Proceedings of the Institute of Acoustics

HIGHER ORDER REVERBERATION STATISTICS FOR NON-UNIFORMLY DISTRIBUTED SCATTERERS

A.J. Fenwick

Maritime Systems Department, DRA Farnborough, GU14 6TD, UK

1. INTRODUCTION

Understanding and describing the behaviour of reverberation, the randomly fluctuating, decaying echoes from scatterers lying in the path of an outgoing sound pulse, is crucial to sonar system studies. Modelling effort has been concentrated on describing the mean power variation as a function of sonar and environmental parameters. The detailed statistics of the reverberation voltage and its envelope have received less attention in sonar than have the corresponding statistics for radar clutter. Yet recently there has been an emphasis in processing research on use of higher order statistics, particularly in non-Gaussian environments.

The dearth of models is probably an indication of the difficulty of the problem and the simplicity of most detection systems. If there are sufficiently many scatterers within the volume enclosed by the effective range and angle resolution cells of the sonar and if the average distribution and target strength of the scatterers is constant, the reverberation is Gaussian and its envelope is Rayleigh. It is observed in practice however, that large amplitude fluctuations occur too frequently. The distribution is said to have a longer tail. Heuristic distributions fitted to the envelope probability density function (pdf) include the Log Normal and Weibull. Physical models are preferable, because they allow the separation of the effects of sonar and environmental parameters. The difficulty with them is in finding an adequate but tractable description of the mechanism producing the non-uniformity of scattering which is implied by the long distribution tails.

The K-distribution, used to model the first order distribution of radar clutter [1] has a physical basis and has become accepted as a standard, at least in the United Kingdom. Crowther [2] has developed a simple analytic model for the first order envelope distribution of non-Gaussian reverberation. It was found to be capable of fitting experimentally observed distributions at three spot frequencies less than 10 KHz with surface parameters which were independent of frequency. Somewhat unexpectedly, given the large difference in sonar cell sizes, it was later found that the model could be fitted to reverberation distributions at 100's of KHz [3]. It will be demonstrated that the basic assumptions of the model are sufficient to develop an analytical model for the multi-dimensional distributions of non-Gaussian reverberation and their higher order moments.

HIGHER ORDER REVERBERATION STATISTICS

2. THE TWO STATE MARKOV MODEL

It is clearly not reasonable to assume a uniform distribution of scatterers under all conditions. Examples are the wave facets which act as specular reflectors, and outcrops of rock in a bottom of sand and mud. The assumption that a large number of scatterers contribute to the signal returned from a cell may still be valid. The average energy scattered to-wards the receiver, however, depends on the target strength of scatterers within the ensonified area or volume. During operation within an area, a variety of conditions will be encountered, and the probability of observing returns of a particular energy will be the average of the probabilities from each of the areas ensonified.

Following Crowther [2] it will be assumed that the sonar resolution cell is defined by top-hat functions in both range and bearing. In addition, to simplify the presentation, it will be assumed that the resolution in range is high, so that the ensonified area at a given range R can be approximated as a line of length $X = \Phi.R$, where Φ is the bearing resolution. A generalisation to resolution cells with arbitrary aspect ratio between down range and cross range dimensions is given in [2] and will be stated at the appropriate point in the development.

In a uniform propagation environment, the average level of reverberation decays systematically. If only a single source of scattered energy is present, i.e. the reverberation originates either from the water volume, the sea floor or the sea surface, the trend is always decreasing, and this is the case assumed in the following. The average power decay is described by $E\{|v(t)|^2\}$, where v(t) is the complex modulation of the received signal at the pulse carrier frequency. The fluctuation is the normalised voltage $x(t) = v(t)/(E\{|v(t)|^2\})^{1/2}$. Its envelope is given by $y(t) = |v(t)|^2/E\{|v(t)|^2\}$.

2.1 The Markov Model for scattering strength variation

The scattering strength is defined for a resolution cell, but the physical properties of the scattering surface change independently of the sonar pulse length, beamwidth and range. It is convenient to think of surface scattering patches where the surface scattering properties are homogeneous. The model described in [2] postulates a three parameter description of the scattering strength variation. This generates a probability density function for surface scattering strength which yields a non-Gaussian probability density function for reverberation by the composition hypothesis.

2.1.1 The first order composition hypothesis. The signal scattered from a single resolution cell has the Gaussian density $f(x) = (2\pi s^2)^{1/2}\exp(-|x|^2/s)$, and the envelope power has the gamma density $f(y) = \exp(-y/s)$, where 's' is the cell scattering strength. Assume that the scattering strength can vary over all positive values. The observed probability densities are the averages over these pdf's of all values of scattering strength and the result is known as a mixture. If f(s) is the pdf for cell scattering strength is assumed to be

HIGHER ORDER REVERBERATION STATISTICS

$$f(y) = \int_0^\infty \exp(-y/s)\, f(s)\, ds \quad . \tag{1}$$

2.1.2 Two valued local scattering strength. It is assumed that the surface has only two scattering states s_0 and s_1. The proportion of high valued areas is ρ, and the ratio between the two scattering strengths is γ. For unity average scattering strength, $s_0 = 1/(1+\rho(\gamma-1))$, and $s_1 = \gamma s_0$.

2.1.3 Markov state transitions. In one dimension, the scattering state is described by a function taking two values, 0 or 1 along the line. Assume that the changes of state take place randomly such that the probability of transition t_{ij} from state 'j' to state 'i' (i,j = 0,1) after traversing an infinitesimal length δx is given by the transition matrix

$$T = [t_{ij}] = \begin{bmatrix} 1-\rho\delta x/L & (1-\rho)\,\delta x/L \\ \rho\delta x/L & 1-(1-\rho)\,\delta x/L \end{bmatrix}$$

where L is a characteristic length scale, defining the effective size of a scattering patch.

2.1.4 Summary of model parameters: The three surface scattering parameters are the contrast between high and low scattering strengths γ, the proportion of high scattering state ρ, and the characteristic length of the patch L.

2.2 First order probability distribution

The distribution for scattering strength is found by finding the proportion β of a line of length X which is in the high state. There are two possible initial states and two final states and there may be one or many transitions between states. The probability of a particular value of β is found by counting up the number of ways in which the interval can be divided between high and low states. Let p_j be the probability that the initial state is 'j', and let $\psi_{ij}(\beta|X)$ be the probability that the proportion of a line of length 'X' in the high state is β given the initial state is 'j' and the final state is 'i'. Dropping the dependence on X, the probability density function of β is given by

$$f(\beta) = \Sigma\Sigma p_j \psi_{ij}(\beta), \tag{2}$$

Suppose $\beta = 1$, which means that the entire resolution cell is in the high state. Then $\psi_{ij}(\beta) = 0$ unless $i = j = 1$. The probability of this occurring is the probability of no transitions over the length of the cell. Consider steps of length of δx. Over the cell length X there are $X/\delta x$ transitions and hence the probability of remaining in the high state is $t_{11}^{X/\delta x} = (1-(1-\rho)\delta x/L)^{X/\delta x}$. In the limit as δx tends to zero, $\psi_{11}(1) = \exp(-X(1-\rho)/L)$, and the probability density is

$$f(1) = p_1 \psi_{11}(1) = \rho\exp(-X(1-\rho)/L). \tag{3}$$

HIGHER ORDER REVERBERATION STATISTICS

Following the same argument, $\psi_{00}(0) = \exp(-\rho X(1-\rho)/L)$, and

$$f(0) = p_0\psi_{00}(0) = (1-\rho)\exp(-X\rho(1-\rho)/L). \quad (4)$$

For arbitrary β, Crowther [2] shows that,

$$f(\beta) = \{(1-\rho)\delta(\beta) + \rho\delta(\beta-1) + 2\lambda^2\rho(\beta\rho + 1 - \beta)I_1(\mu)/\mu + 2\lambda\rho I_o(\mu)\}. \exp\{-[\beta + (1-\beta)\rho]\lambda\} \quad (5)$$

where $\lambda = X(1-\rho)/L$, the ratio of ensonified length to effective patch length,

$$\mu = 2(\rho\beta(1-\beta)/(1-\rho))^{1/2}\lambda, \quad (6)$$

and I_o and I_1 are modified Bessel functions of the first kind. Equations (3) and (4) are obtained by substituting $\beta = 0$ and $\beta = 1$ respectively in (5) and (6).

Under the assumption of high range resolution, $s = [1 + (\gamma-1)\beta]/[1 + (\gamma-1)\rho]$, and hence

$$f(s) = \frac{d\beta}{ds} f(\beta) = [\rho + (\gamma - 1)]^{-1} f(\beta). \quad (7)$$

2.2.1 Two-dimensional scattering cell. Crowther [2] gives a basis for redefining the cell to patch ratio λ in terms of both cell dimensions by $\lambda = 2[(1+X/2L)(1+Y/2L)-1]/(1-\rho)$, where Y is the range resolution and X is the cross range resolution due to the angular beamwidth. This new definition is then used in (5) and (6).

2.3 First order moments

The composition hypothesis allows the moments of the normalised reverberation envelope to be found from the moments of the scattering strength. From (1),

$$E\{y^n\} = n!E\{s^n\} \quad (8)$$

3. MULTI-DIMENSIONAL STATISTICS

The composition hypothesis can be used to propose a model for the observed multi-dimensional statistics of non-Gaussian reverberation. For a set of values scattered from a single set of cells, it may be assumed that the joint distribution is Gaussian [4], but it depends on the variances of the observed values and correlations of pairs of the variables. As before, the distribution of the observed normalised complex voltage samples $x_1, x_2, .., x_n$ is obtained by averaging all cell strengths and

HIGHER ORDER REVERBERATION STATISTICS

$$f(\mathbf{x}) = \int (2\pi)^{n/2} |\mathbf{M}(\mathbf{s})|^{-1/2} \exp(-\mathbf{x}^{H}\mathbf{M}(\mathbf{s})^{-1}\mathbf{x})\, f(\mathbf{s})\, d\mathbf{s} \qquad (9)$$

where M(s) is the covariance matrix of the I and Q components of the samples, f(s) is the joint density of cell scattering strengths and $.^{H}$ is the complex conjugate transpose of a complex vector. By applying the assumptions of the Markov scattering surface, the joint density can be found. The higher order statistics of the process are then found from (9).

3.1 Second order pdf for scattering strength variation

The problem reduces to that of finding the joint probability of proportions of high scattering strength β_1 and β_2 in the two scattering cells. It will shown that the joint probability can be derived from the components $\psi_{ij}(\beta|X)$ which comprise the first order pdf.

Let $\psi_{ij}(\beta_1, \beta_2)$ be the joint probability of high scattering proportions β_1 and β_2 given initial state 'j' on the first interval and final state 'i'on the second. For non-overlapping intervals, separated by length 'd',

$$\psi_{ij}(\beta_1, \beta_2) = \Sigma\Sigma\psi_{ik}(\beta_2)t_{km}(d)\psi_{mj}(\beta_1), \qquad (10)$$

where $t_{km}(d)$ is the probability of being in state 'k' after an interval of length 'd' when the initial state is 'm'. The interval transition probability is found by noting that $\mathbf{T}(d_1+d_2) = \mathbf{T}(d_1)\mathbf{T}(d_2)$. Considering a large number of small equal intervals, and taking the limit as their lengths tend to zero, $\mathbf{T}(d) = [t_{ij}(d)] = \begin{bmatrix} 1-\rho(1-\alpha) & (1-\rho)-\alpha(1-\rho) \\ \rho(1-\alpha) & \rho+\alpha(1-\rho) \end{bmatrix}$, where $\alpha = \exp(-d/L)$.

For overlapping intervals, let Y be the length of a range resolution cell, and let the overlap be of length 'd'. Let the proportion of the overlap in the high scattering state be $\delta\beta$, then

$$\psi_{ij}(\beta_1, \beta_2) = \int\sum\sum \psi_{ik}(\Delta\beta_2|\Delta Y)\, \psi_{km}(\delta\beta|d)\, \psi_{mj}(\Delta\beta_1|\Delta Y)\, d\delta\beta\,, \qquad (11)$$

where $\Delta\beta_i = \beta_i - \delta\beta$, i=1,2 and $\Delta Y = Y-d$. The limits of integration depend on Y, d, β_1 and β_2, but by defining $\psi_{mj}(\beta|Y) = 0$ for $\beta > 1$ or $\beta < 0$, the limits of integration can be set formally to [0,1].

The joint pdf of the proportions of high scattering state in the two cells is given by the general formula

$$f(\beta_1, \beta_2) = \Sigma\Sigma\psi_{ij}(\beta_1,\beta_2)p_j. \qquad (12)$$

The conversion between proportions of high scatter and cell scattering strengths depends on the

HIGHER ORDER REVERBERATION STATISTICS

Jacobian of the transformation from (β_1,β_2) to (s_1,s_2), which, because s_1 is independent of β_2 and s_2 is independent of β_1, reduces to $\frac{\beta_1 \partial\beta}{s_1 \partial s_2}$. Hence $f(s_1,s_2) = [\rho + (\gamma-1)^{-1}]^{-2} f(\beta_1,\beta_2)$.

3.1.1 Moments. The second order moments of particular interest are the cross-correlations of the power envelope. The prediction of the Olshevskii theory is that there is no correlation for time differences of more than a pulse width. In [2] it is shown that this is not so for the Markov surface. The triangular correlation expected from a rectangular pulse occurs only when the surface is homogeneous corresponding in the model to no contrast, $\gamma = 1$, or zero proportion of high state, $\rho = 0$. For non-homogeneous surfaces, the correlation of the envelope provides a means of testing the model, and was used in the analysis reported in [2], to establish the parameters of the scattering surface.

3.2 Probability densities of arbitrary order
It is more convenient to extend the previous results if they are first represented in terms of matrices.

3.2.1 First order. Let $\Psi(\beta) = [\psi_{ij}(\beta)]$, let $\mathbf{p} = [p_j] = [1-\rho, \rho]^T$, and let $\mathbf{u} = [1,1]^T$. The first order density is

$$f(\beta) = \mathbf{u}^T \Psi(\beta) \mathbf{p}. \tag{13}$$

3.2.2 Second order. Let $\Psi(\beta_1,\beta_2) = [\psi_{ij}(\beta_1,\beta_2)]$, let $\mathbf{T}(d) = [t_{ij}(d)]$ and let $\Psi(\beta|Y) = [\psi_{mj}(\beta|Y)]$. For non-overlapping and overlapping intervals respectively,

$$\Psi(\beta_1,\beta_2) = \Psi(\beta_2)\mathbf{T}(d)\Psi(\beta_1). \tag{14}$$

$$\Psi(\beta_1, \beta_2) = \int_0^1 \Psi(\Delta\beta_2|\Delta Y)\, \Psi(\delta\beta|d)\, \Psi(\Delta\beta_1|\Delta Y)\, d\delta\beta \,. \tag{15}$$

The second order density is given by

$$f(\beta_1, \beta_2) = \mathbf{u}^T \Psi(\beta_1, \beta_2) \mathbf{p}. \tag{16}$$

3.2.3 Arbitrary order. The densities of arbitrary order are found by induction from (14) and (15). Defining $\Psi(\beta_1, .., \beta_n)$ as the matrix of probabilities of proportion of high state for specified end conditions, then if the n'th resolution cell does not overlap any of the other n-1 intervals,

$$\Psi(\beta_1, .., \beta_n) = \Psi(\beta_n)\, \mathbf{T}(d_{n-1}) \Psi(\beta_1, .., \beta_{n-1}).$$

HIGHER ORDER REVERBERATION STATISTICS

If the n'th cell overlaps the n-1'th, but no others,

$$\Psi(\beta_1, .., \beta_n) = \int_0^1 \Psi(\Delta\beta_n|\Delta Y_n)\,\Psi(\delta\beta_{n-1}|d_{n-1})\,\Psi(\beta_1, .., \Delta\beta_{n-1}|\Delta Y_{n-1})\,d\delta\beta_{n-1}\ ,$$

where the last term in the integrand represents the probability of high scatter proportion in a shorter n-1'th resolution cell.

In either case, the joint density of the high scatter proportions is given by

$$f(\beta_1, .., \beta_n) = \mathbf{u}^T\Psi(\beta_1, .., \beta_n)\mathbf{p},$$

and the joint density of the scattering strengths by

$$f(s_1, .., s_n) = [\rho + (\gamma\text{-}1)^{-1}]^{-n}\, f(\beta_1, .., \beta_n).$$

3.2.4 Moments. Using the fourth moment rule [5], all higher order moments of jointly Gaussian random variables can be found from the pairwise correlations. If u,v,x,y have the mixture distribution given by (9), then from equation A1.6.6 of [5]

$$E\{uv^*xy^*\} = \int\left(E\{uv^*\}E\{xy^*\} + E\{uy^*\}E\{vx^*\}\right)f(\mathbf{s})\,d\mathbf{s}\ , \qquad (17)$$

since the covariances are functions of the cell scattering strengths. For separated scattering cells on a Markov surface, equation (17) shows that the fourth order moments may be calculated from second order covariances. Experimental verification of this is a first step in validating the extension of the model to higher order statistics.

4. CONCLUSION

The essential steps in extending an experimentally supported model for the first order probability density of non-Gaussian reverberation to multi-dimensional pdf's have been set out. The model is based on a Markov assumption for the variation of scattering strength in a two state scattering surface, whose may be derived from the measurements of first order pdf's and second order envelope correlations. No further parameters are required to extend the model to higher order densities. The calculation of higher order moments has been related to the fourth moment rule for jointly Gaussian variates, and experimental confirmation of this prediction provides a means of validating the model.

5. REFERENCES

[1] E JAKEMAN, 'Statistics of gamma-lorentzian intensity fluctuations', OPTICA ACTA, 27 pp735-741 (1980)
[2] P A CROWTHER, 'FLUCTUATION STATISTICS OF SEA BED ACOUSTIC BACKSCATTER', in 'Bottom Interacting Ocean Acoustics', ed W A Kuperman and F B Jensen, Plenum Press (1980)
[3] P A CROWTHER, A J FENWICK, J POOLE, unpublished results, (1979)
[4] V V OLSHEVSKII, 'Characteristics of Sea Reverberation', Eng. Trans. publ Consultant Bureau, (19 67)
[5] J HUDSON, 'Adaptive Array Principles', Peter Peregrinus, (1981

©British Crown Copyright 1995

On a Choice of the Optimum Sounding Signal for Time-Delay Spectrometry

A. Yaremchuk, Andreyev Acoustics Institute, Moscow, Russia

The paper is aimed for investigation into a choice of the optimum sounding and demodulating signals that are to be used for evaluation of the response function of a transducer (or any communication line) with the best accuracy. It is shown that in a general case the Fourier transform of the sounding signal must be an exponential with a pure imaginary odd phase, monotonously growing (at least in the working band of the transmitter) with frequency. If this be the case, the response function may be reconstructed exactly via assimilation of the received data, with the signal/noise ratio kept unchanged. By means of theory of pseudodifferential operators it is shown that the process of gathering information about a communication line characteristics is in one-to-one correspondence with a simple mechanical system describing a charged particle in an electric field. The analysis of the model shows that an increase of the sounding sweep duration results in improvement of the signal/noise ratio, leading to exact reconstruction of the response function. A one-component demodulating signal provides exact reconstruction only in the case of an infinitely long sounding sweep, but implementation of special two-component demodulating signals enables one to make allowance for finite duration of an experiment. The method may be also applied to the case when there is prior information on the transmitter or noise characteristics, resulting in further specification of the optimum signal. In order to illustrate such performance we consider, by way of example, the case of a response function ceasing by a certain finite instant, obtaining an optimum periodic sounding signal.