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THE STATISTICAL ENERGY ANALYSIS OF COUPLED OSCILLATORS

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1 INTRODUCTION

Statistical energy analysis (SEA) has become an established method for the analysis of the noise and vibration behaviour of complex, built-up structures at higher frequencies [1]. The response is described in terms of the flow of energy through the structure. It is recognized that the properties of the structure are uncertain, and hence a statistical description is required. In principle at least, the structure is assumed to be drawn from an ensemble of similar structures whose properties are random and estimates of the ensemble average response are required.

The earliest approaches to SEA and derivations of the SEA equations concerned systems comprising two coupled oscillators and two coupled sets of oscillators whose properties are random. In [2] it was seen that for broadband excitation the energy flow between two, conservatively coupled oscillators is proportional to the difference in their blocked energies, i.e. the energies when the other oscillator is fixed. This result was then applied to the coupling power between two specific modes of coupled, multi-modal subsystems, which can be regarded as comprising sets of oscillators. In [3] the results were extended to the case of coupled oscillator sets, various assumptions being made to show that the coupling power is proportional to the difference in the mean blocked modal energies of the subsystems. The coupling loss factor was introduced and estimated using wave approaches rather than from the modal approach itself. A more formal approach was developed in [4], which adopted a statistical description for the properties of the sets of oscillators, and ensemble averages of the coupling power found. Finally in [5] it was shown that the coupling power between two oscillators is also proportional to the difference in their actual energies. It is on this base that SEA is founded.

The above analyses involve a few important assumptions and a number of less important ones. One is that the interaction of two oscillators is independent of the presence of the other oscillators in the sets. A second is that the coupling loss factor can be estimated by finding the ensemble average of the coupling parameters for the mode pairs.

Systems of coupled oscillators are revisited in this paper. In particular, the second assumption is removed, and full account is taken of the correlation between the coupling parameters for the mode pairs and the specific energies of those modes. Various observations are made about the qualitative and quantitative features of the behaviour under broadband excitation, some of which differ from those which are commonly assumed within SEA. It is seen that the coupling power or coupling loss factor can be written in terms of the “strength of connection” between the oscillators and a term involving their bandwidths and the separation of their natural frequencies. Equipartition of energy does not occur as damping tends to zero, except in the case where the uncoupled oscillators have identical natural frequencies. For coupled sets of oscillators the coupling loss factor depends on damping: it is proportional to damping at low damping and is independent of damping in the high damping, weak coupling limit. A parameter which describes the strength of coupling is identified: this depends on both the damping and the strength of connection.

The next section concerns the case of two coupled oscillators, and various expressions for powers and energies are derived and discussed. These are then applied to the case of two coupled sets of oscillators in section 3.

2 TWO SPRING-COUPLED OSCILLATORS

Consider the system comprising two spring-coupled oscillators shown in Figure 1. It is assumed that the system is linear. In the Appendix it is shown that if the forces $f_1(t)$ and $f_2(t)$ are random, stationary, statistically independent, band-limited white noise, with spectral densities S_1 and S_2 over some frequency band B, then the time average oscillator energies and input powers are

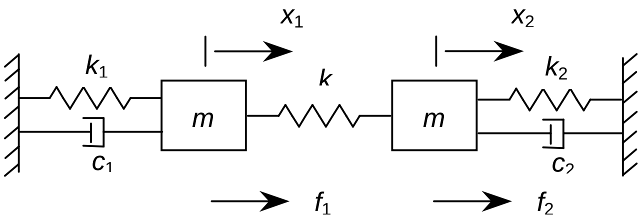

Figure 1 Two, spring-coupled oscillators.

$$E_1 = \left[\frac{\pi S_1}{m_1 \Delta_1} - \frac{k^2 (\Delta_1 + \Delta_2)}{m_1 m_2 \Delta_1} \frac{1}{Q} \left(\frac{\pi S_1}{m_1 \Delta_1} - \frac{\pi S_2}{m_2 \Delta_2} \right) \right] \tag{1}$$

$$E_2 = \left[\frac{\pi S_2}{m_2 \Delta_2} - \frac{k^2 (\Delta_1 + \Delta_2)}{m_1 m_2 \Delta_2} \frac{1}{Q} \left(\frac{\pi S_2}{m_2 \Delta_2} - \frac{\pi S_1}{m_1 \Delta_1} \right) \right] \tag{2}$$

$$P_1 = \frac{\pi S_1}{m_1}, \; P_2 = \frac{\pi S_2}{m_2} \tag{3}$$

where

$$Q = \left(\omega_1^2 - \omega_2^2\right)^2 + \left(\Delta_1 + \Delta_2\right)\left(\omega_2^2 \Delta_1 + \omega_1^2 \Delta_2\right) + \frac{k^2 (\Delta_1 + \Delta_2)^2}{m_1 m_2 \Delta_1 \Delta_2} \tag{4}$$

In the above equations $m_{1,2}$, $c_{1,2}$ and $k_{1,2}$ are the mass, damping constant and stiffness of the corresponding oscillators, and k is stiffness of the coupling spring. $\omega_{1,2} = \sqrt{(k_{1,2} + k)/m_{1,2}}$ are the 'blocked' natural frequencies of the oscillators, i.e., the natural frequency of one oscillator when the other is held stationary, while $\Delta_{1,2} = c_{1,2}/m_{1,2}$ are the half-power bandwidths. (Here the energy of each oscillator is defined as twice the kinetic energy: this is because the strain energy in the coupling spring cannot be unambiguously ascribed to one or other oscillator. There are no significant consequence of this.) The input powers are independent of the natural frequencies, while the energies depend on them though the term Q. The coupling power from one oscillator to the other is

$$P_{12} = \frac{k^2 (\Delta_1 + \Delta_2)}{m_1 m_2} \frac{1}{Q} \left(\frac{\pi S_1}{m_1 \Delta_1} - \frac{\pi S_2}{m_2 \Delta_2} \right) \tag{5}$$

This can be written in terms of the oscillator energy difference as

$$P_{12} = \beta (E_1 - E_2) \tag{6}$$

where the coupling parameter

$$\beta = \left(\frac{k^2}{m_1 m_2}\right) \frac{(\Delta_1 + \Delta_2)}{(\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\omega_2^2 \Delta_1 + \omega_1^2 \Delta_2)} \quad (7)$$

is a constant, independent of the oscillator energies.

Equation (6) is the familiar statement of coupling power proportionality: the coupling power is proportional to oscillator energy difference and energy flows from the oscillator with the higher energy to that with the lower energy. This equation was derived in Ref. [2] using a slightly different approach and forms the basis for the subsequent development of SEA. The coupling loss factor $\eta_{12} = \beta/\omega$ depends on the subsystem uncoupled natural frequencies, and in particular the difference between them through the term $(\omega_1^2 - \omega_2^2)$, the strength of connection (the coupling stiffness k) and the oscillator bandwidths Δ (or, more commonly, their loss factors $\eta = \Delta/\omega$).

In the above, the oscillator natural frequencies are taken as the blocked natural frequencies. This is one manner in which the oscillators can be uncoupled. The uncoupled system can equally be regarded as that in which the coupling spring $k = 0$. Coupling power proportionality holds equally in terms of the free natural frequencies $\omega_{f1} = \sqrt{k_1/m_1}$ and $\omega_{f2} = \sqrt{k_2/m_2}$. Thus in principle it is not relevant whether the oscillators are uncoupled by blocking (or clamping) the interface or by cutting (or freeing) it.

2.1 Discussion

Putting $\Delta = (\Delta_1 + \Delta_2)/2$ and $\omega = (\omega_1 + \omega_2)/2$ then, if the uncoupled frequencies of the oscillators are approximately equal so that $\omega_1 \approx \omega_2 = \omega$, it follows that

$$\beta = \frac{k^2}{2 m_1 m_2 \omega^2} \frac{\Delta}{\delta^2 + \Delta^2}, \quad (8)$$

where $\delta = \omega_1 - \omega_2$. This can equally be written as

$$\beta = \frac{\kappa^2}{2} \frac{\Delta}{\delta^2 + \Delta^2}, \quad (9)$$

where

$$\kappa^2 = \frac{k^2}{m_1 m_2 \omega^2} = \frac{k^2 \omega^2}{k_1 k_2} \quad (10)$$

In equation (9), κ is a measure of the strength of connection between the oscillators, while the second term $\Delta/(\delta^2 + \Delta^2)$ depends on both damping and the separation of the natural frequencies. If the natural frequencies lie within each other's half-power bandwidth, then $\delta < \Delta$, and the term tends to $1/\Delta$: the resonances overlap.

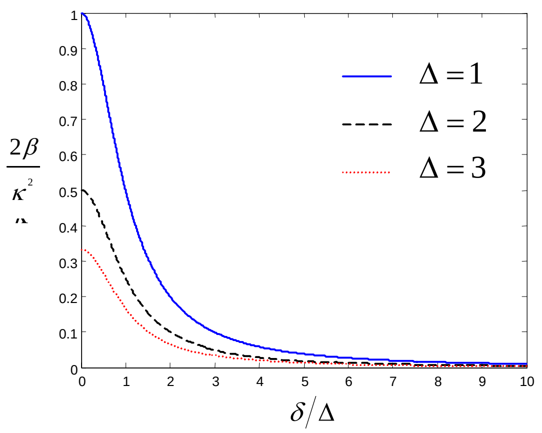

Figure 2. Coupling parameter β as a function of δ and Δ.

If, on the other hand, the resonances do not overlap then $\delta > \Delta$, the term tends to Δ/δ^2, and the coupling power becomes small. The coupling parameter β thus behaves somewhat as a band-pass filter of width Δ. As an example, Figure 2 shows β as a function of δ/Δ for various Δ. The narrow-band filtering effect is clear.

In the very special case where the uncoupled natural frequencies of the oscillators are identical, then $\delta = 0$. In the limit $\Delta \to 0$, then $\beta \to \infty$ and the oscillator energies become equal: this is equipartition of energy. However, if $\delta \neq 0$, then as $\Delta \to 0$, $\beta \to \kappa^2\Delta/2\delta^2$ and the energies are not equal: if only oscillator 1 is excited, i.e. $S_2 = 0$, and the energy ratio becomes

$$\frac{E_2}{E_1} = \frac{\kappa^2}{2\delta^2 + \kappa^2} \tag{11}$$

Thus we can conclude that equipartition of energy does *not* occur in the low-damping limit, except if the oscillator natural frequencies are identical: this is in contrast to what is commonly stated in the SEA community, but has been observed via wave [6] and system-mode approaches [7].

2.2 The ensemble and response statistics

In the situations to which SEA is applied the system properties are not known exactly, but are random. The statistics of the ensemble of two-oscillator systems is defined by the joint probability density function $p(m_1, \omega_1, \Delta_1; m_2, \omega_2, \Delta_2; k; S_1, S_2)$. The statistics of the response quantities can then be found. The expected value of the energy of oscillator 1, for example, is given by

$$\mathrm{E}[E_1] = \int E_1(m_1, \omega_1 ....) p(m_1, \omega_1 ....) dm_1 d\omega_1 .... \tag{12}$$

where $\mathrm{E}[\bullet]$ denotes the expectation.

How the statistics of the ensemble are defined is likely to be difficult in practice. It is reasonable to assume that the oscillators, coupling and excitations are statistically independent so that

$$p = p_1(m_1, \omega_1, \Delta_1)\, p_2(m_2, \omega_2, \Delta_2)\, p_k(k)\, p_{S_1}(S_1)\, p_{S_2}(S_2) \tag{13}$$

Furthermore, small changes in m_1, Δ_1, k, S_1 etc, generally give small changes in the response so that they may be assumed known. (In normal SEA terms, this implies that the 'infinite' input mobility, coupling transmission coefficient and damping levels are known.) The natural frequencies are usually uncertain, however, so that the probability density function reduces to $p_1(\omega_1)\, p_2(\omega_2)$. Finally, small changes in the natural frequencies affect the response primarily through their separation $\delta = (\omega_1 - \omega_2)$, which appears in the denominator of the oscillator energies.

Consequently, it will be assumed that the ensemble is defined so that both ω_1 and ω_2 are random and uniformly probable over some frequency band. If this band of uncertainty is fairly small compared to the bandwidth of excitations, so that "end effects" can be ignored, then it can equally be assumed that ω_1 is known and that $\delta = (\omega_1 - \omega_2)$ is random and uniformly probable in some band Ω. As a result, the ensemble of two, spring-coupled oscillators is defined by

$$p_\delta(\delta) = \begin{cases} 1/\Omega; & (-\Omega/2 \le \delta \le \Omega/2) \\ 0; & \text{otherwise} \end{cases} \tag{14}$$

Under these circumstances the expected values of the input powers and subsystem energies become

$$\bar{P}_1 = \frac{\pi S_1}{m_1}, \ \bar{P}_2 = \frac{\pi S_2}{m_2} \tag{15}$$

$$\bar{E}_1 = \frac{1}{\Omega}\int_\Omega E_1(\delta)\, d\delta, \ \bar{E}_2 = \frac{1}{\Omega}\int_\Omega E_2(\delta)\, d\delta \tag{16}$$

where $\bar{\bullet}$ denotes the ensemble average. From the above it follows that

$$\begin{aligned} \bar{E}_1 &= \frac{\pi S_1}{m_1\Delta_1} - \frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_1} \mathrm{E}\left[\frac{1}{Q}\right]\left(\frac{\pi S_1}{m_1\Delta_1} - \frac{\pi S_2}{m_2\Delta_2}\right) \\ \bar{E}_2 &= \frac{\pi S_2}{m_2\Delta_2} - \frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_2} \mathrm{E}\left[\frac{1}{Q}\right]\left(\frac{\pi S_2}{m_2\Delta_2} - \frac{\pi S_1}{m_1\Delta_1}\right) \end{aligned} \tag{17}$$

where

$$\mathrm{E}\left[\frac{1}{Q}\right] = \frac{1}{\Omega}\int_\Omega \frac{1}{Q}\, d\delta = \frac{\pi}{4\omega^2}\frac{1}{\Omega}\frac{1}{\sqrt{\Delta^2+\kappa^2\left(\Delta^2/\Delta_1\Delta_2\right)}} \tag{18}$$

Here it is assumed that the uncertainty bandwidth Ω is large compared to the half-power bandwidths. The limits of integration can then be replaced by $(-\infty, +\infty)$ to a good approximation.

2.2.1 Discussion

Using the simplification $\Delta_1 \approx \Delta_2 = \Delta$, equation (18) becomes

$$\mathrm{E}\left[\frac{1}{Q}\right] = \frac{\pi}{4\omega^2}\frac{1}{\Omega}\frac{1}{\sqrt{\Delta^2+\kappa^2}} \tag{19}$$

The ensemble average energies can be written in terms of the input powers as

$$\bar{\mathbf{E}} = \mathbf{A}\bar{\mathbf{P}} \tag{20}$$

where

$$\mathbf{A} = \frac{1}{\Delta}\begin{bmatrix} 1-\alpha/\Omega & \alpha/\Omega \\ \alpha/\Omega & 1-\alpha/\Omega \end{bmatrix} \tag{21}$$

is a matrix of energy influence coefficients, and where

$$\alpha = \frac{\pi}{2} \frac{\kappa^2}{\sqrt{\Delta^2 + \kappa^2}} \tag{22}$$

is a parameter that depends on both the damping Δ and the strength of connection κ. Together these indicate the strength of coupling between the two oscillators via the parameter

$$\gamma = \frac{\kappa}{\Delta} \tag{23}$$

If $\gamma < 1$ the coupling is weak while if $\gamma > 1$ the coupling is strong.

From equations (20)-(21) it follows that

$$\bar{\mathbf{P}} = \mathbf{L}\bar{\mathbf{E}} \tag{24}$$

where

$$\mathbf{L} = \mathbf{A}^{-1} = \omega \begin{bmatrix} \eta + \eta_{12} & -\eta_{21} \\ -\eta_{12} & \eta + \eta_{21} \end{bmatrix} \tag{25}$$

is a matrix of damping and coupling loss factors and where

$$\eta_{12} = \eta_{21} = \frac{\alpha/\Omega}{1 - 2\alpha/\Omega} \eta \tag{26}$$

is the coupling loss factor. Note that $\omega\eta_{12}$ depends on damping. At low damping $\eta_{12} = \left(\pi\kappa/\left(2\Omega - \pi\kappa\right)\right)\eta$ is proportional to η, while for high damping it asymptotes to the constant $\omega\eta_{12} = \pi\kappa^2/2\Omega$.

Note that the conclusions drawn above are in stark contrast to those normally drawn in SEA: the coupling loss factor is a function of damping; there is a parameter $\gamma = \kappa/\Delta$ that describes the strength of coupling between the two oscillators; equipartition does not occur in the limit $\Delta \to 0$. The reason for this is that in previous analyses the coupling loss factor was defined from the expected value of the coupling power $\mathrm{E}\left[P_{12}\right] = \mathrm{E}\left[\beta\left(E_1 - E_2\right)\right]$ with the further assumption being made that the coupling term β and the energy difference are statistically independent, so that

$$\mathrm{E}\left[P_{12}\right] = \mathrm{E}\left[\beta\right]\mathrm{E}\left[E_1 - E_2\right] \tag{27}$$

and where it is found that $\mathrm{E}\left[\beta\right]$ is a constant, independent of damping. This is clearly an approximation - β and $\left(E_1 - E_2\right)$ are strongly correlated: if β is large then $\left(E_1 - E_2\right)$ is small, and vice versa.

3 TWO SETS OF OSCILLATORS

Consider now the case of two, spring-coupled subsystems *a* and *b*. As in conventional SEA, each subsystem is regarded as being a set of oscillators and, when coupled, each oscillator in one set shares energy with all the oscillators in the second set.

The subsystems are excited by white noise over some frequency band *B*. Their modal densities are n_a and n_b, so that the number of modes in this band is, on average, $N_{a,b} = n_{a,b}B$. we assume that the total input powers are simply the sums of the powers input to each oscillator, i.e.

$$P_a = \sum_{j=1,N_a} P_j^{(a)}; \quad P_b = \sum_{k=1,N_b} P_k^{(b)} \tag{28}$$

where $P_j^{(a)}$ is the power input to oscillator *j* in set *a*. Similarly the subsystem energies and the coupling power are

$$E_a = \sum_{j=1,N_a} E_j^{(a)}; \quad E_b = \sum_{k=1,N_b} E_k^{(b)}; \quad P_{ab} = \sum_{j=1,N_a} \sum_{k=1,N_b} P_{jk}^{(ab)}; \tag{29}$$

where $P_{jk}^{(ab)}$ is the coupling power between oscillators *j* and *k* in sets *a* and *b*.

The assumption is now made, as in previous approaches, that the coupling power between oscillators *j* and *k* depends on their energy difference in a manner that is independent of the presence of the remaining oscillators. This implies that coupling power proportionality still holds and hence, from equation (5)

$$P_{jk}^{(ab)} = \frac{k_{jk}^2 \left(\Delta_j + \Delta_k\right)}{m_j m_k} \frac{1}{Q_{jk}} \left(\frac{\pi S_j}{m_j \Delta_j} - \frac{\pi S_k}{m_k \Delta_k} \right) \tag{30}$$

while

$$P_j^{(a)} = \frac{\pi S_j}{m_j} \tag{31}$$

Hence the energy in set *a* becomes

$$E_a = \sum_{j=1}^{N_a} \left[\frac{\pi S_j}{m_j \Delta_j} - \sum_{k=1}^{N_b} \frac{k_{jk}^2 \left(\Delta_j + \Delta_k\right)}{m_j m_k Q_{jk} \Delta_j} \left(\frac{\pi S_j}{m_j \Delta_j} - \frac{\pi S_k}{m_k \Delta_k} \right) \right] \tag{32}$$

3.1 The ensemble, response statistics and SEA

The properties of the two sets of oscillators are random and the ensembles from which they are drawn defined by a joint probability distribution function *p* which depends on the oscillator masses, bandwidths and natural frequencies, the coupling spring stiffness and the excitation spectral densities. Clearly the problem is extremely complicated, and estimation of ensemble statistics requires the evaluation of an integral of very high dimension.

As in Section 2, assumptions will be made about the ensemble. The properties of the two sets of oscillators, those of the excitations and those of the coupling are assumed statistically independent. Each oscillator in set a has the same mass m_a, bandwidth Δ_a and ensemble average spectral density S_a, with similar assumptions for set *b*. The coupling stiffness is k_{ab} for all mode pairs. However, the statistics of the natural frequencies, and in particular their spacing, are important. The joint probability density function that defines the ensemble then becomes

$$p_{\omega_a}\left(\omega_1^{(a)},\omega_2^{(a)},\ldots,\omega_{N_a}^{(a)}\right)p_{\omega_b}\left(\omega_1^{(b)},\omega_2^{(b)},\ldots,\omega_{N_b}^{(b)}\right) \qquad (33)$$

Taking the ensemble average of equations (31) and (32) then gives

$$\overline{P_a}=\sum_{j=1}^{N_a}P_j^{(a)}=\frac{N_a\pi S_a}{m_a};\quad \overline{P_b}=\sum_{k=1}^{N_b}P_k^{(b)}=\frac{N_b\pi S_b}{m_b} \qquad (34)$$

$$\overline{E_a}=\frac{N_a\pi S_a}{m_a\Delta_a}-k_{ab}^2\mathrm{E}\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{1}{Q_{jk}}\right]\frac{\left(\Delta_a+\Delta_b\right)}{m_a m_b\Delta_a}\left(\frac{\pi S_a}{m_a\Delta_a}-\frac{\pi S_b}{m_b\Delta_b}\right) \qquad (35)$$

$$\overline{E_b}=\frac{N_b\pi S_b}{m_b\Delta_b}-k_{ab}^2\mathrm{E}\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{1}{Q_{jk}}\right]\frac{\left(\Delta_a+\Delta_b\right)}{m_a m_b\Delta_b}\left(\frac{\pi S_b}{m_b\Delta_b}-\frac{\pi S_a}{m_a\Delta_a}\right) \qquad (36)$$

The expectations in equations (35) and (36) are

$$\mathrm{E}\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{1}{Q_{jk}}\right]=\int\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{1}{Q_{jk}}p_{\omega_a}\left(\omega_1^{(a)}\ldots\omega_{N_a}^{(a)}\right)p_{\omega_b}\left(\omega_1^{(b)}\ldots\omega_{N_b}^{(b)}\right)d\omega_1^{(a)}\ldots d\omega_{N_a}^{(a)}d\omega_1^{(b)}\ldots d\omega_{N_b}^{(b)} \qquad (37)$$

Following the same arguments as those in section 2.2, it is evident that it is the statistics of the natural frequencies that are dominant in determining the ensemble average energies. In particular, the term $1/Q_{jk}$ depends most sensitively on $\delta_{jk}=\omega_j^{(a)}-\omega_k^{(b)}$. For a selected pair of oscillators this is a uniformly distributed random variable, as considered for the case of two coupled oscillators above, so that $\mathrm{E}\left[1/Q_{jk}\right]$ is given by equation (18). However, for the case of coupled sets of oscillators, although $\left(\omega_j^{(a)}-\omega_k^{(b)}\right)$ might be uniformly probable, $\left(\omega_{j+1}^{(a)}-\omega_k^{(b)}\right)$ depends also on the natural frequency spacing statistics of subsystem a, which define the probability density function for $\left(\omega_{j+1}^{(a)}-\omega_k^{(b)}\right)$. In other words, $\left(\omega_{j+1}^{(a)}-\omega_k^{(b)}\right)=\left(\omega_j^{(a)}-\omega_k^{(b)}\right)+\left(\omega_{j+1}^{(a)}-\omega_j^{(a)}\right)$, and while $\left(\omega_j^{(a)}-\omega_k^{(b)}\right)$ may be uniformly probable, $\left(\omega_{j+1}^{(a)}-\omega_j^{(a)}\right)$ usually is not.

Thus the ensemble average energies depend on the modal densities in sets *a* and *b* (which determine the numbers of modes in the band *B*) and the natural frequency spacing statistics of sets *a* and *b* (which determine the correlations between the terms $1/Q_{jk}$ for various mode pairs (j,k)). Common spacing statistics include a Poisson distribution, which implies that $\omega_j^{(a)}$ and $\omega_{j+1}^{(a)}$ are statistically independent. This situation is characteristic of uniform, regular two- or three-dimensional subsystems (i.e., rectangular, or other shapes with a separable geometry). Examples include rectangular plates. Another distribution is one in which the spacing statistics are fully rigid, so that $\omega_{j+1}^{(a)} - \omega_k^{(a)} = 1/n_a$. This implies the natural frequencies are uniformly spaced. An example is that of a uniform, one-dimensional subsystem such as a rod, for which the natural frequency spacing is $\pi c/l$, *c* and *l* being the wave speed and the length. One final example is that of GOE spacing statistics, which are held to be typical of subsystems with large amounts of randomness. The natural frequency spacing is then Rayleigh distributed, but there are also statistics relating second-nearest neighbours, etc.

From the analysis in Section 2.2 it follows that $\mathrm{E}\left[1/Q_{jk}\right]$ depends on the natural frequency separation $\delta_{jk} = \omega_j^{(a)} - \omega_k^{(b)}$ and the mean modal bandwidth, and hence the modal overlap. Further, in the high modal overlap limit, many natural frequencies lie within the half-power bandwidth of each other, and hence the ensemble average energies become independent of the natural frequency spacing statistics. This is in accord with observations.

3.2 Example: Poisson distribution

As an example, consider the case where the natural frequencies are random, uniformly distributed and uncorrelated for both sets of oscillators. The natural frequency spacing statistics are then Poisson distributed. An example is that of two rectangular, uniform plates connected by a point spring. The joint probability density function that defines the ensemble is now

$$p_{\omega_a}\left(\omega_1^{(a)}\right) p_{\omega_a}\left(\omega_2^{(a)}\right) \dots p_{\omega_b}\left(\omega_{N_b}^{(b)}\right) p_{\omega_a}\left(\omega_1^{(a)}\right) p_{\omega_b}\left(\omega_2^{(b)}\right) \dots p_{\omega_b}\left(\omega_{N_b}^{(b)}\right) \tag{38}$$

where for each natural frequency

$$p_{\omega_a}\left(\omega_j^{(a)}\right) = p_{\omega_b}\left(\omega_k^{(b)}\right) = \frac{1}{B} \tag{39}$$

If it is assumed for simplicity that $\Delta_a \approx \Delta_b = \Delta$ then, from equation (19)

$$\mathrm{E}\left[\frac{1}{Q_{jk}}\right] = \frac{\pi}{4\omega^2} \frac{1}{\mathrm{B}} \frac{1}{\sqrt{\Delta^2 + \kappa^2}} \tag{40}$$

while

$$\mathrm{E}\left[\sum_{j=1}^{N_a} \sum_{k=1}^{N_b} \frac{1}{Q_{jk}}\right] = N_a N_b \mathrm{E}\left[\frac{1}{Q_{jk}}\right] \tag{41}$$

Consequently from equations (34) and (35)

$$\overline{E_a} = \frac{1}{\Delta}\left(\overline{P_a} - \frac{N_a N_b}{B} \frac{\pi}{2} \frac{\kappa^2}{\sqrt{\Delta^2 + \kappa^2}} \left(\frac{\overline{P_a}}{N_a} - \frac{\overline{P_b}}{N_b}\right)\right) \tag{42}$$

with a similar expression for $\overline{E_b}$. Putting

$$\alpha = \frac{\pi}{2} \frac{\kappa^2}{\sqrt{\Delta^2 + \kappa^2}} \tag{43}$$

as in the case of two coupled oscillators, then the energies and powers are related by

$$\overline{\mathbf{E}} = \frac{1}{\Delta} \begin{bmatrix} 1-\alpha n_b & \alpha n_a \\ \alpha n_b & 1-\alpha n_a \end{bmatrix} \overline{\mathbf{P}} \tag{44}$$

By inverting this matrix it follows that the coupling loss factors are such that

$$\omega n_a \eta_{ab} = \omega n_b \eta_{ba} = \frac{\alpha n_a n_b}{1-\alpha\left(n_a + n_b\right)} \Delta \tag{45}$$

In summary, coupling power proportionality holds between subsystems *a* and *b*, with the total coupling power being proportional to the difference in the modal energies and the coupling loss factors satisfy the consistency relation (sometimes referred to as reciprocity), conclusions that are consistent with previous analyses. However, the coupling energy flow and the coupling loss factors depend not only on the strength of connection (through κ) but also on the level of damping (through Δ). For high levels of damping

$$\omega n_a \eta_{ab} \approx \frac{\pi \kappa^2}{2} n_a n_b \tag{46}$$

is independent of damping and depends only on the strength of connection (c.f. the coupling transmission coefficient) while for low levels of damping

$$\omega n_a \eta_{ab} \approx \frac{\pi \kappa n_a n_b}{2 - \pi\kappa\left(n_a + n_b\right)} \Delta \tag{47}$$

is proportional to damping. (For a large enough strength of connection it is possible that the denominator in the last equation becomes negative, but in this case the assumptions that each mode pair interacts independently of the others breaks down).

4 CONCLUSIONS

In this paper the SEA of coupled oscillators was revisited. In particular, the assumption was removed that the coupling parameter β and the energy difference for an oscillator pair are uncorrelated. Some of the subsequent conclusions then differ from those of conventional SEA, but are consistent with other observations (e.g. [6,7]),

First, it was seen that equipartition of energy does not occur in the low damping limit except in the very special case where the oscillators have identical natural frequencies. Secondly, the coupling loss factors are seen to depend not only on the strength of connection (the spring parameter κ, the coupling transmission coefficient τ etc.) but also on the level of damping: for large enough damping the coupling loss factors asymptote to a constant, while for low damping they are proportional to the damping, with the constant of proportionality depending on the natural frequency spacing statistics of the oscillator sets.

A coupling parameter $\gamma = \kappa / \Delta$ describing the strength of coupling was found: weak and strong coupling correspond to the cases of small and large γ respectively. Finally the analysis provides a method by which the coupling loss factors can be estimated from modal properties. the practical situations where the behaviour described in this paper is likely to be important are primarily structural applications, where the modal overlap is low, rather than acoustic ones.

5 ACKNOWLEDGEMENTS

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APPENDIX: ANALYSIS OF TWO COUPLED OSCILLATORS

The equations of motion of the two oscillator coupled system shown in Figure 1 are

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + \left(k_1 + k\right)x_1 - kx_2 = f_1\left(t\right);\; m_2\ddot{x}_2 + c_2\dot{x}_2 + \left(k_2 + k\right)x_2 - kx_1 = f_2\left(t\right) \qquad \text{(A1)}$$

where $m_{1,2}$, $c_{1,2}$ and $k_{1,2}$ are the mass, damping and stiffness of the corresponding oscillators, and k is stiffness of the coupling spring. For time-harmonic excitations $f_{1,2}\left(t\right) = F_{1,2}\exp\left(i\omega t\right)$, the amplitudes $x_{1,2}$ of the masses are determined by the frequency response matrix

$$H\left(\omega\right) = \frac{1}{m_1 m_2 D\left(\omega\right)}\begin{bmatrix} m_2\left(\omega_2^2 - \omega^2 + i\omega\Delta_2\right) & k \\ k & m_1\left(\omega_1^2 - \omega^2 + i\omega\Delta_1\right)\end{bmatrix} \qquad \text{(A2)}$$

In the above equation, $\omega_{1,2} = \sqrt{\left(k_{1,2} + k\right)/m_{1,2}}$ are the 'blocked' natural frequencies of the oscillators, i.e., the natural frequency of one oscillator when the other is held stationary, while $\Delta_{1,2} = c_{1,2}/m_{1,2}$ are the half-power bandwidths, while

$$D\left(\omega\right) = \omega^4 - i\omega^3\left(\Delta_1 + \Delta_2\right) - \omega^2\left(\omega_1^2 + \omega_2^2 + \Delta_1\Delta_2\right) + i\omega\left(\omega_2^2\Delta_1 + \omega_1^2\Delta_2\right) + \left(\omega_1^2\omega_2^2 - \frac{k^2}{m_1 m_2}\right) \qquad \text{(A3)}$$

Suppose the forces are random and statistically independent. Their power spectral densities are $S_{f_1}\left(\omega\right)$ and $S_{f_2}\left(\omega\right)$, while the cross spectral density $S_{12}\left(\omega\right) = 0$. The power spectral densities of the velocities of the masses are thus

$$\begin{Bmatrix} S_{V_1} \\ S_{V_2} \end{Bmatrix} = \omega^2 \begin{bmatrix} |H_{11}|^2 & |H_{12}|^2 \\ |H_{12}|^2 & |H_{22}|^2 \end{bmatrix} \begin{Bmatrix} S_{f_1}(\omega) \\ S_{f_2}(\omega) \end{Bmatrix} \tag{A4}$$

It is now assumed that the excitations are band-limited white noise so that $S_{f_1}(\omega)$ and $S_{f_2}(\omega)$ are constants S_1 and S_2 over some frequency band B. The energies of the oscillators are now considered to equal twice the kinetic energies. (The strain energy stored in the coupling spring cannot be unambiguously allocated to one or other oscillators.) The time average energies, the input powers and the coupling power are then

$$E_{1,2} = \int_B 2K'_{1,2}(\omega)\,d\omega; \quad P_{1,2} = \int_B P'_{1,2}(\omega)\,d\omega; \quad P_{12} = \int_B P'_{12}(\omega)\,d\omega \tag{A5}$$

where

$$K'_1 = \frac{1}{4}\frac{m_1\omega^2}{m_1^2 m_2^2 |D(\omega)|^2}\left\{m_2^2\left[\left(\omega^2 - \omega_2^2\right)^2 + \Delta_2^2\omega^2\right]S_1 + k^2 S_2\right\} \tag{A6}$$

$$P'_1 = \frac{\Delta_1}{2m_1|D(\omega)|^2}\omega^2\left[\left(\omega^2 - \omega_2^2\right)^2 + \omega^2\Delta_2^2 + \frac{\Delta_2 k^2}{\Delta_1 m_1 m_2}\right]S_1 \tag{A7}$$

$$P'_{12} = \frac{k^2\omega^2}{m_1^2 m_2^2 |D(\omega)|^2}\left[\frac{1}{2}m_2\Delta_2 S_1 - \frac{1}{2}m_1\Delta_1 S_2\right] \tag{A8}$$

It is now assumed that the bandwidth of excitation B includes both natural frequencies ω_1 and ω_2, and that the damping is light so that the response is dominated by the resonances. The limits of the integrations in the above can then be extended to $(-\infty, +\infty)$. The time average energies and powers then become

$$P_1 = \frac{\pi S_1}{m_1}, \; P_2 = \frac{\pi S_2}{m_2} \tag{A9}$$

$$E_1 = \left[\frac{\pi S_1}{m_1\Delta_1} - \frac{k^2(\Delta_1 + \Delta_2)}{m_1 m_2 \Delta_1}\frac{1}{Q}\left(\frac{\pi S_1}{m_1\Delta_1} - \frac{\pi S_2}{m_2\Delta_2}\right)\right] \tag{A10}$$

$$P_{12} = \frac{k^2(\Delta_1 + \Delta_2)}{m_1 m_2}\frac{1}{Q}\left[\frac{\pi S_1}{m_1\Delta_1} - \frac{\pi S_2}{m_2\Delta_2}\right] \tag{A11}$$

$$Q = \left(\omega_1^2 - \omega_2^2\right)^2 + (\Delta_1 + \Delta_2)\left(\omega_2^2\Delta_1 + \omega_1^2\Delta_2\right) + \frac{(\Delta_1 + \Delta_2)^2 k^2}{m_1 m_2 \Delta_1\Delta_2} \tag{A12}$$