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FAT STRINGS

D J E Nunn

School of Engineering, University of Durham, UK

ABSTRACT

The classical analysis of a vibrating string assumes that the string is perfectly flexible. This implies that the partials will be harmonically related. In practice, however, the finite stiffness of real strings means that the partials become sharper with increasing frequency. This paper examines an extreme case of frequency stretching in which the second partial falls at three times the fundamental frequency. This vibrating system can be thought of as intermediate between a string and a bar, and might be expected to have an intermediate timbre. This can be made using a "fat string", much thicker and shorter than that for a near-harmonic string. I start by giving a theoretical overview of frequency stretching and inharmonicity, and derive the dimensions required for fat strings. I then give synthetic examples of the timbres available. Finally I discuss practical considerations in designing instruments using fat strings, and present prototype instruments using these principles.

1. INTRODUCTION

The displacement y of a string at position x is given by equation (1), where T is the tension, Y is the Young's modulus, S is the cross-sectional area, K is the radius of gyration, and ρ is the mass per unit length.[1] The amplitude of vibration is assumed to be small with respect to the length of the string, and friction is assumed to be negligible. The frequencies depend on the end conditions. For a string pinned at both ends, $y=d^2y/dx^2=0$ at $x=\pm\frac{1}{2}L$, where L is the length of the string. The modal frequencies are then given by equations (2)-(4). Note that the fundamental frequency is not f_0 but $f_1=f_0\sqrt{(1+B)}$. The motion of a string clamped at both ends is more complex; the modal frequencies are given by equation (2) but with f_0 multiplied by a constant. Experiments by Shankland indicate that the behaviour of typical piano bridges lies between the pinned and clamped cases.[7] For simplicity I shall assume in this paper that the equations for the pinned condition are valid.

$$T\frac{d^2y}{dx^2} - YSK^2\frac{d^4y}{dx^4} - \rho.\frac{d^2y}{dt^2} = 0 \qquad (1)$$

General equation of motion for strings.

$$f_n = nf_0\sqrt{1+Bn^2} \qquad (2)$$

where

$$B = \frac{\pi^2 YSK^2}{TL^2} \qquad (3)$$

$$f_0 = \frac{1}{2L}\sqrt{\frac{T}{\rho}} \qquad (4)$$

Pinned strings.

$$B = \frac{\pi^3 Yr^4}{4TL^2} \qquad (5)$$

$$f_0 = \frac{1}{2Lr}\sqrt{\frac{T}{\pi.D}} \qquad (6)$$

Round solid pinned strings.

The dimensionless factor B determines the inharmonicity. The inharmonicity is lowest for long thin low-modulus wires under high tension, and highest for short fat high-modulus wires under low tension. The most common example of this is in the lowest strings of an upright piano, where the length is limited and thus the strings must be very thick. The largest inharmonicity is actually found in the highest strings, but the higher harmonics lie beyond the audible range.[6]

For solid round strings of density D, $S=\pi r^2$, $K=r/2$, and $\rho=DS$, giving equations (5) and (6). These relations do not apply to wound strings, which have lower inharmonicity than solid strings.[2]

The inharmonicity is generally low, in which case equation (2) may be simplified to $f_n=nf_0(1+\frac{1}{2}Bn^2)$. Schuck's investigations[6] of F1 on an upright piano give $B=6.1\times10^{-4}$. Fletcher's measurements[1,2] on an upright give 5.3×10^{-4} for A0, 4×10^{-4} for the lowest solid string and 0.012 for the highest.

There are three techniques for examining simple vibrating systems such as this. The first is to mathematically analyse the governing equations, the second is to develop a computational physical model of the system[4], and the third is to build and measure an actual model.

2. THEORY OF FAT STRINGS

This work relies on an extreme interpretation of the equation for frequency stretching. For sufficiently stiff strings, the second partial reaches three times the fundamental frequency. Solving equation (2) for $f_2/f_1=3$ gives $B=^5/_7=0.714$. The second partial still corresponds to the string vibrating 'in halves', but is at three times the frequency of the first. Such a high value for B requires a string much thicker and shorter than normal. This is what I refer to as a 'fat' string. The third partial is inharmonic, with $f_3/f_1=\sqrt{39}=6.25$.

$$B = \frac{5}{7} \tag{7}$$

$$\frac{r}{L^2} = \frac{2f_1}{\pi}\sqrt{\frac{5D}{3Y}} \tag{8}$$

$$T = \frac{7\pi.Df_1^{\,2}}{3}L^2r^2 = \frac{7\pi^3Y}{20}L^{-2}r^4 \tag{9}$$

Fat round solid pinned strings.

Thus, new timbres can be generated from fat strings. Such a string might have a timbre similar to both a bar and a string. Whether the two harmonic partials and the inharmonic partials fuse sufficiently well to suggest a timbre remains to be determined by simulation and experimentation.

To determine the dimensions for a given f_1, Y, and D, we substitute $B=^5/_7$ into (5) and (6), giving equation (8). There is a choice of r and L. The tension is then given by equation (9); a higher tension gives more efficient transmission of energy and hence a louder sound.

$$\frac{r}{L^2} = 1.628\times10^{-4}.f_1 = \frac{f_1}{6142} \tag{10}$$

$$T \approx 57544.L^2r^2f_1^{\,2} \tag{11}$$

Steel fat round solid pinned strings.

Steel has a density $D=7850\ kg.m^{-3}$ and a modulus typically $Y=2\times10^{11}$ Pa, giving (10) and (11).[3] The examples below assume the use of steel.

The relationship between the parameters required for fat strings is shown in Figure 1. This shows lines of constant tension and constant frequency. The horizontal axis is length and the vertical is radius, and both are logarithmic.

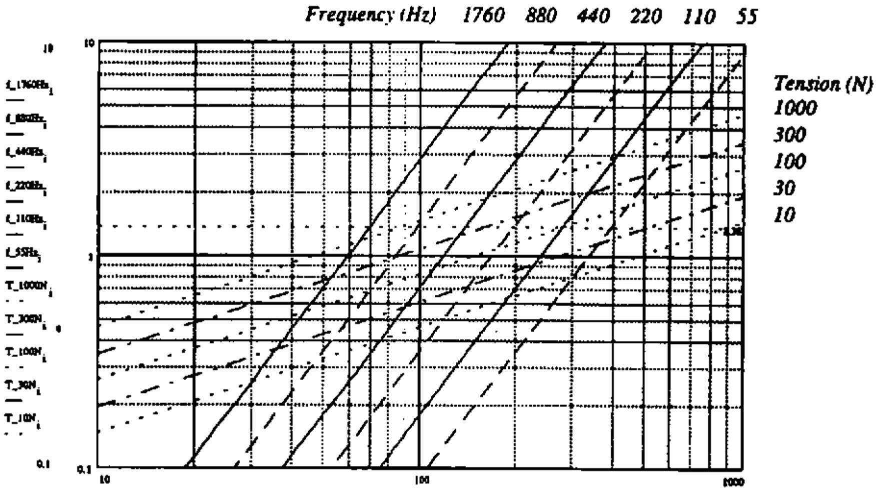

Figure 1 – Frequency and tension of fat steel strings versus string dimensions in millimetres.

3. COMPUTER SIMULATION

To evaluate the timbral quality, tones were synthesised using the above frequency relations. These can be heard via the world-wide web. Audio example 1 uses the first three partials, multiplied by the envelope e^{-4t}-e^{-30t}. In audio example 2, this is transposed and overlaid to form a tune.

Actual piano strings have a much more complex envelope.[1] There are many irregularities in the envelopes of each partial, and for notes up to D#5 (622 Hz) the decay has two distinct slopes. Martin calculates the overall initial slopes as 5.5 dB/s at F1, 15 dB/s at F3, 8.6 dB/s at F5, and 80 dB/s at F7.[5] The higher partials decay faster.

4. EXPERIMENTAL INSTRUMENTS

In order to test the basic principles, a crude monochord was constructed. It was estimated that standard instrument strings would only reach the required degree of inharmonicity by being unacceptably short. Instead, the string was made from a wire coat hanger of an unknown steel alloy. The radius was measured to be 1.38 mm. Using this wire, we use equations (10) and (11) to give the choices shown in table 1.

f (Hz)	l (mm)	T (N)
55	393	51
110	278	102
220	196	204
440	139	409
880	98	817
1760	69	1635

Table 1 – Fat strings using 1.38-mm wire.

The monochord constructed is shown schematically in Figure 2. Trial and error led to a string length of 260 mm. The string is anchored at one end and passes over a basic wooden bridge; tension is applied by suspending weights from the end. Sound picked up by a microphone close to the string is sampled using a PC soundcard.

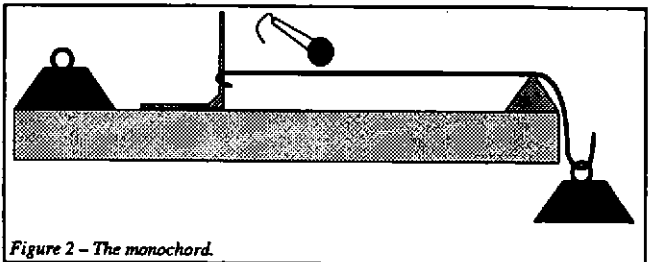

Figure 2 – The monochord.

The fundamental frequency is determined by recording multiple plucks halfway along the string and finding the spectral peaks. The second partial is derived in the same way except the string is plucked at a quarter of its length. Table 2 shows the estimated and measured frequencies of the first two partials.

Tension		f_1		f_2		f_2/f_1		B	
kg	N	estimated	measured	estimated	measured	estimated	measured	estimated	measured
2½	24.5	*92.4*	113.6	*337*	367.4	*3.64*	3.23	*3.391*	1.156
3¾	36.8	*97.5*	125.0	*342*	382.5	*3.51*	3.06	*2.261*	0.808
5	49.0	*102*	132.5	*348*	393.9	*3.40*	2.97	*1.696*	0.672
6¼	61.3	*107*	140.1	*353*	405.3	*3.30*	2.89	*1.357*	0.569
7½	73.6	*111*	143.9	*359*	412.8	*3.22*	2.87	*1.130*	0.546
8¾	85.8	*116*	147.7	*364*	420.4	*3.15*	2.85	*0.968*	0.523

Table 2 – Estimated and measured string parameters.

The string has lower inharmonicity than estimated and so $B={}^5/_7$ is reached at a lower tension. This may derive from a combination of experimental error and inaccurate constants, and warrants further analysis. The closest to $B={}^5/_7$ is achieved with a tension of 49 N, which gives $f_2/f_1=2.97$. Audio examples 3 and 4 are recordings of this string being plucked with a guitar plectrum and a finger. A plectrum is preferable as it gives more energy to the second and higher partials. The partials decay rapidly. The first partial decays at approximately 13 dB/s and the second decays at around 24 dB/s.

5. INSTRUMENT DESIGN

Having discussed the characteristics of fat strings, I now speculate on how instruments could be designed. Stringed instruments can usefully be classified as fixed-length or variable-length. Fixed-length instruments include viols and the guitar family. In principle we could restring such an instrument with fat strings. However, the higher strings become ridiculously thick – the top E string on a guitar would be 45 mm in diameter and would require a tension of 1.36 MN. Equations (8) and (9) show that the tension varies with the sixth power of the length.

It is clear that we must use a much smaller guitar. The string specifications for a one-third-scale guitar with a string length of 217 mm are shown in table 3.

note	f (Hz)	r (mm)	T (N)
B4	329.6	2.527	1880
B3	246.9	1.893	592
G3	196.0	1.503	235
D3	146.8	1.126	74.0
A2	110.0	0.843	23.3
E2	82.4	0.632	7.34

Table 3 – String parameters for one-third-scale guitar.

This gives a total tension of 2812 N. The tension still varies greatly; the higher strings (which are still unfeasibly wide) would be much louder than the lower strings. This is an inevitable consequence of using a fixed string length.

On a harp or a piano, the strings are not the same length. If we design a harp-like instrument, keeping a constant tension of 100 N, we get the dimensions shown in table 4. Figure 3 illustrates these seven strings.

note	f (Hz)	L (mm)	r (mm)
A7	3520	27.4	0.43
A6	1760	43.6	0.54
A5	880	69.1	0.69
A4	440	109.8	0.86
A3	220	174.2	1.09
A2	110	276.6	1.37
A1	55	439.1	1.73

Table 4 – String dimensions for constant-tension fat harp.

This is much more reasonable than using a constant length. However, now the lowest strings are rather thick and the highest are rather short. In practice some compromise between the constant-length and constant-tension strategies may be required. The different strategies used in tables 1, 3, and 4 correspond to different lines in Figure 1.

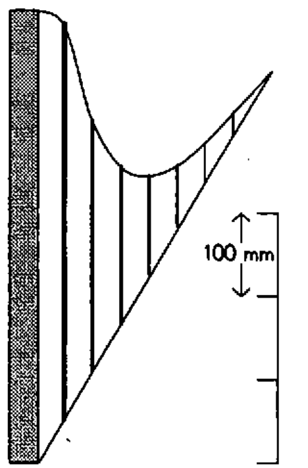

Figure 3 – Fat harp.

6. CONCLUSIONS

I have shown that when a string is sufficiently stiff its second partial reaches a frequency three times that of the first. Whereas a standard string is long, narrow, flexible, and highly tensioned, and a bar is short, thick, rigid, and untensioned, the fat string lies between the two in each respect. This may have potential for new musical instruments.

Future work should include more precise characterisation of the string behaviour, as the experimental and theoretical values do not agree well. In particular, the design of the bridge and anchoring needs to be examined, as the difference between the clamped and pinned conditions increases with the inharmonicity. In addition, the very short decay time means that attention must be paid to the resonator attached to the strings.

7. REFERENCES

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APPENDIX – AUDIO EXAMPLES

The audio examples can be found on http://capella.dur.ac.uk/doug/fatstrings/.

1) Computer-generated fat timbre with three partials at 1, 3, and 6.25. Relative amplitudes are 1, 0.5, and 0.05. The common amplitude envelope for a 220-Hz tone is $e^{-4t} - e^{-30t}$.
2) Melody "Through the Keys" from Bartok's Mikrokosmos played using rate-shifted versions of the above timbre.
3) The monochord (260×1.38 mm, 49 N) plucked with a plectrum. There is a single note plucked at a half of the length, then another plucked at a quarter of the length, followed by a series of plucks from halfway outwards.
4) The same monochord plucked with a finger.