ACOUSTIC PULSE REFLECTOMETRY: SINGULAR SYSTEM ANALYSIS AND REGULARISATION OF THE INVERSE PROBLEM

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1. INTRODUCTION

Although the method of acoustic pulse reflectometry (APR) for the reconstruction of an unknown duct system has been described in previous work (see, for example, Amir et al [1], Sharp and Campbell [2], and Kemp et al [3]), little attention has been given to the *a priori* estimation of the transient input impulse response. However, it is known that the derivation of such a quantity from experimental measurement of the incident and reflected waveforms is an ill-posed problem, as was demonstrated clearly by Sondhi and Resnick [4] for the vocal tract inversion. In section (2) below, the nature of the ill-posedness is recapitulated; section (3) illustrates, with reference to experimental data, the typical consequences for APR methodologies. A solution for the input impulse response, regularised by truncating the singular value decomposition of the convolution matrix, is presented. A stable reconstruction of a stepped-tube system is successfully obtained.

2. ACOUSTIC PULSE REFLECTOMETRY: DEFINITION OF THE ILL-POSED INVERSE PROBLEM

For a linear transmission system, the relationship between the unknown input impulse response, $z(t)$, and the measurable input, $x(t)$, and output, $y(t)$, signals is

$$y(t) = \int_0^t x(t-\tau)z(\tau)d\tau \quad , \qquad 0 \leq t \leq T. \tag{1}$$

The output wave is often described as a convolution between 'source' and 'filter' functions, and, therefore, the determination of the filter, $z(t)$, requires a deconvolution operation. In the Fourier domain, this may be achieved by pointwise division along sample points, with amplified values and large percentage errors entailing as $x(t)$ approaches the limits of experimental resolution. The problem can be quantified more precisely in the discrete time domain, since equation (1) can then be written in terms of sampled vectors, **y**, **z**, as

$$\mathbf{y} = \mathbf{Xz}, \tag{2}$$

where **X** is a lower triangular matrix with elements

$$X_{i,j} = x(i-j) \qquad i \geq j$$
$$= 0 \qquad i < j. \tag{3}$$

The deconvolution then takes the matricial form

$$\mathbf{z} = \mathbf{X}^{-1}\,\mathbf{y}, \tag{4}$$

and the stability of the solution depends on the conditioning of the input matrix, **X**. A condition number can be defined through the singular value decomposition (SVD) of **X**, which yields

$$\mathbf{X} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}', \tag{5}$$

where **U** and **V** are the orthonormal left and right matrices of singular functions^{+}, with columns $\mathbf{u}_j$, $\mathbf{v}_j$ respectively, and a diagonal matrix, **S**, holds, in order of decreasing amplitude, the singular values, *s*(*j*), of the system. An input vector of *N* samples defines a condition number, **cond**=s(1) / s(N), and ill-conditioning, by which small experimental uncertainties in *x*(*t*) propagate through to large errors in the solution of equation (4), is defined by a condition number large with reference to unity. The inverse mapping from measured reflections to filter function is then an inherently 'ill-posed' problem.

3. REGULARISATION OF THE BORE RECONSTRUCTION

Standard APR experimental protocol was applied to a stepped-tube system, and 2048 samples of the incident and reflected waveforms were acquired at a rate of 44.1 kHz, giving a Nyquist limit of 22.05 kHz. The digitised incident pulse was used to define a correlation matrix, **X**, according to equation (3). A decomposition of the resulting 2048 x 2048 matrix yielded the set of singular values shown in figure (1). Due to the vanishingly small amplitude of the higher-order singular values, the condition number was found to be of the order of 10^{20}, indicative of extreme ill-conditioning due to the bandwidth-limited (~200-8,000 Hz) loudspeaker response and low signal-to-noise ratios in the out-of-bandwidth components. (Experimental solutions for this problem are proposed in [7]). A 'truncated' singular value decomposition (TSVD) (see [4], [5]) was immediately suggested as a regularisation procedure; it defines the input impulse response as

$$\mathbf{z} = \frac{1}{s(j)}\sum_{j=1}^{J} \mathbf{y}'\,\mathbf{u}_j\,\mathbf{v}_j\,. \tag{6}$$

Setting *J*=551 delimits a condition number of 43, and yields the transient, *z*(*t*), of figure (2). The slight ripple in the result was expected due to the abrupt truncation of the singular value series. The result is contrasted with the noise-corrupted waveform obtained from unconstrained Fourier deconvolution.

The effect of the truncation on the incident pulse is demonstrated in figure (3). Multiplying together the truncated **U**, **S**, **V** matrices according to equation (5) yields an input waveform from only some 25% of the singular value set, yet the result is scarcely distinguishable from that of the untruncated matrices (that is, within numerical precision, from the original experimental signal). Qualitative comparison of figures (2) and (3) confirms that this small change in the input has entailed a large change in the input impulse response, and that the system is ill-conditioned.

The effect on the bore reconstruction (spatial resolution 3.9 mm) is shown in figure (4). The input impulse response obtained by Fourier methods produces an unstable profile, whereas the solution regularised by TSVD yields a stable and more accurate geometry. The residual ripple in the recovered radius derives from that noted in the input impulse response. Further work will attempt to minimise this problem by Tikhonov weighting of the truncation of the singular value series. The underestimation in the bore at widder radii is predictable from the absence, in the current reconstruction algorithm, of terms accounting for higher duct modes. This issue is currently being addressed by the authors.

4. CONCLUSIONS

The effectiveness of the TSVD in regularisation of the ill-posed inversion in acoustic pulse reflectometry has been demonstrated. The method is complementary to experimental attempts to increase the bandwidth of the input pulse.

5. REFERENCES

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Footnote

+The singular functions form the basis set of the singular system described by equation (1), and are distinct from the Fourier eigenfunctions. They are preferred in the analysis of highly-transient waveforms (see, for example, [5], [6] forthcoming).

Figures

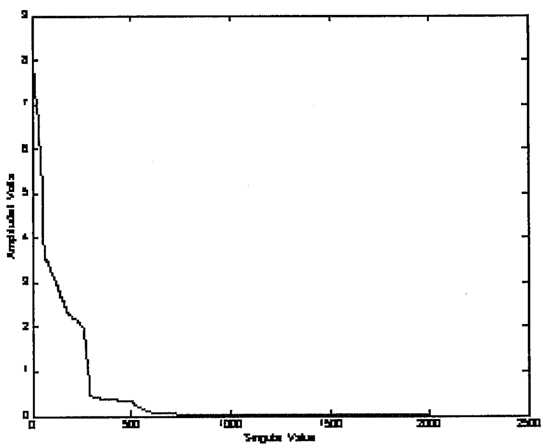

Figure (1). Set of singular values obtained by full decomposition of incident pulse matrix, **X**.

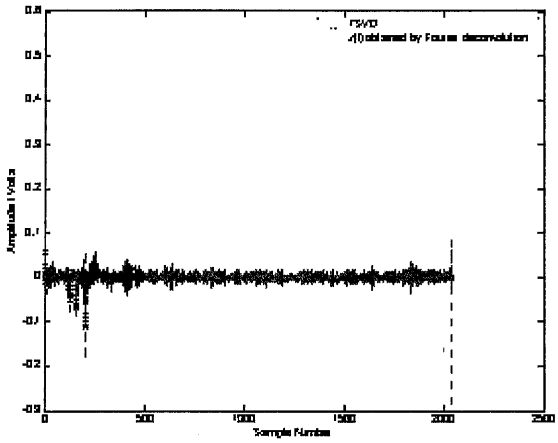

Figure (2). Comparison of $z(t)$ obtained by TSVD and Fourier deconvolution.

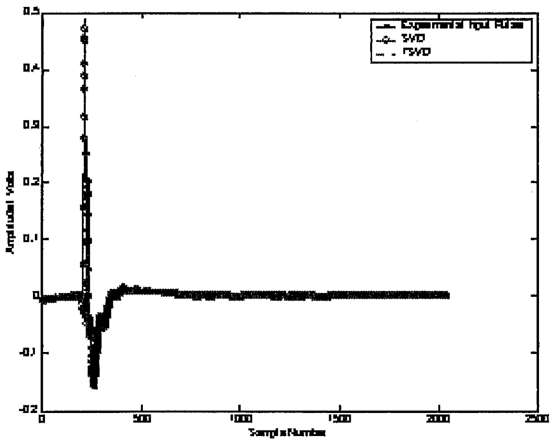

Figure (3). Measured input pulse, and that corresponding to SVD and TSVD.

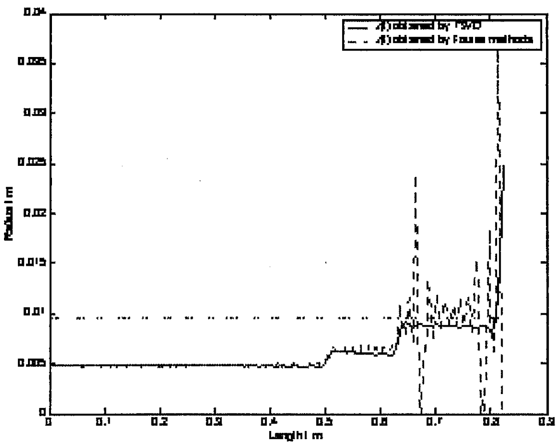

Figure (4). Bore reconstructions in comparison to measurement (horizontal lines).