ACOUSTIC SCATTERING PROBLEMS INVOLVING SMART OBSTACLES

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1. INTRODUCTION

In this paper a mathematical model for an acoustic time dependent scattering problem involving smart obstacles is formulated. Smart obstacles are obstacles that when hit by an incoming acoustic field react in order to pursue an assigned goal. The goal pursued by the smart obstacle considered in this paper in the forward scattering problem is: to appear in a location in space different from its actual location eventually with a shape and boundary impedance different from its actual ones. We call this goal: to appear as a ghost obstacle. The smart obstacle pursues its goal circulating a pressure current (i.e. a quantity whose physical dimension is pressure divided by time) on its boundary. We show that the pressure current necessary to pursue the goal can be determined as the solution of a suitable optimal control problem for the wave equation.

The author and its coworkers have studied similar models for several other classes of smart obstacles in acoustic and electromagnetic scattering (see for example [1]-[6] and the website: http://www.econ.univpm.it/recchioni). The obstacles considered pursue one of the following goals:

1. to be undetectable (i.e.: furtivity problem),
2. to appear with a shape and a boundary impedance different from its actual shape and impedance (i.e.: masking problem),
3. to appear in a location in space different from its actual location eventually with a shape and boundary impedance different from its actual ones (i.e.: ghost obstacle problem).

The direct scattering problems corresponding to 1.-3. have been formulated as optimal control problems for the wave equation (acoustic case) or for the Maxwell equations (electromagnetic case) and the first order optimality conditions for these control problems have been derived applying the Pontryagin maximum principle and solved with appropriate numerical methods on several test problems. Several other approaches to study smart obstacles have been considered in the literature, see for example [7]-[10].

In this paper time harmonic inverse scattering problems involving smart obstacles are also studied.The inverse scattering problem considered is the following: given the knowledge of several far fields generated by the smart obstacle when hit by known incident acoustic fields it reacts with the optimal strategy and the knowledge of the goal pursued by the obstacle find the obstacle (i.e. find the shape, acoustic impedance and spatial location of the obstacle). For simplicity in this paper we limit our attention to the case of the obstacle that tries to be masked when the incoming acoustic field is time harmonic (that is the time harmonic inverse masking problem). Moreover in the inverse problem we assume that the acoustic boundary impedance of the obstacle and of the mask are known. In this case the direct scattering problem is translated in a constrained optimization problem and its solution is characterized as the solution of a set of auxiliary equations, that is a boundary value problem for a system of two Helmholtz equations. The inverse scattering problem is translated in an inverse problem for the system of two Helmholtz equations mentioned above see [11]. Material related to the problems described here is contained in the websites:
http: //www.econ.univpm.it/recchioni/w6, http: //www.econ.univpm.it/recchioni/w13.

2. THE GHOST OBSTACLE OPTIMAL CONTROL PROBLEM

Let $\Omega \subset \mathbf{R}^3, \Omega_G \subset \mathbf{R}^3$ be two bounded simply connected open sets with locally Lipschitz boundaries $\partial\Omega, \partial\Omega_G$ and let $\overline{\Omega}$ and $\overline{\Omega}_G$ be their closures respectively. Let us denote with $\underline{n}(\underline{x}) = (n_1(\underline{x}), n_2(\underline{x}), n_3(\underline{x}))^T \in \mathbf{R}^3, \underline{x} \in \partial\Omega$ the outward unit normal vector to $\partial\Omega$ in $\underline{x} \in \partial\Omega$. Since Ω has a locally Lipschitz boundary, $\underline{n}(\underline{x}), \underline{x} \in \partial\Omega$, exists almost everywhere, similar statements hold for the outward unit normal vector to $\partial\Omega_G$. Furthermore let Ω_G be such that $\Omega_G \neq \emptyset$ and $\overline{\Omega} \cap \overline{\Omega}_G = \emptyset$. We assume that Ω and Ω_G are characterized by constant acoustic boundary impedances $\chi \geq 0$ and $\chi_G \geq 0$, respectively. The case $\chi = +\infty$ and/or $\chi_G = +\infty$ (i.e.: the case of acoustically hard obstacles) can be treated with simple modifications of the formulae presented here. We refer to $(\Omega;\chi)$ as the obstacle and to $(\Omega_G;\chi_G)$ as the ghost obstacle. We consider an acoustic incident wave $u^i(\underline{x},t), (\underline{x},t) \in \mathbf{R}^3 \times \mathbf{R}$, propagating in a homogeneous isotropic medium in equilibrium at rest with no source terms present that satisfies the wave equation with wave propagation velocity $c > 0$ in $\mathbf{R}^3 \times \mathbf{R}$.
Finally we denote with $u^s(\underline{x},t), (\underline{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}$ and with $u_G^s(\underline{x},t), (\underline{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}_G) \times \mathbf{R}$, the waves scattered respectively by the obstacle $(\Omega;\chi)$ and by the ghost obstacle $(\Omega_G;\chi_G)$ when hit by $u^i(\underline{x},t), (\underline{x},t) \in \mathbf{R}^3 \times \mathbf{R}$.
The scattered acoustic field $u^s(\underline{x},t), (\underline{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}$ is defined as the solution of the following exterior problem for the wave equation:

$$\Delta u^s(\underline{x},t) - \frac{1}{c^2}\frac{\partial^2 u^s}{\partial t^2}(\underline{x},t) = 0, \quad (\underline{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}, \qquad (1)$$

with the boundary condition:

$$-\frac{\partial u^s}{\partial t}(\underline{x},t) + c\chi \frac{\partial u^s}{\partial \underline{n}(\underline{x})}(\underline{x},t) = \\ = g(\underline{x},t), \quad (\underline{x},t) \in \partial\Omega \times \mathbf{R}, \qquad (2)$$

where $g(\underline{x},t)$ is given by:

$$g(\underline{x},t) = \frac{\partial u^i}{\partial t}(\underline{x},t) - c\chi \frac{\partial u^i}{\partial \underline{n}(\underline{x})}(\underline{x},t), \quad (\underline{x},t) \in \partial\Omega \times \mathbf{R}, \qquad (3)$$

the boundary condition at infinity:

$$u^s(\underline{x},t) = O\left(\frac{1}{r}\right), r \to +\infty, \quad t \in \mathbf{R}, \qquad (4)$$

and the radiation condition:

$$\frac{\partial u^s}{\partial r}(\underline{x},t) + \frac{1}{c}\frac{\partial u^s}{\partial t}(\underline{x},t) = o\left(\frac{1}{r}\right), \quad r \to +\infty, t \in \mathbf{R}, \qquad (5)$$

where $r = \|\underline{x}\|$, $\underline{x} \in R^3, \Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $c > 0$ is the wave propagation velocity and $O(\cdot)$ and $o(\cdot)$ are the Landau symbols. We note that $g(\underline{x},t), (\underline{x},t) \in \partial\Omega \times \mathbf{R}$ is defined almost everywhere and that the boundary condition (2) can be adapted to deal with the limit case of the acoustically hard obstacles, i.e. $\chi = +\infty$. The obstacle $(\Omega;\chi)$ that scatters the field u^s solution of (1), (2), (3), (4), (5) is called passive obstacle. The field $u_G^s(\underline{x},t), (\underline{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega_G}) \times \mathbf{R}$ scattered by the (passive) ghost obstacle is defined as the solution of (1), (2), (3), (4), (5) when in the problem defined above we replace Ω with Ω_G and χ with χ_G. Note that we always consider the ghost obstacle as a passive obstacle.

We consider the following problem:

Ghost Obstacle Problem: Given an incoming acoustic field $u^i(\underline{x},t), (\underline{x},t) \in \mathbf{R}^3 \times \mathbf{R},$ an obstacle $(\Omega;\chi)$, a ghost obstacle $(\Omega_G;\chi_G)$ choose a pressure current circulating on $\partial\Omega$ for $t \in \mathbf{R}$ in such a way that the wave scattered by $(\Omega;\chi)$ when hit by the incoming acoustic field u^i appears, outside a given set containing Ω and Ω_G, "as similar as possible" to the wave scattered in the same circumstances by the ghost obstacle $(\Omega_G;\chi_G)$.

Remember that a pressure current is a quantity whose physical dimension is: pressure divided by time.

Our goal is to model the ghost obstacle problem as an optimal control problem introducing a control variable $\psi(\underline{x},t), (\underline{x},t) \in \partial\Omega \times \mathbf{R},$ that is a pressure current acting on the boundary of the obstacle. To this aim, we replace the boundary condition (2) with the following boundary condition:

$$-\frac{\partial u^s}{\partial t}(\underline{x},t) + c\chi \frac{\partial u^s}{\partial \underline{n}(\underline{x})}(\underline{x},t) = g(\underline{x},t) + \qquad (6)$$
$$+(1+\chi)\psi(\underline{x},t), (\underline{x},t) \in \partial\Omega \times \mathbf{R}.$$

Let Ω_ε be a bounded simply connected open set containing Ω and Ω_G with Lipschitz boundary $\partial\Omega_\varepsilon$ and let $ds_{\Omega_\varepsilon}, ds_{\partial\Omega}$ be the surface measures on $\partial\Omega_\varepsilon$ and $\partial\Omega$ respectively.

We choose the following cost functional:

$$F_{\lambda,\mu,\varepsilon}(\psi) = \int_{\mathbf{R}} dt \left\{ \int_{\partial\Omega_\varepsilon} (1+\chi)\lambda\left(u^s(\underline{x},t) - u_G^s(\underline{x},t)\right)^2 ds_{\partial\Omega_\varepsilon} + \qquad (7)\right.$$
$$\left. + \int_{\partial\Omega} (1+\chi)\mu\varsigma\psi^2(\underline{x},t) ds_{\partial\Omega} \right\},$$

where $\lambda \geq 0, \mu \geq 0$ are adimensional constants such that $\lambda + \mu = 1,$ and ς is a nonzero positive dimensional constant. We model the ghost obstacle problem via the following optimal control problem:

$$\min_{\psi \in C} F_{\lambda,\mu,\varepsilon}(\psi), \qquad (8)$$

subject to the constraints (1), (4), (5) and (6).

This is a legitimate mathematical model of the ghost obstacle problem. In fact the minimization $F_{\lambda,\mu,\varepsilon}$ makes small $u^s - u_G^s$ for $(\underline{x},t) \in \partial\Omega_\varepsilon \times \mathbf{R}$, that is makes small $u^s - u_G^s$ for $(\underline{x},t) \in (\mathbf{R}^3 \setminus \Omega_\varepsilon) \times \mathbf{R}$ and makes small the "size" of the pressure current used while the constraints (1), (4), (5), (6) guarantee the satisfaction of the dynamic conditions associated to the problem considered.

The set C is the space of the admissible controls that we leave undetermined. The obstacle $(\Omega;\chi)$ that generates the scattered field u^s solution of (8), (1), (4), (5), (6) is called smart or active obstacle.

Note that in (7) the choice $\Omega_G \subset \Omega, \Omega_\varepsilon = \Omega$ gives the masking problem and that the choice $\Omega_G = \phi,$ $\Omega_\varepsilon = \Omega$ gives the furtivity problem.

3. THE FIRST ORDER OPTIMALITY CONDITIONS

Let us make the following assumptions: let (r,θ,ϕ) be the usual spherical coordinate system in $\mathbf{R}^3$ with pole in the origin, let B be the sphere with center the origin and radius one and let ∂B be its boundary, we assume that:

(a) the boundary of the obstacle Ω is a starlike surface with respect to the origin, that is Ω and $\partial\Omega$ can be represented as follows:

$$\Omega = \left\{\underline{x} = r\hat{\underline{x}} \in \mathbf{R}^3 \middle| 0 \le r < \xi(\hat{\underline{x}}), \hat{\underline{x}} \in \partial B\right\}, \quad (9)$$

$$\partial\Omega = \left\{\underline{x} = r\hat{\underline{x}} \in \mathbf{R}^3 \middle| r = \xi(\hat{\underline{x}}), \hat{\underline{x}} \in \partial B\right\}, \quad (10)$$

where $\xi(\hat{\underline{x}}) > 0, \hat{\underline{x}} \in \partial B$, is a single valued function defined on ∂B that is assumed sufficiently regular for the manipulations that follow;

(b) the sets Ω_ε and $\partial\Omega_\varepsilon$ can be represented as follows:

$$\Omega_\varepsilon = \left\{\underline{x} = r\hat{\underline{x}} \in \mathbf{R}^3 \middle| 0 \le r < (\xi(\hat{\underline{x}}) + \varepsilon), \hat{\underline{x}} \in \partial B\right\}, \varepsilon > 0, \quad (11)$$

$$\partial\Omega_\varepsilon = \left\{\underline{x} = r\hat{\underline{x}} \in \mathbf{R}^3 \middle| r = \xi(\hat{\underline{x}}) + \varepsilon, \hat{\underline{x}} \in \partial B\right\}, \varepsilon > 0. \quad (12)$$

for a suitable choice of $\varepsilon > 0$.

Note that the assumptions (*a*) and (*b*) are only one of many other possible choices of assumptions that can be made to guarantee the satisfactory solution of the model (8), (1), (4), (5), (6). This choice is made just to fix the ideas and to keep the exposition simple.

Under the assumptions (*a*) and (*b*), applying the Pontryagin maximum principle the optimal state trajectory $\tilde{u}^s$ and the corresponding adjoint variable trajectory $\tilde{\varphi}$ satisfy the necessary first order optimality conditions associated to the optimal control problem (8), (1), (4), (5), (6), that is they are the solution of the following exterior problem for a system of two coupled wave equations:

$$\Delta\tilde{u}^s(\underline{x},t) - \frac{1}{c^2}\frac{\partial^2\tilde{u}^s}{\partial t^2}(\underline{x},t) = 0, \; (\underline{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}, \quad (13)$$

$$\tilde{u}^s(\underline{x},t) = O\left(\frac{1}{r}\right), \quad r \to +\infty, \; t \in \mathbf{R}, \quad (14)$$

$$\frac{\partial\tilde{u}^s}{\partial r}(\underline{x},t) + \frac{1}{c}\frac{\partial\tilde{u}^s}{\partial t}(\underline{x},t) = o\left(\frac{1}{r}\right), \quad r \to +\infty, \; t \in \mathbf{R}, \quad (15)$$

$$-\frac{\partial\tilde{u}^s}{\partial r}(\underline{x},t) + c\chi\frac{\partial\tilde{u}^s}{\partial\underline{n}(\underline{x})}(\underline{x},t) = g(\underline{x},t) - \\ -\frac{(1+\chi)}{2\mu\varsigma}\tilde{\varphi}(\underline{x},t), (\underline{x},t) \in \partial\Omega \times \mathbf{R}, \quad (16)$$

$$\Delta\tilde{\varphi}(\underline{x},t) - \frac{1}{c^2}\frac{\partial^2\tilde{\varphi}}{\partial t^2}(\underline{x},t) = 0, \; (\underline{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}, \quad (17)$$

$$\tilde{\varphi}(\underline{x},t) = O\left(\frac{1}{r}\right), \quad r \to +\infty, \; t \in \mathbf{R}, \quad (18)$$

$$\frac{\partial\tilde{\varphi}}{\partial r}(\underline{x},t) + \frac{1}{c}\frac{\partial\tilde{\varphi}}{\partial t}(\underline{x},t) = o\left(\frac{1}{r}\right), \quad r \to +\infty, \; t \in \mathbf{R}, \quad (19)$$

$$-\frac{\partial\tilde{\varphi}}{\partial t}(\underline{x},t) - c\chi\frac{\partial\tilde{\varphi}}{\partial\underline{n}(\underline{x})}(\underline{x}) =$$

$$-2\lambda(1+\chi)f_{\varepsilon}\left(\frac{x}{\|x\|}\right)\left(\tilde{u}^{s}\left(x+\varepsilon\frac{x}{\|x\|},t\right)-u_{G}^{s}\left(x+\varepsilon\frac{x}{\|x\|},t\right)\right),(x,t)\in\partial\Omega\times\mathbf{R},$$

(20)

$$\lim_{t\to-\infty}\tilde{u}^{s}(\underline{x},t)=0,\underline{x}\in\mathbf{R}^{3}\setminus\overline{\Omega}, \qquad (21)$$

$$\lim_{t\to+\infty}\tilde{\varphi}(\underline{x},t)=0,\underline{x}\in\mathbf{R}^{3}\setminus\Omega, \qquad (22)$$

where $f_{\varepsilon}(\underline{x}/\|\underline{x}\|), x\in\partial\Omega$ is the function defined by:

$$f_{\varepsilon}\left(\frac{x}{\|x\|}\right)=f_{\varepsilon}(\hat{\underline{x}}(\theta,\phi))=\frac{\upsilon_{\varepsilon}(\theta,\phi)}{\upsilon(\theta,\phi)},\underline{x}\in\partial\Omega,\hat{\underline{x}}=\frac{x}{\|x\|}\in\partial B, \qquad (23)$$

$0\le\theta\le\pi, 0\le\phi<2\pi$,

$$\upsilon(\theta,\phi)=\xi\sqrt{\left(\frac{\partial\xi}{\partial\theta}\right)^{2}\sin^{2}\theta+\left(\frac{\partial\xi}{\partial\phi}\right)^{2}+\xi^{2}\sin^{2}\theta}, \qquad (24)$$

$0\le\theta\le\pi, 0\le\phi<2\pi$,

$$\upsilon_{\varepsilon}(\theta,\phi)=(\xi+\varepsilon)\sqrt{\left(\frac{\partial\xi}{\partial\theta}\right)^{2}\sin^{2}\theta+\left(\frac{\partial\xi}{\partial\phi}\right)^{2}+(\xi+\varepsilon)^{2}\sin^{2}\theta},$$

$0\le\theta\le\pi, 0\le\phi<2\pi$. (25)

The relation between $\tilde{\varphi}$ and the optimal control $\tilde{\psi}$ solution of problem (8), (1), (4), (5), (6) is the following one:

$$\tilde{\psi}(\underline{x},t)=-\frac{1}{2\mu\varsigma}\tilde{\varphi}(\underline{x},t),(\underline{x},t)\in\partial\Omega\times\mathbf{R}. \qquad (26)$$

Let us point out that we have:

$$ds_{\partial\Omega}=\upsilon(\theta,\phi)d\theta d\phi, 0\le\theta\le\pi, 0\le\phi<2\pi, \qquad (27)$$

and

$$ds_{\partial\Omega_{\varepsilon}}=\upsilon_{\varepsilon}(\theta,\phi)d\theta d\phi, 0\le\theta\le\pi, 0\le\phi<2\pi. \qquad (28)$$

4. NUMERICAL SOLUTION OF THE EXTERIOR PROBLEM (13) – (22)

Numerical methods to solve the exterior problem (13)-(22) have been developed in [5], [6]. These methods belong to the class of the operator expansion methods and are highly parallelizable. Some numerical experiments proving the validity of the control problem proposed as mathematical model of the ghost obstacle problem are shown in the website http://www.econ.univpm.it/recchioni/w11.

5. THE INVERSE MASKING PROBLEM

Let us describe the data defining the direct masking problem. Let $\Omega \subset \mathbf{R}^3$ and $\Omega_M \subset \mathbf{R}^3$ be bounded simply connected open sets with locally Lipschitz boundaries $\partial\Omega, \partial\Omega_M$ respectively and let $\overline{\Omega}, \overline{\Omega}_M$ be their closures. Let us denote with $\underline{n}(\underline{x}) = (n_1(\underline{x}), n_2(\underline{x}), n_3(\underline{x}))^T \in \mathbf{R}^3$, $\underline{x} \in \partial\Omega$ the outward unit normal vector to $\overline{\Omega}, \overline{\Omega}_M$. We assume that Ω and Ω_M are characterized by constant acoustic boundary impedances $\chi, \chi_M, \chi \geq 0, \chi_M \geq 0$ respectively and that $\Omega_M \subseteq \Omega$. We refer to the couple $(\Omega;\chi)$ as the smart obstacle and to the couple $(\Omega_M;\chi_M)$ as the mask. We assume that the origin of the coordinate system belongs to Ω_M .We assume that $(\Omega;\chi)$ is a smart obstacle that pursues the goal of appearing as the mask $(\Omega_M;\chi_M)$ and that the mask $(\Omega_M;\chi_M)$ is a passive obstacle. The behaviour of this smart obstacle can be modelled as the solution of an optimal control problem that due to the fact that we consider only time harmonic incoming acoustic fileds reduces to the optimization problem that follows. Since we deal with time harmonic fields we formulate the problem using directly the Helmholtz equation without going through the wave equation formulation. Given the incoming time harmonic acoustic wave u^i the obstacle $(\Omega;\chi)$, the mask $(\Omega_M;\chi_M)$, such that $\Omega_M \subseteq \Omega$, choose a pressure current circulating on $\partial\Omega$ in order to minimize a cost functional that measures the "magnitude" of the pressure current used and the "magnitude" of the "difference" between the wave scattered by the smart obstacle $(\Omega;\chi)$ and the wave scattered by the mask $(\Omega_M;\chi_M)$ when hit by u^i.

The incoming wave u^i is the spatial part of an incoming acoustic time harmonic plane wave, that is:

$$u^i(\underline{x}) = u^i_{\omega,\underline{\alpha}}(\underline{x}) = e^{\iota\omega(\underline{x},\underline{\alpha})/c}, \underline{x} \in \mathbf{R}^3 \qquad (29)$$

where $\iota \in C$ is the imaginary unit, $c > 0$ is the wave propagation velocity, $\omega / c = k$ is the wave number and $\underline{\alpha} \in \mathbf{R}^3$ is an unit vector. The direction $\underline{\alpha}$ is the propagation direction of the incoming field. Let us denote with $u^s_{\omega,\underline{\alpha}}(\underline{x})$ the (spatial part of the) acoustic field scattered by the obstacle $(\Omega;\chi)$ when hit by u^i and with $u^s_{M,\omega,\underline{\alpha}}(\underline{x})$ the (spatial part of the) acoustic field scattered by the $(\Omega_M;\chi_M)$ in the same circumstances.

To model the smart obstacle we need to act on the boundary via a control variable that we assume to be time harmonic and given by $\widetilde{\psi}(\underline{x},t) = \exp(-\iota\omega\, t)\psi(\underline{x}) \in \partial\Omega, t \in \mathbf{R}$, that is, we replace the boundary condition (6) with the following one:

$$\iota\omega u^s_{\omega,\underline{\alpha}}(\underline{x}) + c\chi \frac{\partial u^s_{\omega,\underline{\alpha}}}{\partial \underline{n}(\underline{x})}(\underline{x}) = (1+\chi)\psi(\underline{x}) + b_{\omega,\underline{\alpha}}(\underline{x}),\ \underline{x} \in \partial\Omega, \qquad (30)$$

where $b_{\omega,\underline{\alpha}}(\underline{x})$ is the function analogous of g in (2) similarly the wave equation (1) must be substituted with the Helmholtz equation for the space dependent part of the unknown scattered field and the boundary conditions at infinity (4), (5) must be substituted with the Sommerfeld radiation condition at infinity. The function $\widetilde{\psi}$ is a time harmonic pressure current, i.e. a quantity given by pressure divided by time, and its space dependent part ψ appears in the boundary condition (30) satisfied by the scattered field generated by the smart obstacle. Note that both $\widetilde{\psi}$ and ψ are in general complex valued functions, this is due to the fact that we work in the time harmonic framework, so that the term "pressure current" is somehow abused. We determine the function that corresponds to the optimal reaction of the smart obstacle $(\Omega;\chi)$ as the solution of the following problem:

$$\min_{\psi \in C} F_{\omega,\underline{\alpha},\lambda,\mu}(\psi) \qquad (31)$$

subject to the appropriate constraints (i.e. the Helmoholtz equation and the boundary condition (30) and the sommerfeld radiation condition at infinity). The cost functional $F_{\omega,\underline{\alpha},\lambda,\mu}$ is given by:

$$F_{\omega,\underline{\alpha},\lambda,\mu}(\psi)=\int_{\partial\Omega} ds_{\partial\Omega}(\underline{x})(1+\chi)$$
$$\left\{\lambda\left|u^{s}_{\omega,\underline{\alpha}}(\underline{x})-u^{s}_{M,\omega,\underline{\alpha}}(\underline{x})\right|^{2}+\mu\varsigma|\psi(\underline{x})|^{2}\right\}. \quad (32)$$

Note that minimizing the functional (32) corresponds to making small the control variable employed and the difference $u^{s}_{\omega,\underline{\alpha}}-u^{s}_{M,\omega,\underline{\alpha}}$ on $\partial\Omega$ and as a consequence on $\mathbf{R}^3\setminus\overline{\Omega}$. This last fact translates mathematically the goal of the obstacle $(\Omega;\chi)$ that is the goal of $(\Omega;\chi)$ of appearing as the mask $(\Omega_M;\chi_M)$.

Note that in the optimization problem (31), with the appropriate constraints $u^{s}_{\omega,\underline{\alpha}}$ depends on ψ through the boundary condition (30).

Reasoning as in [11] for the constrained optimization problem mentioned above we can write the first order necessary optimality condition as the following auxiliary boundary value problem:

$$\left(\Delta u^{s}_{\omega,\underline{\alpha}}+\frac{\omega^2}{c^2}u^{s}_{\omega,\underline{\alpha}}\right)(\underline{x})=0,\ \underline{x}\in\mathbf{R}^3\setminus\overline{\Omega}, \quad (33)$$

$$\left(\Delta \varphi_{\omega,\underline{\alpha}}+\frac{\omega^2}{c^2}\varphi_{\omega,\underline{\alpha}}\right)(\underline{x})=0,\ \underline{x}\in\mathbf{R}^3\setminus\overline{\Omega}, \quad (34)$$

$$\iota\omega u^{s}_{\omega,\underline{\alpha}}(\underline{x})+c\chi\frac{\partial u^{s}_{\omega,\underline{\alpha}}}{\partial\underline{n}(\underline{x})}(\underline{x})+$$
$$+\frac{(1+\chi)}{2\mu\varsigma}\varphi_{\omega,\underline{\alpha}}(\underline{x})=b_{\omega,\underline{\alpha}}(\underline{x}),\ \underline{x}\in\partial\Omega, \quad (35)$$

$$\iota\omega\varphi_{\omega,\underline{\alpha}}(\underline{x})-c\chi\frac{\partial\varphi_{\omega,\underline{\alpha}}}{\partial\underline{n}(\underline{x})}(\underline{x})+$$
$$+2\lambda(1+\chi)\left(u^{s}_{\omega,\alpha}(\underline{x})-u^{s}_{M,\omega,\underline{\alpha}}(\underline{x})\right)=0,\underline{x}\in\partial\Omega, \quad (36)$$

with the following conditions at infinity:

$$\frac{\partial u^{s}_{\omega,\underline{\alpha}}(\underline{x})}{\partial r}-\iota\frac{\omega}{c}u^{s}_{\omega,\underline{\alpha}}(\underline{x})=o\left(\frac{1}{r}\right),\ r\to+\infty, \quad (37)$$

$$\frac{\partial\varphi_{\omega,\underline{\alpha}}(\underline{x})}{\partial r}+\iota\frac{\omega}{c}\varphi_{\omega,\underline{\alpha}}(\underline{x})=o\left(\frac{1}{r}\right),\ r\to+\infty. \quad (38)$$

Let $u^{s}_{\omega,\underline{\alpha}}(\underline{x})$, $\varphi_{\omega,\underline{\alpha}}(\underline{x})$, $\underline{x}\in\mathbf{R}^3\setminus\Omega$ be the solution of (33)-(38). The function $\varphi_{\omega,\underline{\alpha}}(\underline{x})$, $\underline{x}\in\mathbf{R}^3\setminus\Omega$ solution of (33)-(38) is an auxiliary function related to the optimal control $\psi(\underline{x})=\hat{\psi}(\underline{x})$, $\underline{x}\in\partial\Omega$ solution of (31), with the appropriate constraints by the following relation:

$$\hat{\psi}(\underline{x}) = -\frac{1}{2\mu\varsigma}\varphi_{\omega,\underline{\alpha}}(\underline{x}),\ \underline{x} \in \partial\Omega. \qquad (39)$$

The function $\hat{\psi}$ is the space dependent part of the pressure current that must circulate on the boundary of $(\Omega;\chi)$ to obtain the optimal masking effect, that is to obtain the effect that the field scattered by $(\Omega;\chi)$ resembles in $\mathbf{R}^3 \setminus \Omega$ the field scattered by the mask $(\Omega_M;\chi_M)$ as much as possible in the mathematical model chosen. Now we can formulate the following mathematical model for the direct masking problem:
Acoustic Time Harmonic Masking Direct Problem: given the acoustic incoming plane wave (29), the obstacle $(\Omega;\chi)$, the mask $(\Omega_M;\chi_M)$ and the parameters λ,μ,ς, appearing in (32) find the scattered acoustic field $u^s_{\omega,\underline{\alpha}}$ and the optimal control given by $\hat{\psi}(\underline{x})$ solution of (31) with the appropriate constraints or alternatively find $u^s_{\omega,\underline{\alpha}}, \varphi_{\omega,\underline{\alpha}}$ solution of (33)-(38) and determine $\hat{\psi}(\underline{x})$ from (39).
The formulation of a time harmonic inverse problem for a smart obstacle $(\Omega;\chi)$ that pursues the goal of appearing as the assigned mask $(\Omega_M;\chi_M)$ is based on a first step that consists in the reconstruction of the mask. Note that to reconstruct the mask we can solve numerically the time harmonic passive inverse problem. Obviously the mask will be determined with an accuracy that depends on the ability of the smart obstacle to mask itself, that is on the values of the parameters λ,μ and ς appearing in (31).
The inverse masking problem can be translated into the following two steps:

Step 1: from the knowledge of the far fields associated to the acoustic fields scattered by the smart obstacle when the optimal pressure current is employed for several incoming waves with different incident directions and/or wave numbers and of the acoustic boundary impedance χ_M of the mask, determine the boundary $\partial\Omega_M$ of the mask;

Step 2: from the knowledge of the mask $(\partial\Omega_M;\chi_M)$, of the masking parameters λ, μ and ς and of the acoustic boundary impedance χ of the smart obstacle determine the boundary $\partial\Omega$ of the smart obstacle.
Step 1 is the usual inverse obstacle scattering problem for the Helmholtz equation and can be formulated as an optimization problem as shown for example in [11]. An alternative way to approach the inverse masking problem is contained in [12] where the so called Herglotz function method to solve inverse problems for the Helmholtz equation is generalized to the problems involving smart obstacles.
The data of Step 2 are the mask $(\Omega_M;\chi_M)$ and the parameters λ,μ and ς that define the cost functional (32) and the acoustic boundary impedance χ of the obstacle Ω.
For simplicity in the following of this section we assume that the obstacle and the mask are acoustically soft (i.e. $\chi = \chi_M = 0$).
First of all we note that when χ=0 from equations (30), (36) we can deduce the following equation:

$$\begin{aligned}&\left(2\lambda + 2\mu\varsigma\omega^2\right)u^s_{\omega,\underline{\alpha}}(\underline{x}) = 2\lambda u^s_{M,\omega,\underline{\alpha}}(\underline{x}) - \\ &-\iota\omega 2\mu\varsigma b_{\omega,\underline{\alpha}}(\underline{x}),\ \underline{x}\in\partial\Omega,\end{aligned} \qquad (40)$$

and that $u^s_{M,\omega,\underline{\alpha}}(\underline{x}), \underline{x} \in \partial\Omega$ can be obtained from the results of Step 1 solving the direct scattering problem for the passive obstacle $(\partial\Omega_M;\chi_M)$. Note that equation (40) satisfied on the boundary $\partial\Omega$ is a condition on $u^s_{\omega,\underline{\alpha}}$ when $\partial\Omega$ is known or is a condition on $\partial\Omega$ when $u^s_{\omega,\underline{\alpha}}$ is known. Equation (40) is used in this last sense when solving the inverse problem. In [11], [12] several numerical examples of solution of inverse problems involving smart obstacles are presented. These examples are obtained applying the inverstion procedures to synthetic data resulting from the numerical solution of the direct scattering problems. The results shown in [11], [12] are encouraging and suggest that the problem of solving inverse problems involving smart obstacles deserves further investigations.

6. EXTENSION AND CONCLUSIONS

The work presented can be extended to a new class of smart obstacles that pursue the following goal:

4. one of the goals specified in the Introduction restricted to a definite band in the frequency space.

We can conclude that the idea of modelling the smart obstacles using optimal control problems or constrained optimization problems is an interesting idea. Moreover the work developed until now with the model proposed can be profitably extended in several directions such as the study of closed loop controls, finite horizon controls, or the study of several inverse problems involving smart obstacles. These are challenging mathematical questions whose solution can be very valuable in practical applications.

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