ARRAY MODELING OF NON-RAYLEIGH REVERBERATION

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ABSTRACT

After matched filtering and beamforming, active sonar reverberation is often observed to have a non-Rayleigh distributed envelope, particularly for broadband transmit waveforms and large aperture arrays. Statistical analysis and modelling of the reverberation envelope has predominantly occurred after beamforming where the number of scattering elements contributing to reverberation is taken to be proportional to the array beamwidth. In this paper, a non-Rayleigh reverberation model is developed for an array of hydrophones in the farfield of the seafloor scattering. The effective number of scattering elements ($\tilde{\alpha}$), which is the shape parameter of the moment-matched *K* distribution, is derived for the conventional beamformer output. A narrowband approximation is developed for cases where the steering vector of the array does not change significantly over the band of the transmit waveform. An analysis of the conventional beamformer output for a towed array example illustrates that $\tilde{\alpha}$ from the array model is greater than that obtained from the 3-dB beamwidth, well approximated when the 6-dB beamwidth is used, and depends on the multipath structure and seafloor scattering statistics. High elevation-angle arrivals near the array endfire that contribute to beams at azimuths closer to broadside are seen to increase $\tilde{\alpha}$ in these beams with a commensurate reduction in $\tilde{\alpha}$ in the beams nearer to endfire. A technique for simulating non-Rayleigh array reverberation data is developed accounting for both multipath propagation and anisotropic seafloor scattering statistics.

1. INTRODUCTION

Active sonar reverberation is often assumed to follow a Rayleigh probability distribution owing to a large number of scatterers and a central limit theorem argument. However, when broadband transmit waveforms and large arrays reduce the size of the sonar resolution cell and therefore the number of independent scattering elements that contribute to reverberation, the envelope statistics can become non-Rayleigh. The *K* distribution may be used to represent such reverberation with the shape parameter related to the number of scattering elements in the resolution cell, a quantity assumed to be proportional to array beamwidth and inversely proportional to transmit waveform bandwidth [1].

In this paper, a model is developed to represent non-Rayleigh reverberation at the hydrophone level rather than after beamforming. The array response to scattering from the seafloor is formulated as the superposition of the response to angular sectors originating at the phase centre of the array and spaced so that they are several times denser than the minimum azimuthal beamwidth of the array. This allows straightforward simulation of non-Rayleigh array data and more accurate evaluation of the statistics of the conventional beamformer output in conjunction with ray-based propagation in a shallow water environment. A towed-array example is used to illustrate the differences between array modelling and assuming a *K*-distribution shape parameter proportional to beamwidth.

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2. ARRAY MODELING

For ease of exposition, an array model for non-Rayleigh reverberation will be developed and analyzed for a shallow water environment with a constant sound speed, range-independent bathymetry, and a ray based solution to the wave equation. Extension to more complicated environments follows readily [2]. It is further assumed that the array is in the farfield of the seafloor scattering and that the source and receiver are co-located.

Owing to multipath propagation, reverberation observed at two-way travel time t impinges on an array from a number of regions on the seafloor. As shown in [2,3] this may be approximated by modelling the response of the seafloor to direct-path-only propagation and subsequently applying a time-varying finite impulse response (FIR) filter to account for the multipath arrivals. Define $x(t,\theta)$ as the complex envelope of the impulse response of the seafloor to insonification of an angular sector at azimuth θ with width $d\theta$ where only direct-path propagation is considered from the source to the scattering patch to the phase center of the receiving array. If a_p for $p=1,\ldots,P$ are the amplitudes of the P multipath and τ_p are the corresponding delays that represent the reverberation at time t as defined in [2,3], then

$$y(t,\theta) = R_0(t) * \sum_{p=1}^{P} a_p x(t-\tau_p,\theta) \quad \textbf{(1)}$$

approximates the reverberation measured at the phase centre of the array from the azimuthal sector at θ after matched filtering where $R_0(t)$ is the autocorrelation function of the transmit waveform. In order to account for spatial non-stationarity in the statistical response of the seafloor, the statistics of $x(t,\theta)$ can vary with t and θ. Azimuthally dependent bathymetry and oceanography can be incorporated by forcing the multipath characterization to depend on θ in addition to time.

Let $\mathbf{d}(\theta,\phi,f)$ be the plane-wave steering vector at frequency f for an array comprising N hydrophones pointing to azimuth θ and elevation angle ϕ. In this paper, bold is used to denote vector or matrix variables, a superscript ^{T} the transpose operator and a superscript ^{H} the conjugate transpose operator. The reverberation received by the array may then be characterized in the frequency domain as

$$\mathbf{Y}(f) = S_0(f)\int_0^{2\pi} X(f,\theta)\sum_{p=1}^{P} a_p e^{-j2\pi f\tau_p}\mathbf{d}(\theta,\phi_p,f)d\theta \quad \textbf{(2)}$$

where $S_0(f)$ is the Fourier transform of $R_0(t)$, $X(f,\theta)$ is the Fourier transform of $x(t,\theta)$ and ϕ_p is the elevation angle of the pth multipath arrival from azimuthal angle θ. From this representation, it will be possible to perform a statistical analysis of the output of a conventional beamformer and to simulate non-Rayleigh reverberation at the hydrophone level.

Under direct-path-only propagation, the shape parameter α of the *K* distribution has been shown to be proportional to the number of scattering elements in a sonar resolution cell [1] and is a useful indicator of whether the reverberation envelope statistics are Rayleigh-like or more heavy-tailed. In this context, large values of α represent Rayleigh-like statistics while small values are indicative of heavier tailed reverberation. Although not explicitly necessary, it will be assumed in this paper that $x(t,\theta)$ is *K*-distributed with a shape parameter that varies with time as described in [1-3].

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3. CONVENTIONAL BEAMFORMER STATISTICAL ANALYSIS

Application of a conventional beamformer (CBF) pointing to azimuth ψ_a and elevation ψ_e (i.e., direction $\psi = (\psi_a, \psi_e)$) with weight vector $\mathbf{w}(\psi, f)$ to the array data described in (2) results in the beam output

$$Y(f,\psi) = S_0(f)\int_0^{2\pi} X(f,\theta)\sum_{p=1}^{P} a_p e^{-j2\pi f\tau_p}\mathbf{w}^H(\psi,f)\mathbf{d}(\theta,\phi_p,f)d\theta = \int_0^{2\pi} X(f,\theta)C(f,\theta,\psi)d\theta \quad (3)$$

and the corresponding time domain signal

$$y(t,\psi) = \int_0^{2\pi}\int_{-\infty}^{\infty} x(t-s,\theta)c(s,\theta,\psi)dsd\theta \quad (4)$$

where $c(t,\theta,\psi)$ is the inverse Fourier transform of

$$C(f,\theta,\psi) = S_0(f)\sum_{p=1}^{P} a_p e^{-j2\pi f\tau_p}\mathbf{w}^H(\psi,f)\mathbf{d}(\theta,\phi_p,f) \quad (5)$$

Extending the results of [3], which derive the equivalent shape parameter [4] of a sum of independent but differently distributed *K*-distributed random variates, the equivalent shape parameter of the beam output $y(t,\psi)$ in (4) for a general $c(t,\theta,\psi)$ may be shown to be

$$\tilde{\alpha}(t,\psi) = \frac{\left[\int_0^{2\pi}\int_{-\infty}^{\infty}\alpha_0(t-s,\theta)\left|c(s,\theta,\psi)\right|^2 dsd\theta\right]^2}{\int_0^{2\pi}\int_{-\infty}^{\infty}\alpha_0(t-s,\theta)\left|c(s,\theta,\psi)\right|^4 dsd\theta} \approx \frac{\left[\int_0^{2\pi}\alpha_0(t,\theta)\int_{-\infty}^{\infty}\left|c(s,\theta,\psi)\right|^2 dsd\theta\right]^2}{\int_0^{2\pi}\alpha_0(t,\theta)\int_{-\infty}^{\infty}\left|c(s,\theta,\psi)\right|^4 dsd\theta} \quad (6)$$

where $\alpha_0(t,\theta)$ is the shape parameter of $x(t,\theta)$ and the approximation is obtained by assuming that $\alpha_0(t-s,\theta)$ varies slowly over the extent of $c(s,\theta,\psi)$ which is approximately the multipath extent $\max_p \tau_p - \min_p \tau_p$. The units of $\alpha_0(t,\theta)$ may be taken as the number of scattering elements per second per radian. Although these results are most easily derived assuming that $x(t,\theta)$ is a white random process in both t and θ, it is sufficient that the scattering elements be smaller than the smallest resolution cell in consideration, defined by the smallest beamwidth of the array and largest bandwidth of the transmit waveform.

If the transmit waveform has a narrow enough bandwidth for the array steering vector to not change over its frequency band, then a narrowband approximation to $C(f,\theta,\psi)$ of (5) may be written as

$$C(f,\theta,\psi) = S_0(f)\sum_{p=1}^{P} a_p b_p(\psi,\theta)e^{-j2\pi f\tau_p} \quad (7)$$

where $b_p(\psi,\theta) = \mathbf{w}^H(\psi,f_c)\mathbf{d}(\theta,\phi_p,f_c)$ is the response of the beamformer pointing to ψ at center frequency f_c to a signal arriving from the direction of the pth multipath from angle θ. The

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inverse Fourier transform of $C(f,\theta,\psi)$ is then the sum of scaled and delayed waveform autocorrelation functions, which simplifies (6) to

$$\tilde{\alpha}(t,\psi)=\frac{1}{W}\cdot\frac{\left[\int_0^{2\pi}\sum_{p=1}^{P}\alpha_0(t-\tau_p,\theta)\left|a_p b_p(\psi,\theta)\right|^2 d\theta\right]^2}{\int_0^{2\pi}\sum_{p=1}^{P}\alpha_0(t-\tau_p,\theta)\left|a_p b_p(\psi,\theta)\right|^4 d\theta} \quad \textbf{(8)}$$

where it is assumed that $\alpha_0(t,\theta)$ varies slowly over the length of $R_0(t)$ ($\approx$ one over the waveform bandwidth W), that the waveform autocorrelation function is well approximated by a rectangular pulse of width $1/W$, and that the multipath do not interfere with each other. The last assumption implies that the bandwidth is large enough to isolate each multipath. Lower bandwidths may be considered by recharacterizing the multipath filter into a sum of orthogonal components as described in [3] resulting in a different set of amplitudes and delays and a slight modification to (8) to account for the width of each orthogonal component.

Evaluation of (6) and (8) for direct-path-only propagation in an isotropic scattering environment for a horizontal line array comprising N hydrophones spaced every d meters shows that the narrowband approximation for the equivalent shape parameter is accurate for bandwidths

$$W \le \left[\frac{d}{c}\frac{\left(\sum_{i=1}^{N} w_i\right)^2}{\sum_{i=1}^{N} w_i^2}\right]^{-1} \quad \text{or, equivalently,} \quad \frac{W}{f_d} \le 2\frac{\sum_{i=1}^{N} w_i^2}{\left(\sum_{i=1}^{N} w_i\right)^2} \quad \textbf{(9)}$$

where w_i are the beamformer shading coefficients, c is the speed of sound, and $f_d = c/(2d)$ is the design frequency of the array. The term in the brackets in the first equation of (9) may be loosely described as the time it takes a wavefront arriving from endfire to cross the array, which is one characterization of when a waveform is narrowband with respect to array signal processing [5]. Equation (9) also accounts for non-uniform shading of the array, which effectively reduces the array aperture. Above these bandwidths, the narrowband approximation first underestimates $\tilde{\alpha}(t,\psi)$ by up to ten percent at broadside (less than one percent at endfire) and then overestimates $\tilde{\alpha}(t,\psi)$ with greater disparity as N increases.

The result of (8), though it is for the limited case of a narrowband waveform with enough bandwidth to resolve all multipath, is still representative of the change in $\tilde{\alpha}(t,\psi)$ arising from multipath propagation and will be used in the following section to illustrate the effect of array modelling of non-Rayleigh reverberation as opposed to beamformer output modeling in a towed-array example.

4. TOWED-ARRAY EXAMPLE

To illustrate the statistical analysis of the previous section, consider an omni-directional source with a co-located line array receiver with $N=16$ equally spaced hydrophones with a design frequency $f_d=1200$ Hz and a transmit waveform center frequency $f_c=1000$ Hz placed at 40 m depth in a water column 100 m deep. The parameters characterizing the isovelocity soundspeed and range-independent bathymetry environment are found in Fig. 1. Let the array be

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beamformed using a Hanning weighted conventional beamformer $\mathbf{w}(\psi, f) = \mathbf{W}\mathbf{d}(\psi, 0, f)$ where $\mathbf{W} = \mathrm{diag}\{w_i\}$ is a diagonal matrix containing the Hanning weights and the elevation angle is assumed to be zero. The seafloor is assumed to produce reverberation with a constant shape parameter $\alpha_0 = 0.1W180/\pi$ scattering elements per second per radian at travel time $t = 10$ s. The narrowband constraint of (9) dictates that $W \le 247$ Hz, which in turn requires all multipath to be separated by at least 4 ms for (8) to hold. For the aforementioned waveguide, the closest multipath within 20 dB of the strongest path are separated by 4.3 ms, so the narrowband assumption is valid and these results are valid for $234 < W < 247$ Hz.

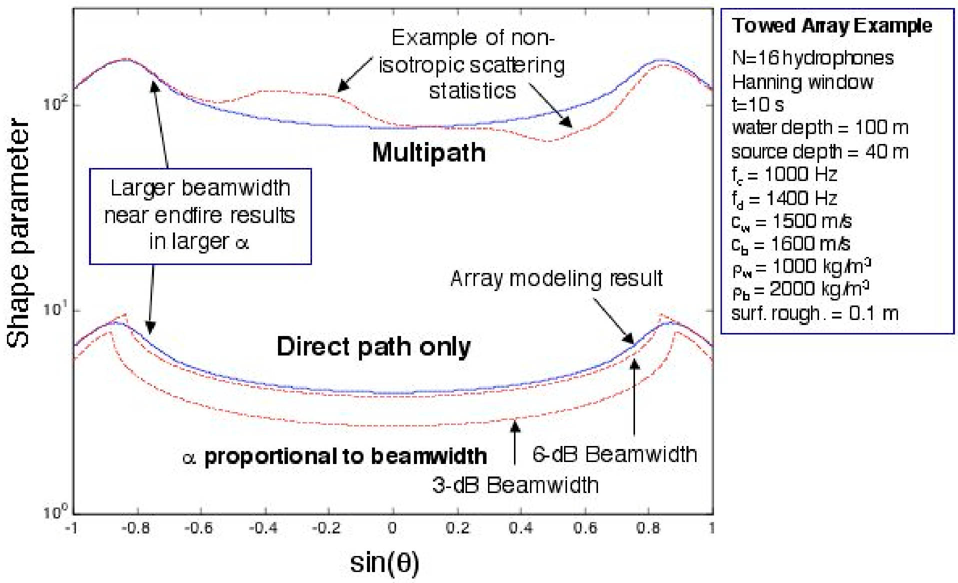

Figure 1: Equivalent shape parameter of CBF beam output for direct path propagation from model and proportional to beamwidth and for multipath propagation with isotropic and non-isotropic scattering statistics.

The equivalent shape parameter of (8) for direct-path-only propagation from long range scattering (i.e., the elevation angle is zero) is shown in Fig. 1 and compared with that estimated from the 3-dB and 6-dB beamwidths of the beamformer which are herein defined as the total angle in degrees where the beampattern is within the specified range of its peak value. Note that this may come from the mainlobe, sidelobes, or grating lobes of the beampattern. The shape parameter is then estimated by the beamwidth times $2\alpha_0/W$, which assumes that the seafloor only contributes at this time from the left and right azimuths. For example, using the 3-dB beamwidth at broadside of 13.6 degrees results in a shape parameter estimate $= 13.6 \cdot 2 \cdot 0.1 = 2.72$ scattering elements. Each of these methods illustrates that the shape parameter increases away from broadside to the array owing to the larger number of scattering elements contributing to the wider beams found toward endfire until the two sides of the conical beampattern begin to meet and reduce the total beamwidth near endfire. The ratio between the results of the array modelling and α-proportional-to-beamwidth techniques is presented in Fig. 2 where it is seen that the 3-dB beamwidth method underestimates the array-based model by 30% on average in this example. The 6-dB beamwidth approach, which is within 4% of the array modelling result on average, provides a more accurate approximation.

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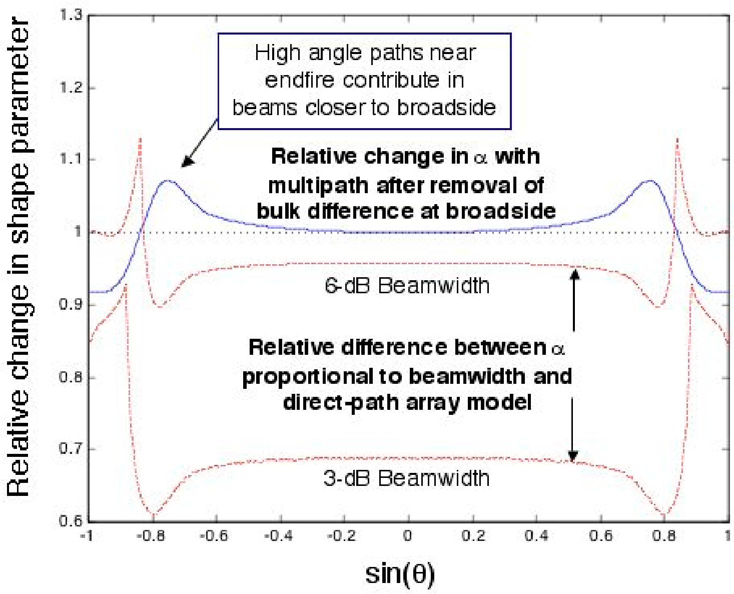

Figure 2: Relative difference in equivalent shape parameter of CBF beam output between model and shape parameter proportional to beamwidth for direct path propagation and between direct path and multipath propagation after removal of the bulk difference at broadside.

The model incorporating multipath for travel time $t = 10$ s is also shown in Fig. 1 along with an example of non-isotropic scattering where $\alpha_0(t,\theta)$ is varied with θ with more Rayleigh-like scattering near $\sin\theta = -0.3$ and less Rayleigh-like scattering near $\sin\theta = 0.6$. As is expected, multipath propagation results in reverberation that is more Rayleigh-like than the direct-path response with a shape parameter nearly twenty times larger. The effect of incorporating multipath at the array level is seen in Fig. 2 where high angle paths near endfire are seen to contribute to beams closer to broadside to the array resulting in a smaller shape parameter (relative to direct-path-only propagation) near endfire and a larger shape parameter slightly closer to broadside. Near broadside, the net change is essentially constant as the higher angle paths enter in the same beam as the lower angle paths. This effect is more pronounced at shorter travel times and less pronounced at later travel times owing to the decreasing influence of high angle paths on reverberation as travel time increases.

5. SIMULATING ARRAY DATA

The array reverberation model described by (2) provides a means for simulating reverberation at the hydrophone level rather than the beam output. As in [2], it is assumed that the aforementioned characterization of multipath is stationary over some small time span allowing a frequency domain implementation of the filter described in (1). The array data may be formed by first simulating the seafloor response arising from direct-path propagation to the phase centre of the receiving array from angular sector θ,

$$\mathbf{x}_\theta = \left[x(t,\theta) \quad x(t+T_s,\theta) \quad \cdots \quad x(t+(M-1)T_s,\theta)\right]^T \qquad (10)$$

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where T_s is the sampling rate of the basebanded signal and there are M time samples in the processing block. Let the M-by-M matrix $\mathbf{U}$ represent the discrete Fourier transform (DFT); that is, the DFT of $\mathbf{x}_\theta$ is $\mathbf{X}_\theta = \mathbf{U}^H \mathbf{x}_\theta$ with inverse $\mathbf{x}_\theta = \mathbf{U}\mathbf{X}_\theta$. By defining

$$\mathbf{e}_\theta(f) = \sum_{i=1}^{P} a_p e^{-j2\pi f \tau_p} \mathbf{d}(\theta, \phi_p, f) \qquad \textit{(11)}$$

and $\mathbf{E}_\theta = [\mathbf{e}_\theta(f_1) \;\; \mathcal{L} \;\; \mathbf{e}_\theta(f_M)]$ with $f_k = f_c + (k - 1 - M/2)/(MT_s)$, the array data from angular sector θ may be characterized as

$$\mathbf{y}_\theta = \mathbf{E}_\theta \operatorname{diag}\{\mathbf{X}_\theta\}\mathbf{U}^T \qquad \textit{(12)}$$

which has row dimension N representing each hydrophone output and column dimension M representing the data in the processing block. Approximating the integral over θ as a finite summation yields

$$\mathbf{y} = d\theta \left\lfloor \sum_{i=1}^{L} \mathbf{E}_{\theta_i} \operatorname{diag}\{\mathbf{X}_{\theta_i}\} \right\rfloor \mathbf{U}^T \qquad \textit{(13)}$$

where $\theta_i = (i-1)2\pi/L$ and $d\theta = 2\pi/L$. It is recommended that L be chosen large enough to allow at least three and preferably five or more angular sectors within the narrowest beam of the array (e.g., that found at broadside).

If the transmit waveform has a narrow enough bandwidth for the steering vector to remain constant over its frequency band, then (12) simplifies to

$$\mathbf{y}_\theta = \mathbf{e}_\theta(f_c)\mathbf{X}_\theta^T \mathbf{U}^T = \mathbf{e}_\theta(f_c)\mathbf{x}_\theta^T \qquad \textit{(14)}$$

and the integration over θ may be efficiently described as

$$\mathbf{y} = d\theta [\mathbf{e}_{\theta_1}(f_c) \;\; \mathcal{L} \;\; \mathbf{e}_{\theta_L}(f_c)][\mathbf{x}_{\theta_1} \;\; \mathcal{L} \;\; \mathbf{x}_{\theta_L}]^T. \qquad \textit{(15)}$$

6. CONCLUSION

An array model for non-Rayleigh reverberation has been developed that accounts for ray-based propagation in a shallow water environment, provides straightforward simulation of array data, and provides more accurate statistical analysis of the conventional beamformer output. The statistical analysis indicated that the *K*-distribution shape parameter for reverberation observed on an equi-spaced line array is well approximated by that estimated from the 6-dB array beamwidth and illustrated that propagation effects such as high-angle multipath arrivals can increase or decrease the shape parameter compared with direct-path only propagation. A simplification to the modelling, analysis, and simulation was developed for narrowband transmit waveforms along with a characterization of what bandwidths satisfy the narrowband constraint in terms of the statistics of a conventional beamformer output from an equi-spaced line array.

The modelling and analysis presented in this paper provide the means for assessing the effect of array beamforming on the statistics of reverberation in a shallow water environment and may lead to novel array design methodologies or new beamforming algorithms that account for non-Rayleigh reverberation.

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