FIELD AND SURFACE RECONSTRUCTIONS FOR 3D ROUGH SURFACE PROBLEMS

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INTRODUCTION

Acoustic scattering problems by rough surfaces arise frequently in the natural world and the study of the direct and especially the inverse problem is of great interest in the sciences. A well-known example is given by the outdoor sound propagation and noise control. Here, we study the inverse scattering problem where the unknown scatterer is a surface which is a non-local perturbation of an infinite plane surface, compare Figure 1. The task is the reconstruction of the total acoustic field and the shape of the scatterer from remote measurements of a time-harmonic acoustic field.

Our goal is the reconstruction of the shape of the surface by recovering the total field from the knowledge of the values of the total field on a measurement plane. We further assume to know the location of some point source (incident field) and that the rough surface is sound-soft (Dirichlet boundary condition). We then consider a single layer approach to reconstruct the unknown scatterer. Using single layer potentials for the solution of shape reconstruction problems was first suggested by Kirsch-Kress (see [1]) in the case of bounded obstacles. The basic idea of this method is the reformulation of the inverse problem in an optimization problem, where the field is approximated by a single layer potential and the unknown surface is simultaneously fitted as a zero curve of the sum of scattered and incident field. The key difficulty of the inverse rough surface scattering problem is the non-compactness of the scatterer and the non-compactness of the associated single layer potential. This significantly changes the mathematical analysis of the optimization problem and generates problems for convergence and stability.

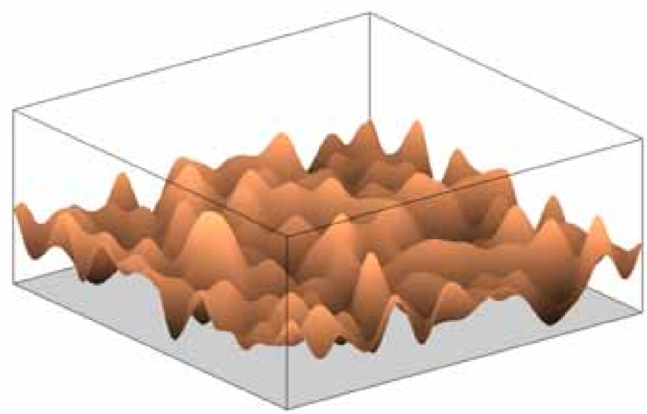

Figure 1: Illustration of typical rough surface.

Here we develop the Kirsch-Kress Method for rough surfaces. In particular, we carry out the full analysis of the Kirsch-Kress method for rough surfaces using a semi-finite approach. We show the existence of an *optimal* surface and the *convergence* of the method towards a *solution of the inverse problem*. In the last section, we present reconstructions of a finite section of the unknown scatterer. Here, we extend recent results of [2], to the inverse problem. The numerical results prove the feasibility of the method.

THE ROUGH SURFACE SCATTERING PROBLEM

The Direct Problem

Before we start with the inverse problem we first need to describe and study the direct problem. Here, the scattering surface Γ is given by the graph of a function $f : R^2 \to R$ and the domain of (sound) propagation is defined by

$$\Omega = \Omega_f = \left\{ x \in R^3 : x_3 > f(x_1, x_2) \right\}. \qquad (1)$$

The incident field u^i is given by a point source located in the point $z \in R^3$ above the scattering surface defined by

$$\Phi(x,z) = \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x-z|} \qquad (2)$$

We want to find the scattered field $u^s = u - u^i$ such that the total field u is a solution of the Helmholtz equation

$$\Delta u + k^2 u = 0, \qquad (3)$$

in Ω_f. Here, k denotes the wave number and it is either positive or possesses positive real and imaginary part. We further assume that the surface is sound-soft, i.e. the total field vanishes

$$u = 0 \qquad (4)$$

on the boundary Γ. For the completeness of the analysis we require that the scattered field satisfies the bound

$$|u^s(x)| \leq c \qquad (5)$$

in the domain of propagation. In the case of a real wave number, we further require the limiting absorption principle. This means that the solution for the real wave number is a pointwise limit of the solution where we replace the zero imaginary part of the wave number with a sufficiently small imaginary part. For the reformulation of the direct problem via a well-posed boundary integral equation we refer to [3] and [4].

For the inverse problem we assume to know the total field on a finite measurement plane $\Gamma_{h,A}$ defined by

$$\Gamma_{h,A} = \left\{ x \in R^3 : x_3 = h, x_1 \leq A, x_2 \leq A \right\} \qquad (6)$$

Measurements can be carried out for example by a synchronized array of microphones or by some time-domain measurements with one microphone which is moved along the surface where well-defined incident pulses are repeated in a controlled way. Frequency domain data with full phase information is then obtained by Fourier transform.

Our inverse problem is formulated as follows: Suppose we know the incident field $u^i = \Phi(.,z)$ and the scattered field u^s on the measurement plane $\Gamma_{h,A}$ and that the unknown scatterer is sound-soft. Then, find the total field u and the unknown scatterer such that $u - u^i$ coincides with the measurement values for $x \in \Gamma_{h,A}$.

THE KIRSCH-KRESS METHOD FOR THE INVERSE PROBLEM

For the solution of this problem we first subtract a mirrored fundamental solution from our measured data, i.e. given the data u^s on $\Gamma_{h,A}$ we define v by

$$u^s(x) = v - \Phi(x,(z_1, z_2, -z_3)) \tag{7}$$

and seek a representation of v as a single layer potential, i.e.

$$S\varphi(x) = v(x) \text{ for every } x \in \Gamma_{h,A} \tag{8}$$

where S is defined as the single layer potential

$$S\varphi(x) = \int_{\Gamma_t} G(x,y)\varphi(y)ds(y)\,. \tag{9}$$

and the kernel $G(x,y)$ is the half-space Greens function given by

$$G(x,y) = \Phi(x,y) - \Phi(x,(y_1, y_2, -y_3)),$$

for $x = (x_1, x_2, x_3), y$ are points in R^3 which are not equal. Here, the integration domain is a planar test surface, which we choose such that the plane lies below the unknown scatterer. Since S is a compact operator from $L^2(\Gamma_t) \to L^2(\Gamma_{h,A})$ the above equation is ill-posed (instable) and we need to employ some regularisation strategy. To solve $S\varphi(x) = v(x)$ on $\Gamma_{h,A}$ we apply the classical Tikhonov regularisation for stabilization, i.e. we solve

$$\alpha S^*\varphi + S^*S\varphi = S^*v \tag{10}$$

on $\Gamma_{h,A}$ with a regularisation parameter $\alpha > 0$. The approximation of the total field is then given by

$$u = u^i + u^s = S\varphi + G(.,z). \tag{11}$$

This solves the *field reconstruction* problem. For *shape reconstruction* we then need to use the boundary condition. The zeros of the exact total field represent the location of the scattering surface in case of Dirichlet boundary condition. Therefore, we seek the scattering surface as a minimum surface of the total field u.

The original realization of the Kirsch-Kress method employs the minimization of the approximated field

$$\| u^i + u^s \| = \| S\varphi + G(.,z) \| \tag{12}$$

in the L^2-norm over some suitable set of admissible surfaces Γ and the simultaneous minimization of the Tikhonov functional

$$J_{\alpha,B}(\varphi) = \| SP_B\varphi - v \|^2 + \alpha \| P_B\varphi \|^2 . \tag{13}$$

Here, we use the projection operator P_B which truncates a function on R^2 to its values inside the disk

$$D_B = \{x \in R^3 : |x| = \max\{x_1, x_2\} \leq B\} . \quad (14)$$

This means that we work with densities φ which are compactly supported, which can be seen as a semi-finite approach due to the finite support of the density. To obtain a rigorous mathematical convergence analysis the idea of Kirsch-Kress is to consider the functional

$$\mu(\varphi, \Gamma) = \| SP_B\varphi - v \|^2 + \alpha \| P_B\varphi \|^2 + \| S\varphi + G(.,z) \|^2_{L^2(\Gamma)} \quad (15)$$

and to look for a pair of a curve Γ and a density φ such that this functional is minimized. We note that the minimization over the surfaces Γ takes place in the last term in the norm defined by

$$\| w \|_{L^2(\Gamma)} = (\int_\Gamma | w(x) |^2 \, dx)^{1/2} \quad (16)$$

for a complex valued function $w : \Gamma \to C$.

Our results on the convergence of the scheme can be summarized as follows: We can show that such an 'optimal' pair minimizing (15) exists for every α > 0. Furthermore, given a null-sequence of α_n , every accumulation point of these pairs (which yields accumulation points of surfaces Γ) solves the inverse problem. Mathematical details are worked out in [6].

NUMERICAL REALIZATION

We now turn to the description of the numerical realisation. First, we need the measurement data on a measurement plane. To derive this data we can use the iterative solver investigated in [2], based on either the finite section approach or the multi-section approach. The solution of the direct problem evaluated on a plane will be the start information which we need to solve the inverse problem.

We remark that the problem quickly becomes large. With a minimum of 5 grid points per wavelength and an area of 200 wave length we have already 1 million unknown parameters on the surface patch. Here, we carried out tests with 1-5.000 unknowns.

In the following example we choose

- the location of the point source in (-3, 0, 15),
- the measurements plane to be in the cross section (-10,10)x(-10,10) and in the height of 7.5 with a discretization of 25x26 equidistant grid points,
- the wave number is k=1

The data setting for the reconstruction is

- the height of the test surface h = 0.5,
- the parameter $\alpha = 10^{-6}$,
- the evaluation domain for the reconstructed surface/total field.

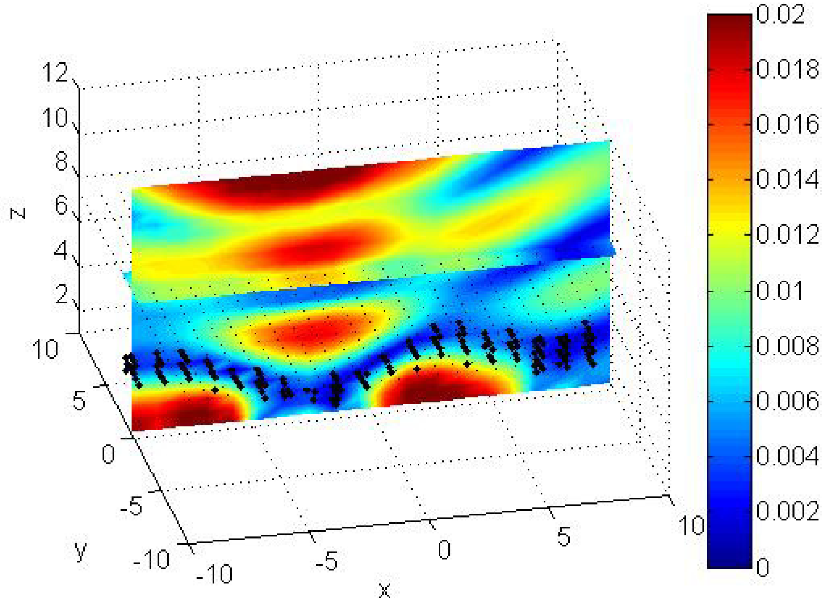

Figure 2:We show both a field and a shape reconstruction for some surface part. The measurements are indicated by the plane parallel to the x,y-plane. These measurements are used to derive the total field shown. We indicate the minimization of the total field by the black dotted curve.

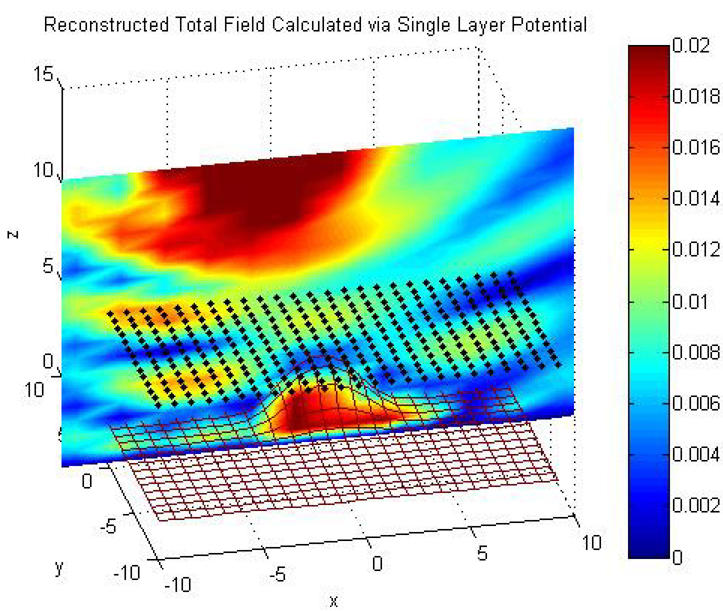

Figure 3: Reconstruction test for some local variation of the surface. It can be observed that the quality of field reconstruction deteriorates as expected when we move away from the region between the measurement surface and the scattering surface.

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