RESPONSE STATISTICS OF UNCERTAIN STRUCTURES

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1 INTRODUCTION

The responses of nominally identical, complex, built-up structures vary from one realisation of the structure to the next. These variations are due to many factors including manufacturing variability and changes in environmental conditions. These variations cause differences in the dynamic frequency response between individual realisations. Inconsistencies in the measurement process will also influence the measured response.

At higher frequencies modelling techniques should consider parameter uncertainty. At high frequencies stochastic methods such as Statistical Energy Analysis are used to calculate average response energy in frequency bands. Such methods lose detailed response information, but in turn are intrinsically less affected by small variations in the properties of the structure. However, low frequency techniques such as Finite Element Analysis are based on modelling a single realisation of the structure and as such take no account of product variability. Ideally an ensemble of different realisations of the structure would be modelled and a corresponding set of response data generated. But computational costs increase significantly with the complexity of the structure being considered and the frequency range of interest. Even for a relatively modest structure this cost is prohibitively high, making it impracticable to simulate a batch of components.

Various techniques are being investigated to include variability in the modelling process. However, in order to model the vibration response of structures with uncertain properties, some knowledge is required as to the nature and statistics of the typical variability found in manufactured components. Variability in the physical properties of a system will lead to variability in the system response. The statistical distribution of both the system physical properties and the system response and their relationship are of interest. Currently very little published data is available on the statistical distribution of variability within manufactured components or the relationship between the inconsistencies in the physical properties and the statistics of the measured response. This paper investigates the variability in the mass and natural frequency of a single complex component (a vehicle alloy wheel rim) and the variability in structure-borne and air-borne response of a complex built-up structure (an automotive vehicle).

In order to gain further insight into the propagation of variability through a system from the physical properties to the response distribution, a complementary simple idealised single degree of freedom system is also considered. Two analytical methods are used, a transformation of variables technique[1] and a power series expansion with moment generation. These are used in conjunction with Monte Carlo simulations. Kompella and Bernhard[2] discuss these methods with reference to the prediction of statistical variation of multiple-input-multiple-output system response.

2 VARIABILITY STATISTICS IN NOMINALLY IDENTICAL STRUCTURES

In order to gain some knowledge of the typical measured variability, the statistics of the response of two different types of structure are investigated. Firstly, a vehicle alloy wheel rim is considered, followed by measured data from two automotive vehicles.

2.1 Statistics of Properties of a Single Complex Component

This section considers the statistical analysis of the first four natural frequencies and mass of an alloy wheel. The measurements were taken and reported by Brown and Gear[3], who measured these properties for a set of 79 nominally identical alloy wheel rims.

Figure 1 shows the distribution of the data together with Gaussian distributions with the same mean and variance. A chi-squared $\left(\chi^2\right)$ analysis[4] was conducted to test the goodness-of-fit of various probability distributions to the data sets.

In general a lognormal distribution closely approximates a Gaussian distribution for data sets where the standard deviation is small compared to the mean value, such that $\sigma/\mu \ll 1$, where σ and μ are the standard deviation and mean respectively. This is certainly the case for the above data and hence the samples fit both distributions equally well. Either a lognormal or a Gaussian distribution was found to be a 'good' fit (i.e. with a χ^2 probability of between 5-95%) for the first, third and fourth mode. Neither distribution was a likely fit to the second mode nor to the distribution of the mass; both of these have a high number of samples close to the mean with a low and sporadic spread. Depending on the manufacturing and finishing process, the mass might have been expected to be a good fit to a Gaussian distribution but this was not found to be the case.

Brown and Gear[3] examined the cross correlation between the natural frequencies and the mass of the wheel, but no correlation was found. They also examined the cross correlations between the natural frequencies and found some correlation between the first and second mode. It was surmised that this could be due to the mode shapes for both modes being similar, but this didn't appear to be valid for the third and fourth mode which, although similar in mode shape, didn't display any significant cross correlation.

2.2 Response Statistics for a Complex Built-up Structure

A statistical analysis of the response distribution for a set of nominally identical, complex, built-up structures has been performed. The aim of the investigation was to better understand the probability distribution of the variability of the structures' responses. The example considered is of an automotive vehicle; results are available for two different vehicle models, the Isuzu Rodeo (98 nominally identical vehicles) and the Isuzu Pickup (57 nominally identical vehicles). The data used for this investigation is reported in Bernhard and Kompella[5-7]. For each vehicle set both structure-borne and air-borne frequency response functions (FRFs) were available. Some statistical analysis of the same data had previously been carried out by Kompella and Bernhard[2], this study extends that work and looks for any variation in the response distribution with frequency. The FRFs are shown in Figure 2.

Both vehicles were tested using an identical test procedure; a brief summary will be given here but for further details see References[5-7]. Vehicles were tested outside in a quiet environment. Structure-borne FRFs were measured using an impact hammer on the wheel hub to provide the excitation. A loudspeaker situated outside the vehicle at the front left wheel position provided acoustic excitation for the air-borne FRFs. Band-limited random noise was used as the acoustic source. Interior microphones at the drivers and passengers ear locations were used as response transducers. The air-borne and structure-borne FRFs provide good data between 125Hz-1kHz and 30-500Hz respectively.

During the vehicle testing, in order to assess the measurement process variability, a reference vehicle of each type was tested repeatedly throughout the test schedule. This provides an indication of the measurement procedure variability compared to the population variability. Figure 3 shows a comparison of the population mean, standard deviation and normalised standard deviation as a function of frequency together with those of the reference vehicle.

As can be seen from Figure 3 the reference vehicle mean is approximately equal to that of the population indicating that the reference vehicle is a typical sample from the population. Also the population standard deviation is in general twice that of the reference vehicle. This indicates that the variation in the sample population is predominately due to vehicle variability and not measurement variability. Also worth noting in Figure 3 is the normalised standard deviation: there is a trend towards increased normalised standard deviation at higher frequencies, as the modal overlap increases.

Further analysis included basic statistical analysis of the response distributions (mean, standard deviation, skew and kurtosis) and these were examined to look for trends relating to the mean level of the response. Techniques such as band averaging were also investigated. Standard probability distributions were fitted to the response distribution at each frequency line and a χ^2 test was used to test the goodness-of-fit of the distribution to the sample data set. The more interesting results will be discussed here.

Figure 4 shows FRFs and a Gaussian distribution fitted to two example frequency lines at 52.5Hz and at 375Hz. A Gaussian distribution was found to be a good fit to the linear response data over most of the frequency range. Some difference was noted between the goodness-of-fit at low frequencies compared to higher frequencies. At higher frequencies both the modal overlap and the variability are high. A summary of the results for the χ^2 test is given in Table 1. For comparison a summary of the goodness-of-fit of a lognormal distribution is shown in Table 2. The results are presented as the percentage of frequency lines for which the χ^2 probability is between 5-95% which is considered to be a good fit, above 95% which is considered to be a poor fit and below 5% which is considered to be an improbably good fit. A dummy data set of pseudo-random numbers, known to come from a Gaussian distribution, was generated. This was used to indicate the typical results that can be expected from a random Gaussian data set of similar size to the sample data sets. It can be seen that for the lognormal distribution 12-20% less of the frequency range was a ‘good fit’ (with a χ^2 of 5-95%), and a corresponding 12-20% more of the frequency range was a ‘poor fit’, when compared to the goodness-of-fit to a Gaussian distribution.

A detailed analysis of the low/high frequency ranges displayed some differences in the χ^2 probabilities. For three of the four sets of FRFs, between 3-7% more of the frequency lines at low frequencies were found to be a better fit to a Gaussian distribution, than those at high frequencies. For the structure-borne FRF the low frequency range was considered to be 30-300Hz, high frequency as 300-500Hz. For the air-borne FRF the low frequency range was considered to be 125-500Hz, high frequency as 500-1000Hz.

In summary, the discrete frequency response statistics can be described well by a Gaussian distribution, and much less well by a lognormal distribution.

In a very complex built-up structure with many degrees of freedom, such as an automotive vehicle, then the distribution of the response may be influenced by the central limit theorem[4]. The central limit theorem describes the case where samples are taken from a population with an arbitrary probability distribution. As the number of samples increases the distribution of the resultant set tends towards a Gaussian distribution irrespective of the distributions of the individual random parameters. This may be the type of effect being seen in complex built up structures, where the probability distributions of the components may be a random distribution dependent on the individual component characteristics and manufacturing process. But the effect on the built-up structure may be to tend towards a Gaussian distribution of the response, particularly where the response isn’t dominated by a small number of degrees of freedom.

3 VARIABILITY IN A SIMPLE IDEALISED SYSTEM

To gain a better understanding of the effects and propagation of variability through a system, a simple idealised single degree of freedom (SDOF) example was considered. A physical analogy would be a simple component, which due to manufacturing inconsistencies, exhibited small random variations in its properties, leading to a range of responses within a batch of components. The statistics of the resulting response is examined at low frequencies (well below the resonance frequency of the system), at resonance and at high frequencies (well above the resonance frequency of the system). For this study the system properties were assumed to be independent; in practice this is unlikely to be the case as variability in a component is likely to affect more than one system property. A further assumption made is that variability is only introduced into a single system property at any one time. When varying a system property it is assumed that the statistics of the variation are known. However, there is no limitation in the analysis methods used as to the type of distribution for the random variation. In this study two variability distributions are used, Gaussian and Rayleigh; both have zero mean and specified variance.

Three methods are explored for estimating the response statistics: Monte Carlo simulations, a perturbational approach and a change of variable technique. These techniques are outlined below.

3.1 Monte Carlo Simulations

Monte Carlo simulations have become progressively more routine with the increase in computing power and the decrease in computing costs. The method can be summarised in the following steps. The probability density functions for the input parameters are established. Random values are generated from each of these input parameter probability distributions. These values are used to calculate an exact output response for that particular set of random input values. Another set of random input values are generated from the input parameter probability distributions, and again the exact output response is calculated. This process is repeated a number of times to generate a statistical set of output response data. The accuracy of the resultant simulated output response data set is dependent on the number of simulations carried out.

This method has the advantage of being simple to use and, provided the functional relationship between the input and output parameters can be expressed mathematically, there are no limitations to the complexity of the system that can be modelled. However, the method is inflexible and cannot account for any changes to the system. It also requires significant amount of computing time, and this requirement increases for the number of simulations required.

Various Monte Carlo simulations were performed on a SDOF system, for the combination of variable mass, stiffness, damping and natural frequency, at low frequencies, high frequencies and around resonance. For each simulation 10,000 random data sets were used to generate the resultant response population. These response distributions were then used to compare to the following techniques discussed below.

3.2 Response Statistics Estimation through a Perturbational Approach

The moments of the system response can be related to the moments of the system inputs or in this case the physical properties of the system. This method is sometimes referred to as the 'Generation of Moments' method. An example of the calculation is shown below for the case of variable stiffness at low frequencies.

Assume the system stiffness k is allowed to vary by a small random amount ε, where ε is a normally distributed random variable with zero mean $(\mu_\varepsilon = 0)$ and low variance $(\sigma_\varepsilon << 1)$, such that

$$k = k_0(1+\varepsilon) \tag{1}$$

where k_0 is the nominal unperturbed stiffness of the system. The magnitude-squared receptance of a single degree of freedom system is given by

$$|\alpha(\omega)|^2 = \left|\frac{X}{F}\right|^2 = \left|\left(\frac{1}{k - m\omega^2 + i\omega c}\right)^2\right| = \left|\left(\frac{1}{k}.\frac{1}{1-\frac{\omega^2}{\omega_n{}^2} + i2\zeta\frac{\omega}{\omega_n}}\right)^2\right| \tag{2}$$

where X is the magnitude of the response, F is the magnitude of the harmonic excitation force at frequency ω, m and c the system mass and viscous damping constant respectively, ω_n is the natural frequency of the system and ζ is the viscous damping ratio. At low frequencies where $\omega/\omega_n << 1$, this can be approximated by

$$|\alpha(\omega)|^2 \approx \frac{1}{k^2} \tag{3}$$

using Equation (1) this becomes

$$|\alpha(\omega)|^2 \approx \frac{1}{k_0{}^2}\frac{1}{(1+\varepsilon)^2} \tag{4}$$

This can be expanded in a general power series to give

$$k_0{}^2|\alpha(\omega)|^2 \approx 1 - 2\varepsilon + 3\varepsilon^2 - 4\varepsilon^3 + 5\varepsilon^4 + .. + \frac{(2+r-1)!}{r!(2-1)!}(-1)^r\varepsilon^r + .. \tag{5}$$

If higher order terms are ignored, the expected or mean value of $k_0{}^2|\alpha(\omega)|^2$ is given by

$$k_0{}^2E\left[|\alpha(\omega)|^2\right] \approx 1 - 2E[\varepsilon] + 3E\left[\varepsilon^2\right] - 4E\left[\varepsilon^3\right] + 5E\left[\varepsilon^4\right] - .... \tag{6}$$

The moments of ε depend on the statistical distribution of the variability. For example, if ε is assumed to be Gaussian distributed such that the probability density function for ε is given by

$$p(\varepsilon) = \frac{1}{\sigma_\varepsilon\sqrt{2\pi}}e^{\frac{-\varepsilon^2}{2\sigma_\varepsilon{}^2}} \tag{7}$$

the first few moments of ε are given by

$$E[\varepsilon] = 0 \tag{8}$$

$$E\left[\varepsilon^2\right] = \frac{\sigma_k{}^2}{k_0{}^2} = \sigma_\varepsilon{}^2 \tag{9}$$

$$E\left[\varepsilon^3\right] = 0 \tag{10}$$

$$E\left[\varepsilon^4\right] = 3\frac{\sigma_k{}^4}{k_0{}^4} = 3\sigma_\varepsilon{}^4 \tag{11}$$

Therefore Equation (6) can be re-written to give the mean of the magnitude-squared receptance at low frequencies, such that

$$\mu_{|\alpha(\omega)|^2} \approx E\left[\left|\alpha(\omega)\right|^2\right]_{low\_freq} \approx \frac{1}{k_0^{\,2}}\left(1+3\sigma_\varepsilon^{\,2}+15\sigma_\varepsilon^{\,4}+...\right) \tag{12}$$

This process can be repeated for high frequencies and at resonance, for variable stiffness, mass and damping, and for any chosen variability distribution for which the moments are known or can be calculated. This method can also be used to estimate the variance of the response using the same principles.

However, there are several limitations on the practical use of this method. The power series expansion requires certain limits for convergence. It can be shown that for the estimations of the response statistics around resonance, the variability must be small compared to the half-power bandwidth of the system, so that

$$\varepsilon << \frac{\Delta\omega}{\omega_n} \tag{13}$$

where $\Delta\omega$ is the half-power bandwidth. So that the power series expansion converges for all members of the Monte Carlo simulations it is therefore necessary that the standard deviation of ε is small compared to the half-power bandwidth, $\sigma_\varepsilon << \Delta\omega/\omega_n$. Otherwise for the regions away from the resonance frequency

$$\varepsilon << 1 \tag{14}$$

for convergence. Another limitation is the restricted information that can be obtained using this method. Although the mean and the variance of the response can be estimated, it cannot be used to determine the distribution of the response. Finally, this method is also limited to simple systems where the relationship between the system response and the input parameters can be expressed in a form suitable for a power series expansion.

A comparison was carried out between the estimations of mean response and response variance from this perturbational approach and the results from Monte Carlo simulations. Evaluations were carried out for each case of variable mass, stiffness and damping, at low frequency, high frequency and at resonance. Each scenario was repeated for both a Gaussian and Rayleigh distributed variable input property. The maximum error between the Monte Carlo simulations and the estimates from the perturbational method was 3%. Increasing the number of Monte Carlo simulation runs and incorporating higher terms in the power series expansion were shown to reduce this error.

3.3 Response Distribution From Transformations of the Probability Density Functions

If the functional relationship between two random variables is assumed to be single valued, and the probability density of one of the variables is known then the probability density function of the other can be determined[1,4]. For example, given two random variables x and y, for which

$$y = f(x) \tag{15}$$

where the probability density function of x is given by $p(x)$, the probability density function for y is

$$p(y) = \frac{p(x)}{\left|\frac{dy}{dx}\right|} = \frac{p(x)}{\left|f'(x)\right|} \tag{16}$$

If each value of y corresponds to n values of x, then

$$p(y) = \frac{np(x)}{\left|f'(x)\right|} \tag{17}$$

As an example at low frequencies $\left|\alpha(\omega)\right|^2$ can be approximated as

$$\left|\alpha(\omega)\right|^2_{low\_freq} \approx \frac{1}{k_0^{\,2}\left(1+\varepsilon\right)^2} \tag{18}$$

If $p(\varepsilon)$ is given by Equation (7) then

$$p(y) = \frac{e^{-\frac{\left(\sqrt{\frac{1}{yk_0^{\,2}}}-1\right)^2}{2\sigma_x^{\,2}}}}{y^{3/2}k_0 2\sigma_x\sqrt{2\pi}} \tag{19}$$

As can be seen from Equation (19) the form of the probability density function of the response distribution is somewhat similar to that of a Gaussian distribution. This process can be performed for any chosen variability distribution for which the probability density function is known. If the variable input property or physical parameter has a Gaussian distribution then the following observations can be made. In general the system response near resonance, for any varying system property that controls the resonance frequency (mass, stiffness or natural frequency), the distribution is a one-sided probability density function. For the regions where the system properties affect the response level the distribution can be seen to approximate a Gaussian distribution (e.g. variable stiffness at low frequencies, or variable damping at resonance).

If the variable input property has a Rayleigh distribution then the following comments can be made. As with the Gaussian example, the system response near resonance for any varying system property that controls the resonance frequency (mass, stiffness or natural frequency) the distribution is a one-sided probability density function. For the regions where the system properties affect the response level there are two slightly different distributions. For variable natural frequency where the level of the response is affected (i.e. high frequencies) the response distribution is approximately Rayleigh. But, for the other variable properties (stiffness, mass, damping) where they control the level of the response the resultant distribution is similar to a Rayleigh distribution but with a right-hand skew instead of a left-hand skew.

Tables 3 and 4 summarise the response probability distributions for a Gaussian distributed and Rayleigh distributed input property.

The limitations of this method are that the relationship between the input variables and the system response is relatively simple. As this relationship becomes more complicated then the integrals must be evaluated numerically. However, when applied to a simple system this method does give an analytical solution to the probability distribution of the system response.

A comparison was carried out between the estimated mean and variance of the response distribution using this method to that obtained from the Monte Carlo simulations. The maximum error between the two methods was 2.5%, this was shown to reduce by increasing the number of Monte Carlo simulation runs.

4 CONCLUSIONS

The measured variability in the mass and natural frequency of a single complex component (a vehicle alloy wheel rim) and the variability in structure-borne and air-borne response of a complex built-up structure (an automotive vehicle) have been investigated. The goodness-of-fit of the response distributions was examined using a χ^2 test. In the case of the alloy wheel rim due to the low standard deviation of the distribution of the modes compared to their mean values, either a lognormal or a Gaussian distribution was found to be a good fit for three of the first four modes. The mass of the wheel rim was found to be a poor fit to a Gaussian distribution. In the case of the automotive vehicles, two vehicles were examined, the Isuzu Rodeo and the Isuzu Pickup. The distribution of the response for both the airborne and structure-borne FRF was shown to be a good fit to a Gaussian probability density function for approximately 80% of the frequency range. A slightly better fit was obtained at low frequencies where up to 7% more of the frequency lines were a good fit to a Gaussian distribution when compared to those at higher frequencies. It can be concluded that, for these examples the response distribution can be well described by a Gaussian distribution. The fit to a lognormal distribution was substantially lower.

In order to study the propagation of variability through a system from the physical properties to the response distribution, a simple idealised single degree of freedom system was considered. Two analytical methods were investigated, a transformation of variables technique and a power series expansion with moment generation. These were used in conjunction with Monte Carlo simulations to examine the effectiveness of each method. The conclusions from this investigation are as follows. Both techniques considered can be used to estimate the statistics of the response provided the statistics of the input variable are assumed or known. The accuracy of each method is independent of the input distribution. The response distribution has two distinct behaviours; near resonance the response is a one-sided distribution, away from resonance it is controlled by the input distribution. There are limitations on both techniques that the system output random variable is a relatively simple function of the input random variables; this limits the usefulness of the methods for more complex structures.

5 ACKNOWLEDGEMENTS

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Gaussian Fit to Data Set	Test Details	% Good Fit	% Poor Fit	% Improbably Good Fit	Comments
Pickup structure-borne FRF	57 vehicles 30-500 Hz	81%	19%	~0.1%	- Doesn't display low/high freq differences
Pickup air-borne FRF	57 vehicles 125-1000Hz	83%	17%	0%	- Slight low/high freq differences - Fit better at low freq ~4.5%
'Dummy' generated data set, Gaussian distributed	57 vehicles 30-500 Hz	90%	10%	~0.5%	- Obviously no low/high freq differences
Rodeo structure-borne FRF	98 vehicles 30-500 Hz	77%	22%	1%	- Low/high freq differences - Fit better at low freq ~7%
Rodeo air-borne FRF	98 vehicles 125-1000Hz	86%	14%	~0.5%	- Slight low/high freq differences - Fit better at low freq ~3%
'Dummy' generated data set, Gaussian distributed	98 vehicles 30-500 Hz	92%	6%	2%	- Obviously no low/high freq differences

Table 1: Goodness of fit to a Gaussian distribution

Lognormal Fit to Data Set	Test Details	% Good Fit	% Poor Fit	% Improbably Good Fit	Comments
Pickup structure-borne FRF	57 vehicles 30-500 Hz	69.1%	30.8%	0.1%	- No significant low/high freq differences (2.3% better at high freq)
Pickup air-borne FRF	57 vehicles 125-1000Hz	63.7%	36.2%	0.1%	- Fit better at low frequencies 4.4%
'Dummy' generated data set, Gaussian distributed	57 vehicles 30-500 Hz	80.1%	19.4%	0.5%	- Obviously no low/high freq differences
Rodeo structure-borne FRF	98 vehicles 30-500 Hz	64.4%	35.6%	0%	- No significant low/high freq differences (1.1% better at high freq)
Rodeo air-borne FRF	98 vehicles 125-1000Hz	65.2%	34.5%	0.3%	- Fit better at low frequencies 16%
'Dummy' generated data set, Gaussian distributed	98 vehicles 30-500 Hz	81.5%	19%	0.5%	- Obviously no low/high freq differences

Table 2: Goodness of fit to a lognormal distribution

Table 3: Summary of response p.d.f $p\left(\left|\alpha(\omega)\right|^2\right)$ where $k_0 = 3450.23$ N/m, $m_0 = 5.47$ kg, $c_0 = 12.07$ Ns/m, $\sigma_\varepsilon = 0.01$, averages = 10000, Gaussian distributed input variable

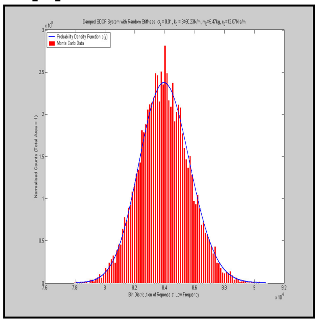
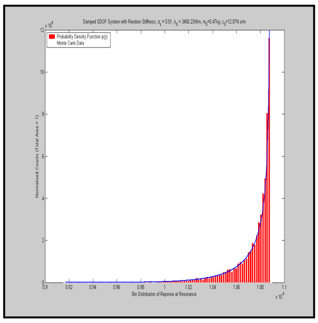
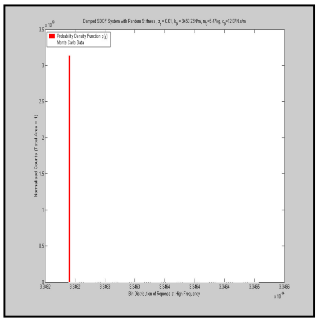
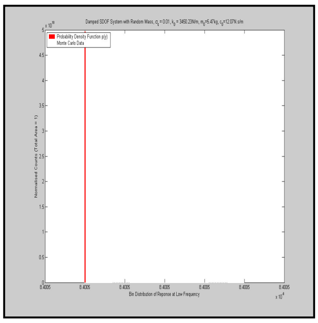
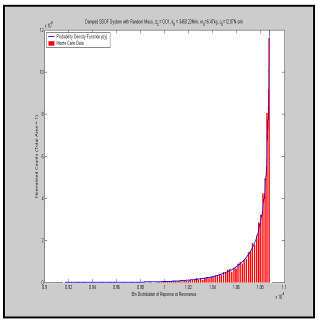
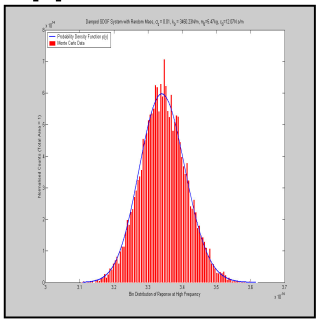
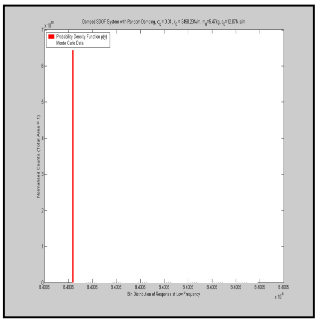
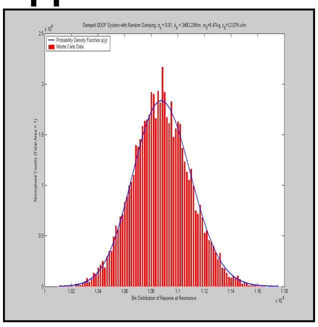
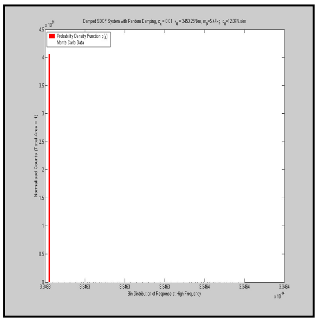
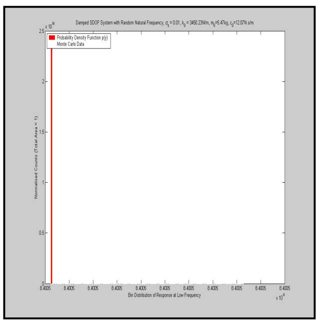
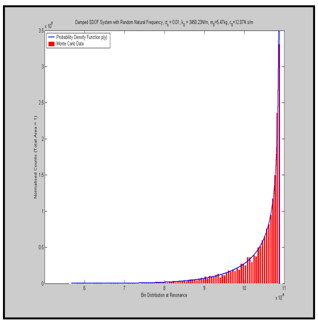
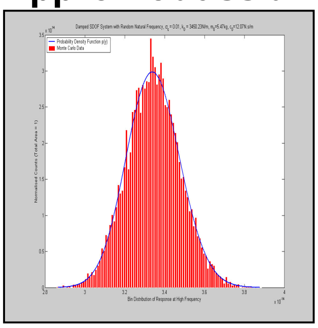
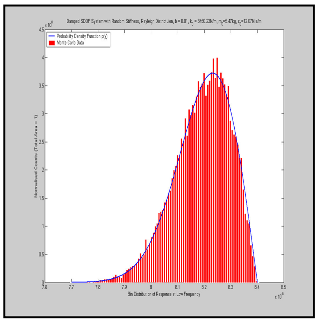
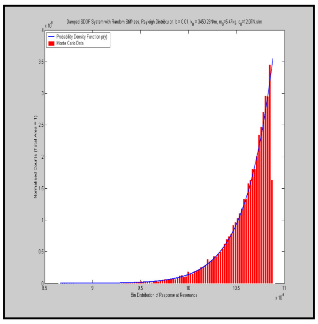
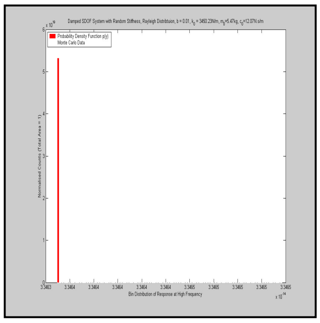
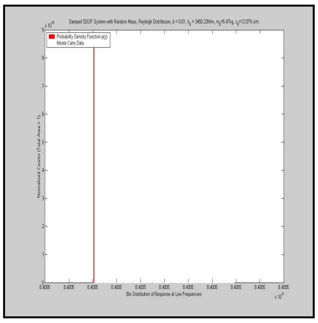
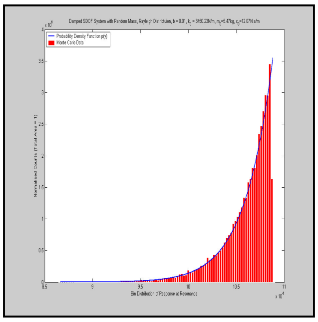
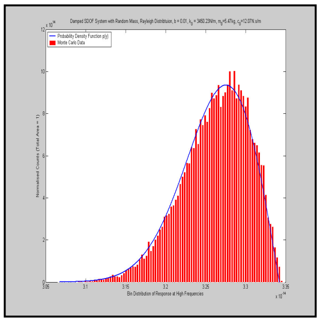
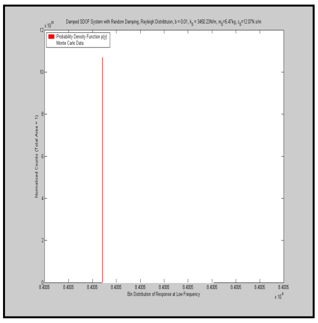
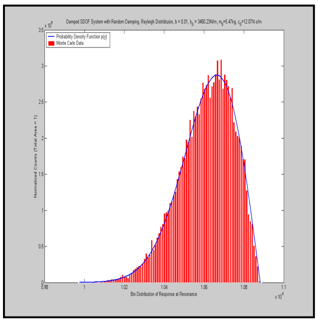
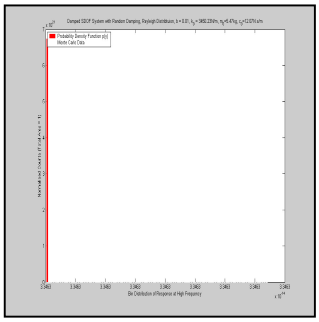
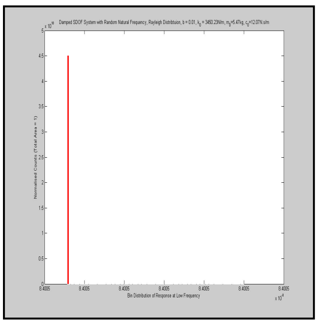
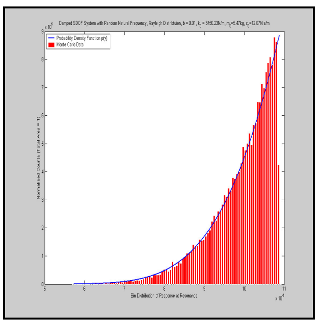
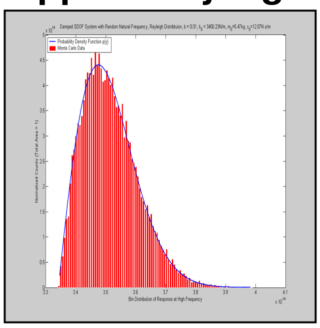
Gaussian Distributed Input Variable	$\omega << \omega_{ref}$	$\omega \approx \omega_{ref}$	$\omega >> \omega_{ref}$
Variable Stiffness	Approx Gaussian 	One Sided 	
Variable Mass		One Sided 	Approx Gaussian 
Variable Damping		Approx Gaussian 	
Variable Natural Frequency		One Sided 	Approx Gaussian 

Table 4: Summary of response p.d.f $p\left(\left|\alpha(\omega)\right|^2\right)$ where $k_0 = 3450.23$ N/m, $m_0 = 5.47$ kg, $c_0 = 12.07$,Ns/m $b = 0.01$, averages = 10000, Rayleigh distributed input variable

Rayleigh Distributed Input Variable	$\omega << \omega_{ref}$	$\omega \approx \omega_{ref}$	$\omega >> \omega_{ref}$
Variable Stiffness	**Approx Rayleigh Reflected** 	**One Sided** 	
Variable Mass		**One Sided** 	**Approx Rayleigh Reflected** 
Variable Damping		**Approx Rayleigh Reflected** 	
Variable Natural Frequency		**One Sided** 	**Approx Rayleigh** 

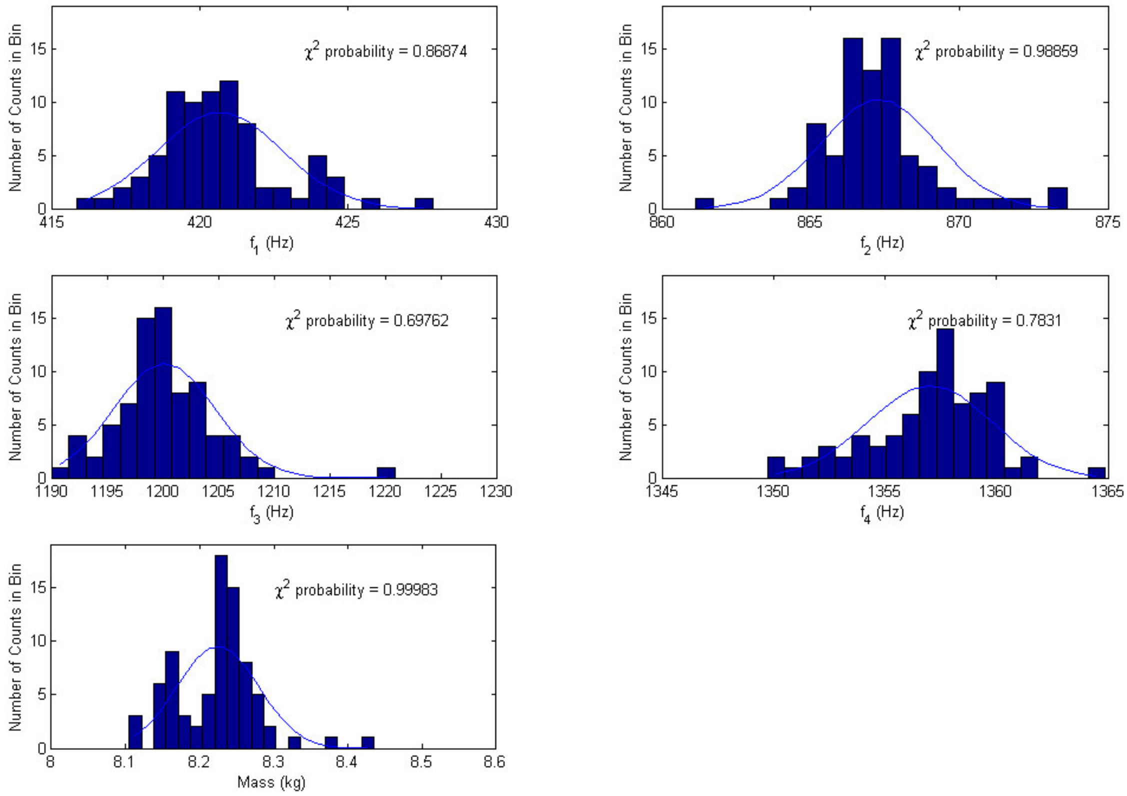


Figure 1: The frequency distribution of the first four modes and the total mass of the alloy wheels. The equivalent sample numbers from the corresponding Gaussian distributions with the same mean and standard deviation as in the original data set are also shown.

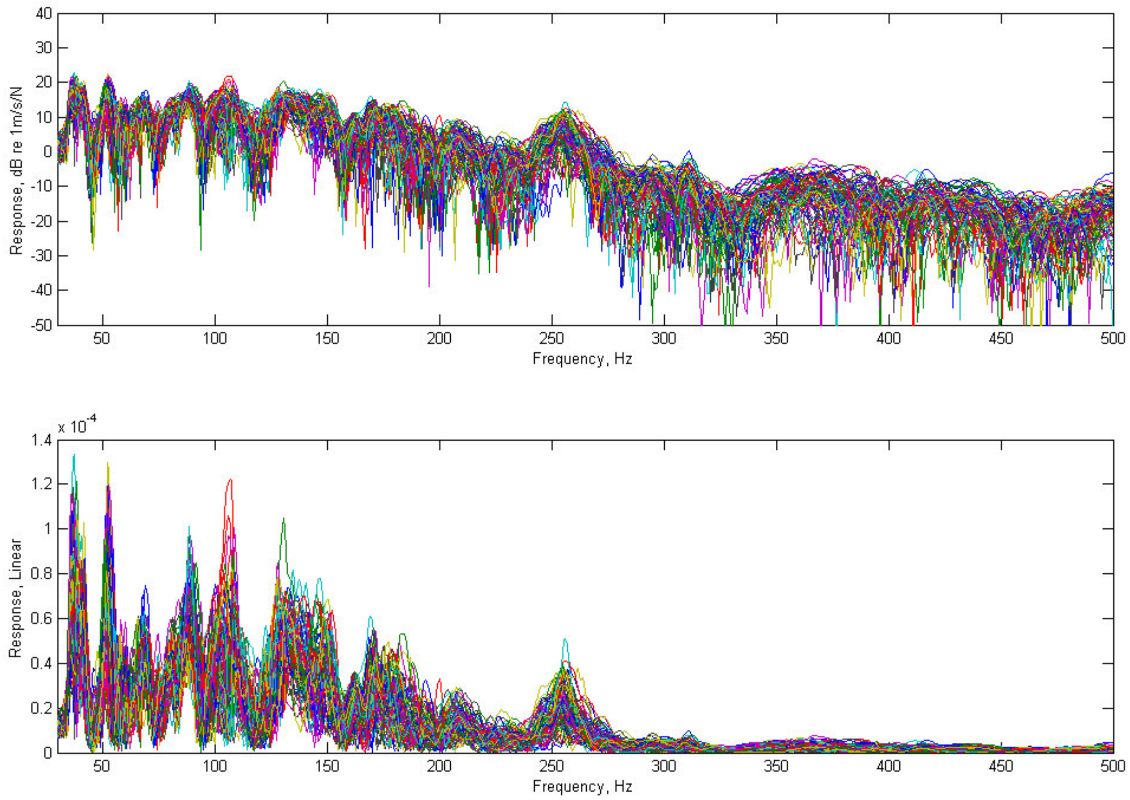


Figure 2: Vehicle FRFs, Rodeo Structure-borne FRFs: (a) dB and (b) linear response of 98 vehicles

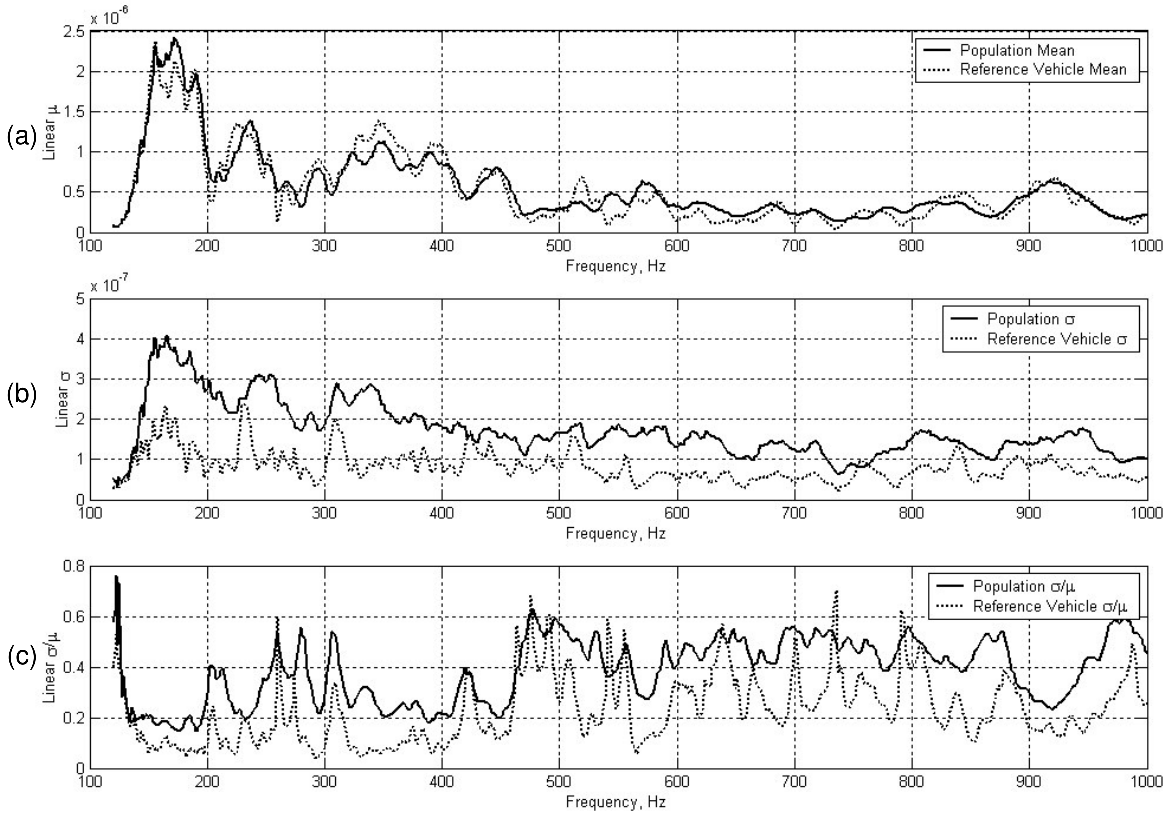


Figure 3: Comparison of Population Statistics to Reference Vehicle Statistics, Rodeo Vehicle Air-borne FRFs: (a) mean vs. frequency, (b) standard deviation vs. frequency, (c) normalised standard deviation

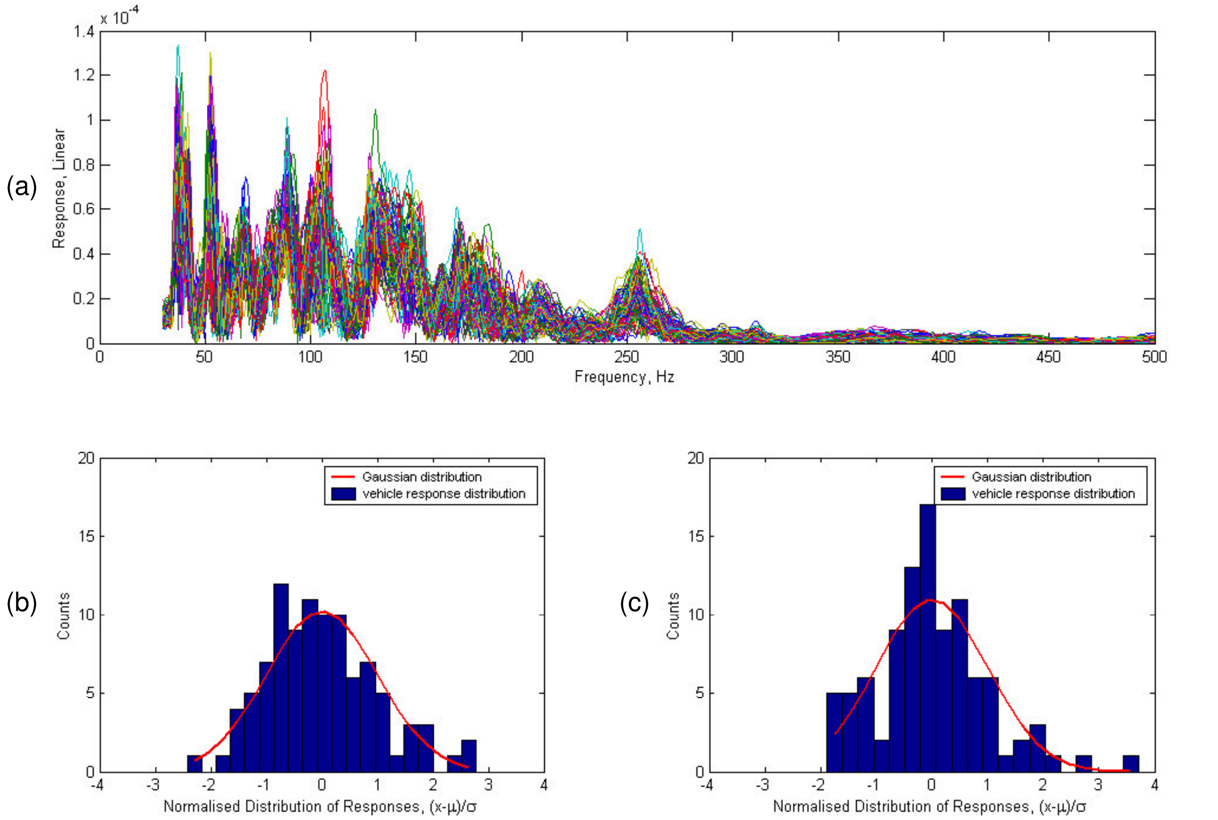


Figure 4: (a) Rodeo Structure-borne FRF, Distribution of Response at (b) 52.5 Hz and (c) 375Hz, Gaussian distributions fitted.