required. This requirement increases with the order of the polynomial shape-fitting model applied to the active section of the array.

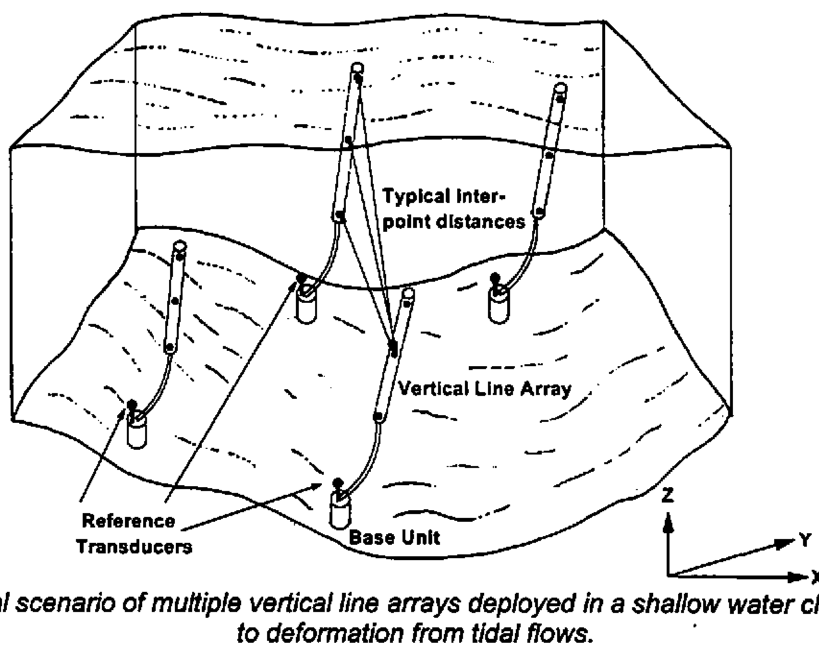

Figure 1. Typical scenario of multiple vertical line arrays deployed in a shallow water channel and subject to deformation from tidal flows.

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2. PRINCIPLES OF MULTIDIMENSIONAL SCALING METHODS.

Multidimensional Scaling (MDS) techniques are statistical methods originally developed to provide low dimensional geometric representation of multivariable data samples. Their basic principle of operation involves assigning a numerical value to represent the *dissimilarity* between individual sample entities, or measuring an actual distance between such, and then reducing this distance data to a multi-dimensional Euclidean space in which the sample entities are represented as a point, with the distance between such points matching as closely as possible the dissimilarity between corresponding entities. The nature of the application of MDS methods is such that the techniques have to be suitably robust and capable of accommodating measurement errors or even partially compensating for missing data.

The MDS algorithms require as a starting point a *(nxn)* matrix of the entire set of inter-transducer distances, measured for *n* transducer elements. If all of these measurements could be made accurately, then the co-ordinate positions of each transducer could be determined using standard linear techniques. However, in the underwater application envisaged, the presence of currents, the approximation to the sound speed and the effects of multipath may all introduce errors into the distance measurements, rendering the standard methods inadequate. MDS provides an alternative technique which is capable of producing good estimates of the relative co-ordinate positions even in the presence of such errors.

MDS methods can only recover relative co-ordinate positions ("local geometry") from a distance data matrix, so in order to establish the absolute positions of the underwater arrays ("global geometry"), a fixed reference co-ordinate set of known transducer positions is required. The matrix of recovered co-ordinates can then be registered in an optimal way to this reference set.

3. METHODOLOGY.

In order to determine the co-ordinates of the array transducers, it is first necessary to measure the entire set of inter-transducer distances. For a system of n transducers, it is only actually necessary to measure $n(n-1)/2$ inter-transducer distances. However, in the presence of ocean currents, and with Gaussian measurement noise, the measured value of d_{ij} may not equal d_{ji}, (where d_{ij} is the measured distance between transducers i and j). Since each transducer is able to transmit and receive, it is possible to measure every inter-transducer distance and thereby form a complete matrix of $n(n-1)$ distances. The MDS methods will then provide overall averaging to minimise the effect of errors and current distortion.

Having obtained the distance matrix, the MDS procedure operates in three distinct phases. Initially, an operation known as Classical Scaling or Principle Co-ordinate Analysis [1],[2] provides an primary estimate of the co-ordinate positions. This first stage is able to yield reasonably accurate estimates of the transducer positions and has been shown to be fairly robust against errors in the distance matrix [3].

The secondary phase of the MDS process makes use of non-metric type algorithms, originally developed to scale data of a non-numerical nature. In this instance however, the data represents Euclidean distances and the least squares approach to refining an approximation of the co-ordinate space can be effectively applied. The algorithm is based on the classic MDS algorithm described by Kruskal for non-metric scaling [4], and iteratively improves the fit of the approximated co-ordinate space to the original measured distance data by minimising a stress function describing the "goodness of fit".

Having obtained a set of relative transducer co-ordinates, with a satisfactory level of coherence to the measured distance data, it remains to register the entire co-ordinate space to a fixed reference frame of known positions. To facilitate this, at least four of the transducers must be of known absolute location, and their corresponding recovered co-ordinate positions translated, rotated and dilated to match as closely as possible the known positions. The techniques used are known as Procrustes operations [5], and are classical procedures used to perform rigid-body motions and uniform scaling. The same operations are then applied to all the remaining recovered transducer positions to produce the final estimate of the vertical array co-ordinates.

3.1 STAGE 1: PRINCIPLE CO-ORDINATE ANALYSIS.

Principle co-ordinate analysis enables the construction of a geometric model of a *p*-dimensional subspace in which entities can be represented as points such that the difference between the measured Euclidean distance d_{ij}, from points *i* to *j* in the data subspace and the corresponding distance $\hat{d}_{ij}$ in the reconstructed subspace is minimised. This method, originally proposed by Torgerson in 1952 and described in detail in references [1],[2] provides an exact analytical solution to the minimisation of $L = \sum_i \sum_j \left(d_{ij} - \hat{d}_{ij}\right)^2$.

For the array positioning problem, we are applying the constraint that the reconstructed subspace is to be in 3-dimensions, although this will only be of concern where errors in the distance measurements produce a solution requiring a subspace of dimension other than three.

The process uses the matrix of *squared* inter-point distances, **D**, between points with co-ordinates, p_i, to regenerate the covariance matrix **Q**, in which:

D is given as $\begin{bmatrix} 0 & \|p_1 - p_2\|^2 & \|p_1 - p_3\|^2 & \cdots \\ \|p_2 - p_1\|^2 & 0 & & \\ \|p_3 - p_1\|^2 & & \ddots & \\ \vdots & & & 0 \end{bmatrix}$

with each element of **D** being the square Euclidean distance between points r and s, $d_{rs}^2 = \sum_{j=1}^{3} (p_{rj} - p_{sj})^2$,

and **Q** is given as **Q=PP'**, where **P** is the (*n*x3) matrix of co-ordinates to be determined, thus

Q = $\begin{bmatrix} p_1p_1 & p_1p_2 & p_1p_3 \\ p_2p_1 & p_2p_2 & \vdots \\ \vdots & . & . \end{bmatrix}$, the elements of **Q** defined as $q_{rs} = \sum_{j=1}^{3} p_{rj}p_{sj}$.

Each element of the matrix **Q** can be obtained from matrix **D** using:

$$q_{rs} = -\tfrac{1}{2}(d_{rs}^2 - d_{r\bullet}^2 - d_{\bullet s}^2 + d_{\bullet\bullet}^2)$$

Here, the components $d_{r\bullet}^2$, $d_{\bullet s}^2$ and $d_{\bullet\bullet}^2$ are defined as follows:

$$d_{r\bullet}^2 = \frac{1}{R}\sum_r d_{rs}^2$$

$$d_{\bullet s}^2 = \frac{1}{S}\sum_s d_{rs}^2$$

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$$d_{\bullet\bullet}^2 = \frac{1}{RS}\sum_r\sum_s d_{rs}^2$$

Since, apart from measurement errors and current distortion, **Q** is symmetric, by use of spectral decomposition we may write **Q=TΛT'**, where Λ is a diagonal matrix of the eigenvalues of **Q** and **T** is the matrix whose columns are the eigenvectors of **Q**. While **Q** is positive definite, the entries of Λ are all positive and thus have real roots, $\Lambda^{1/2}$. Thus we have **Q=TΛT'** = **T**$\Lambda^{1/2}\Lambda^{1/2}$**T'** = **PP'**, and so we can reconstruct **P** from the matrix of inter-transducer distances using **P=T**$\Lambda^{1/2}$. The resulting co-ordinate space has a zero-mean.

Having determined the eigenvalues and corresponding eigenvectors of **Q**, the eigenvalues are ranked in decreasing order. The eigenvalues represent orthonormal axes and are so ranked that the dimensions are of decreasing importance. The eigenvalues of a symmetric matrix are all real, thus for a *p*-dimensional matrix, there ought to be *n-p* values which are zero; for the 3-dimensional situation addressed here, there should be only 3 non-zero eigenvalues. If errors exist in the **D** matrix measurements, this will correspond to other non-zero eigenvalues and so the best approximation to the co-ordinate space will be found from the first 3 ranked eigenvalues.

For the 3-dimensional co-ordinate space describing the vertical arrays, we have:

$\lambda_i = (\lambda_1, \lambda_2, \lambda_3)^T$, the vector of eigenvalues,
$\Psi_1 = (a_1, a_2, a_3, \ldots, a_n)^T$, the eigenvectors corresponding to λ_1,
$\Psi_2 = (b_1, b_2, b_3, \ldots, b_n)^T$, the eigenvectors corresponding to λ_2,
$\Psi_3 = (c_1, c_2, c_3, \ldots, c_n)^T$, the eigenvectors corresponding to λ_3,

and the matrix of approximate co-ordinate positions of the transducers, $p_i = (\sqrt{\lambda_1}a_i, \sqrt{\lambda_2}b_i, \sqrt{\lambda_3}c_i)^T$. The dimensions represented by a,b and c correspond with the x,y,z dimensions although the mapping will not be computed until stage 3 of the process.

3.2 STAGE 2: ITERATIVE REFINEMENT OF THE APPROXIMATE POSITIONS.

Having obtained an initial estimate of the transducer co-ordinates, this may be further improved upon should the original distance data be contaminated by measurement noise or current distortions. This stage, based on the non-metric MDS methods described by J.B.Kruskal [4], should in fact be used with some caution from the outset to determine the final solution. However, the technique can be very successfully applied to improve the accuracy of the output of stage 1.

The degree of agreement between the recovered estimates of the transducer co-ordinates and the original measured data is expressed as the STRESS of the configuration, defined as:

$$S = \sqrt{\frac{S^*}{T^*}} = \sqrt{\frac{\sum (d_{ij} - \hat{d}_{ij})^2}{\sum d_{ij}^2}}$$

where the values d_{ij} are the measured distance data, and $\hat{d}_{ij}$ are the calculated distances between the recovered co-ordinates. Thus the best fitting configuration in 3-dimensions is that which minimises the stress function. Kruskal's method is a numerical solution to this problem, utilising the "steepest descent" approach to iteratively move the co-ordinate estimates step by step in the direction determined by evaluating the (negative) partial derivatives of the function S.

An acknowledged difficulty with this procedure is that of local minima in the solution space, which may or may not be the true overall minimum. If this stage was commenced using an arbitrary estimate of the solution, this could present a significant problem. However, since the estimate provided by stage 1 can generally be regarded as reasonably accurate except in the event of gross error, the minimum arrived at from this approach is most likely to be close to or equal the true minimum. Further refinements may be made by performing a significant number of iteration to search for other local minima. The local minima displaying the lowest stress value may then be selected.

The algorithm requires the configurations, both the measured data and the recovered co-ordinate inter-point distances, to be normalised such that the root-mean-square distance between the points equals one. At every step of the gradient descent process, starting from the initial estimate of the positions obtained from stage 1, p_{ij}, moving down the gradient g with a predetermined step-size α, we arrive at the new configuration p'_{ij}. The step size is varied from one iteration to the next to reduce the number of iterations required to reach the solution. The revised position estimates are of course normalised between each iteration. Thus,

$$p'_{ij} = p_{ij} + \frac{\alpha}{mag(g)} g_{ij}$$

with $mag(g)$ being the relative magnitude of the gradient g for the normalised configuration x_{ij} equal to:

$$mag(g) = \sqrt{\frac{1}{n}\sum_{i,j} g_{ij}^2}$$

The negative gradient, g_{ij}, is calculated according to the following, as described by Kruskal:

$$g_{kl} = S\sum_{i,j}(\delta^{ki} - \delta^{kj})\left[\frac{d_{ij} - \hat{d}_{ij}}{S^*} - \frac{d_{ij}}{T^*}\right]\frac{(p_{il} - p_{jl})}{d_{ij}}$$

where δ^{ki} and δ^{kj} denote Kronecker symbols, with $\delta^{ki} = 1$ if $k = i$ and 0 otherwise, and

$$S^* = \sum(d_{ij} - \hat{d}_{ij})^2,\ \ T^* = \sum d_{ij}^2,\ \ S = \sqrt{S^*/T^*}$$

The variation in the step size α is discussed in some detail in [4]. As the computation proceeds, the value of stress will become successively smaller until no further improvement can be obtained, at which point the configuration may be considered to be the final solution.

3.3 STAGE 3: PROCRUSTES OPERATIONS.

Procrustes operations [2],[5] are again drawn from the domains of statistical analysis where they are used as tools for the comparison of the similarity between two configuration subspaces. The statistical measure of the degree of coincidence between two such subspaces, **A** and **B**, is the sum of the squared distances between their corresponding points:

$$M^2 = \sum_{i=1}^{n}\left\{\sum_{j=1}^{3}(a_{ij} - b_{ij})^2\right\}$$

Before calculating this statistic, it is necessary to first translate, rotate and dilate the two configurations to a position of best fit relative to each other. Such rigid-body operations are known are Procrustes operations, the terminology being derived from a mythological Greek innkeeper, Procrustes, who stretched or removed traveller's limbs so that they would fit into his bed. Such operations do not disturb the internal relationships between the points of either configuration under comparison. Having performed these operations, the value of M^2 provides a measure of the "lack of fit" between the two subspaces. The procedure involves fixing one configuration, the known reference positions, and performing rigid-body operations to the other to match to it. Since in this application we have only four such reference points, we are only able to determine the operations necessary to match the four corresponding recovered positions. However, as long as these reference points are judiciously positioned, the extrapolation of the rigid-body operations, used to align to these four reference points, to the entire set of co-ordinate positions provides an accurate re-orientation of the whole subspace.

The following three steps detail the operations required to match configuration **B**, the recovered positions, to that of **A**, the reference set, in such a way that M^2 is minimised.

Step 1: Matching under translation.
It can be shown [2] that the best fit of the two configurations under translation occurs when both the systems have the *same* centroid. This can be achieved simply by mean-centring both **A** and **B** configurations so that the centroids of each coincide with the axes origin. The centroid of **A** has the co-ordinates $(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ for 3-dimensions, where $\bar{a}_j = \frac{1}{n}\sum_{i=1}^{n} a_{ij}$. Similarly for **B**.

Step 2: Matching under rotation.
Assuming that **A** and **B** are now mean-centred, we can then say that any rotation of **B** relative to configuration **A** can be expressed by an orthogonal matrix **R**, and the resulting rotated configuration as **BR**. Expressing the statistic M^2 as

$$M^2 = \mathrm{trace}\{(\mathbf{A} - \mathbf{B})(\mathbf{A} - \mathbf{B})'\}$$

then, after rotation by **R**, the resulting value of M^2 becomes

$$M^2 = \mathrm{trace}(\mathbf{AA}' + \mathbf{BB}' - 2\mathbf{AR}'\mathbf{B}').$$

Therefore, to minimise the value of M^2 we must choose **R** to maximise **2AR'B'**. It can be shown that the required rotation matrix **R** is given by **R** = **VU'**, where **UΣV'** is the singular value decomposition of the matrix **A'B**.

Step 3: Matching under dilation.
The remaining step of these operations is to rescale the co-ordinates of configuration **B**, now **BR**, by a scaling factor ν, to become ν**BR**, such that M^2 is again minimised.

Thus M^2 becomes:

$$M^2 = \nu^2\mathrm{trace}(\mathbf{BB}') - 2\nu\,\mathrm{trace}(\mathbf{AR}'\mathbf{B}') + \mathrm{trace}(\mathbf{AA}')$$

By differentiation of this expression we can see that M^2 is a minimum value when

$$\nu = \frac{\mathrm{trace}(\mathbf{AR}'\mathbf{B}')}{\mathrm{trace}(\mathbf{BB}')}$$

Using the result from step 2, we can show that trace(**AR'B'**) = trace(Σ), hence the optimum scale factor ν becomes:

$$\nu = \frac{\mathrm{trace}(\Sigma)}{\mathrm{trace}(\mathbf{BB}')}.$$

Step 4: Registering the entire configuration.
Having found the optimal rotation matrix **R** and scaling factor ν, it remains to apply these to the *entire* set of recovered co-ordinates. This done, the whole system, which as a result of the MDS techniques in stages 1 and 2 is produced mean-centred about the origin, must be translated to the final orientation. This translation is defined simply by the difference between the centroid of the entire system and the centroid of **A**, the reference set of four positions.

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4. SIMULATED EXPERIMENTAL RESULTS.

In order to validate the robustness and accuracy of the MDS positioning methods described previously, a series of numerical simulations have been conducted to assess their performance in a realistic underwater channel scenario. The geometry of the simulated array positions accords with the envisaged application of a cluster of vertical line arrays (VLA's) deployed in a shallow water tidal channel. A set of four VLA's are considered; each array being 30m in length with a telemetry transducer mounted top, middle and bottom. The arrays are deployed on a 105m tether cable from a sea-bed mounted base, which also house a transducer, of *known* geometric position. These four base transducers act as the reference positions to which the recovered co-ordinates are registered. A uniform current is approximated as flowing along the channel, introducing non-reciprocity in the sound speed measurements resulting in errors in the corresponding distances between transducers. The arrays are thus subject to deflection in position and attitude by the current flow, however, the recovered co-ordinates of the transducers will enable this to be accurately determined.

To evaluate the performance of the MDS algorithms, firstly the inter-transducer distances are calculated from the set of known co-ordinates of the VLA's and simulated experimental measurement error introduced to this data by adding Gaussian noise. To model the effects of a uniform tidal current, the sound velocity distortion is approximated as:

$$\tilde{\mathbf{c}} = \mathbf{c} + \gamma(\cos\theta)$$

where γ represents the current vector and θ the angle between the current vector and the direct path between two transducers. Simulations have been conducted for tidal currents flowing parallel to the y-axis, i.e. along the channel, with magnitudes varying from 0 ms^{-1} to 0.5 ms^{-1}, in steps of 0.1 ms^{-1}. Furthermore, for each current model, simulations have been carried out with noise added to the ideal distance data, with zero mean and standard deviations ranging from 0 to ±1.0 m of measurement error, in steps of ±0.1 m. The arrays have been assumed to be stationary for the period in which all distance measurements have taken place. Since the four base transducers are required to be of known position, it is assumed that the inter-point distances for these has been calculated exactly and is thus not subject to current distortion or measurement noise.

An important consideration for the accuracy of the MDS result is the placement of the four reference transducers to which the recovered configuration must be registered. To provide maximum accuracy, each transducer should ideally lie along each orthogonal axis, with the fourth located at the origin. The separation of these should be as great as possible to maximise the variance in each orthogonal plane. A feature of MDS is that the ability of the procedure to recover the co-ordinates along a particular axis in the presence of error depends on the amount of variance of the configuration in that axis; the greater the variance of points along that axis, the better the accuracy. In the simulation, the co-ordinates of the reference positions are shown in table 1.

The accuracy of the recovered co-ordinates is assessed by comparison with the known VLA configurations. The root mean square position error of the whole configuration of transducers is used to indicate the success of the operations, according to:

$$\frac{1}{n}\sum_{i=1}^{n}\sqrt{\left\{\sum_{j=1}^{3}\left(p_{ij}-\hat{p}_{ij}\right)^{2}\right\}}$$

where p_{ij} is the known transducer co-ordinate, and $\hat{p}_{ij}$ the recovered transducer co-ordinate.

The standard deviation of the position error is also determined, over the whole recovered configuration. These parameters have been found for each simulated scenario (i.e. variations in ocean current velocity and measurement noise), and for each of six possible VLA configurations. (Six different configurations have been examined to determine if the relative variance of the x, y and z co-ordinates of the transducers has any significant effect on the MDS technique.)

x co-ordinate	y co-ordinate	z (depth) co-ordinate
0	0	200
0	200	200
-100	0	200
10	-50	185

Table 1. Positions of the Reference Transducers.

To obtain an accurate statistical trend through the simulations, each combination of configuration, ocean current and measurement noise was repeated five times, and the graphs below show the average of those five results.

Figure 2 shows the mean position error rising in a linear relationship with the distance measurement noise variance, and the results show that for a measurement noise variance of ±1.0m the mean error between the recovered and true transducer positions is approximately 1.0m, regardless of VLA configuration. It can also be seen from figure 2 that the standard deviation of the position error rises in an approximately linear manner with measurement noise, up to a level of approximately ±0.6m for a noise variance of ±1.0m, across the whole configuration result.

The robustness of the MDS technique with regard to the non-reciprocity of distance measurement due to ocean currents is demonstrated by figure 3. Here it can be seen that the worst case error arising from a current of $5ms^{-1}$ is only 0.0025m ±0.001m. These measurements are for data with no added measurement noise, which as was shown in figure 2 turns out to be the limiting factor.

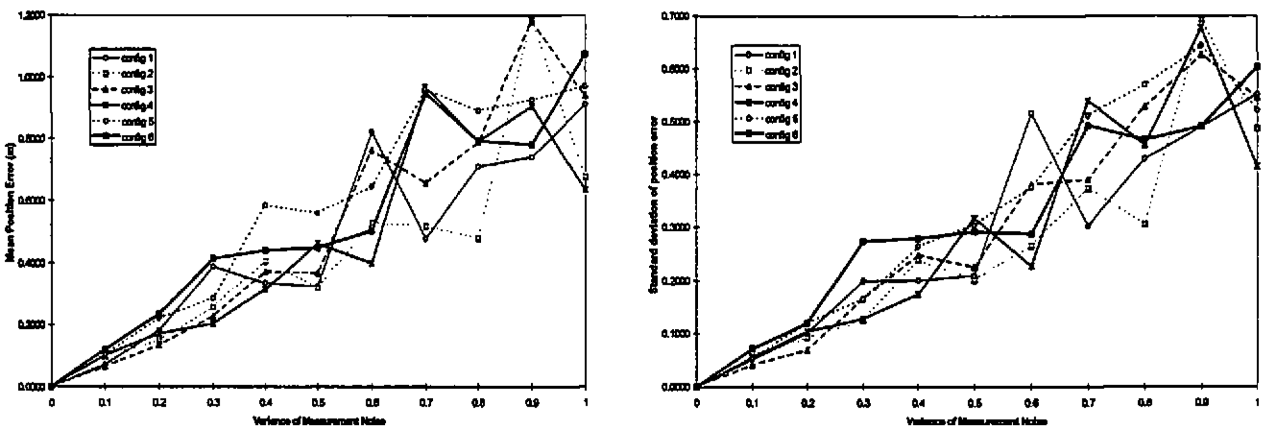

Figure 2. Mean and standard deviation of position error resulting from distance measurements corrupted by additive Gaussian noise of variance 0 - ±1m.

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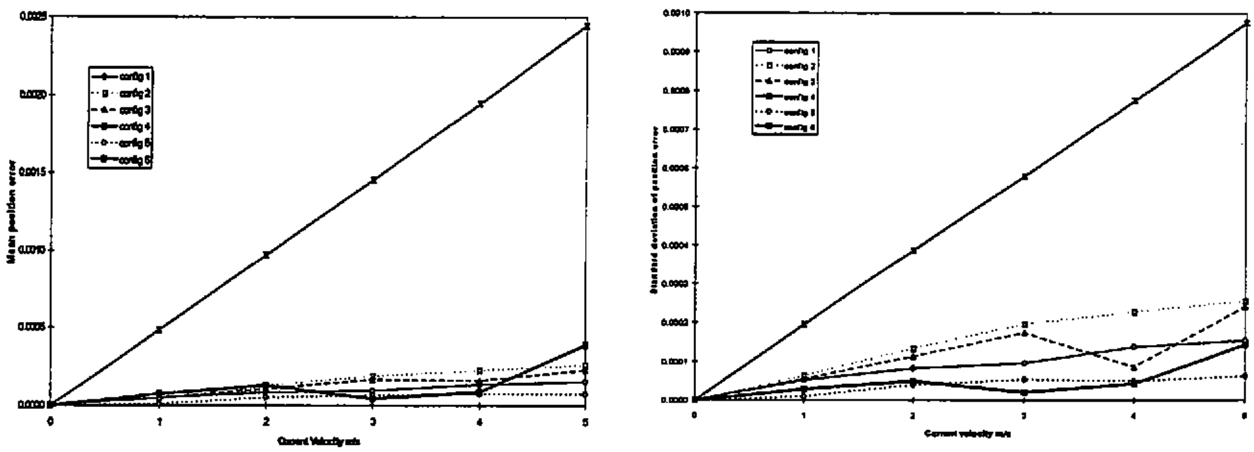

Figure 3. Mean and standard deviation of position error resulting from simulated ocean current distance measurement distortion, for currents 0 - 5 ms^{-1}.

5. DISCUSSION AND SUMMARY.

The task of determining the 3-D positions of a cluster of VLA's is demonstrated here to be a suitable application for Multi-Dimensional Scaling. The techniques used within MDS provide an alternative to traditional methods of position fixing and have been shown to be appropriate in an environment in which ocean currents and distance measurement errors may render alternative approaches inadequate.

The MDS method described within this paper consisted of three stages: an initial position approximation following classical scaling techniques, a least squares refinement of the approximation and a final orientation of the recovered configuration to align with a known reference frame.

Numerical simulations have investigated the performance of the MDS techniques with both ocean currents causing non-reciprocity in distance measures and super-imposed Gaussian measurement noise. Even in the presence of significant errors caused by these factors, the MDS methods have proved able to resolve the 3-D positions of the array sensors to a mean position error of less than 1m even with ±1.0m measurement noise variance. In summary therefore, we conclude that Multi-Dimensional Scaling algorithms provide a robust alternative as a method for 3-D positioning.

6. REFERENCES.

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