AN INTRODUCTION TO A GENERALISED THEORY FOR SOUND FIELD REPRODUCTION

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1 INTRODUCTION

The reproduction of a target sound field is a problem of relevance in many branches of acoustics. This subject has recently acquired increasing importance for the audio industry. This is mainly due to the fact that technological progress during recent years has lead to the diffusion of multi-channel audio systems. The availability of such systems has opened new horizons for the implementation of acoustical theories aimed at the reproduction of a target sound field using an array of loudspeakers. Theories like Wave Field Synthesis [1], [2], [3] and High Order Ambisonics [4], [5], [6] as well as other more recent approaches such as for example [7], [8], [9] and [10] have been proposed and successfully applied to the realisation of this task.

A novel method for the reconstruction of a target sound field with an array of loudspeakers has been proposed and discussed by the authors in previous publications [11], [12]. This method starts from the initial assumption that a target sound field is given on the boundary of a three dimensional control volume, and that an array of loudspeakers can be modelled as an ideal continuous distribution of monopole sources arranged on the boundary of a region of three dimensional space which contains the control volume. The strength of these secondary sources, allowing the reconstruction of a given target field on the interior of this region (or on a subset of it), are computed by solving an integral equation of the first kind. As the latter constitutes an ill-posed problem, the use of a regularisation scheme can be used in order to compute stable solutions. Some of the most relevant results of this theory are briefly recalled here.

The aim of this paper is to present an analytical study of two special cases of the sound field reproduction theory discussed above. These two cases concern two special geometries of the control volume and of the layer of the secondary sources. In the first case analysed the boundary of the control volume and the layer of secondary sources are two concentric spheres. This special geometry allows the calculation of the singular value decomposition of the integral operator involved and this leads to an analytical expression for the function describing the strength of the secondary sources. The second case analysed concerns the situation when the boundary of the control volume and the layer of secondary sources coincide. This special situation allows a re-formulation of the mathematical problem involved, leading to an interesting analogy between this theory of sound field reproduction and the theory of acoustic scattering. The latter demonstrates the relationship of the strength of the secondary sources to the normal derivative of a scattered sound field.

The determination of the strength of the secondary sources involves, as described above, the solution of an ill-posed inverse problem. As discussed previously in [11], [12], the ill-conditioning of the problem can be related to the behaviour of the singular values of the operator involved in the integral equation to be solved. The analytical calculation of the singular values undertaken in this paper sheds some light on the dependence of the ill-conditioning of the problem on the geometry of the problem. It is shown that the roll-off of the singular values is in general exponential and can be related to the distance between the boundary of the control surface and of the layer of secondary sources, and this decay becomes linear when the two surfaces coincide.

2 THE PROBLEM OF SOUND FIELD RECONSTRUCTION

Let V and Λ be two simply connected subsets of the three dimensional space, with smooth boundaries (of class C^2) ∂V and $\partial\Lambda$, respectively. The symbol $\overline{V}$ represents the closure of V, that is $\overline{V} = V \oplus \partial V$. The assumption is also made, that the two subsets introduced are bounded and that $V \subseteq \Lambda$, that is V is contained in Λ. Figure 1 shows a two dimensional diagrammatic representation of the reference geometry. V is referred to as the control volume, while the boundary $\partial\Lambda$ is referred to as the secondary source layer. In what follows, the vector $\mathbf{x}$ is used to identify any location within V, while the vector $\mathbf{y}$ identifies a point on $\partial\Lambda$. The conventions $x = \|\mathbf{x}\|$ and $\hat{\mathbf{x}} = \mathbf{x}/x$ are also used here, and identify the norm and the direction associated with vector $\mathbf{x}$, respectively.

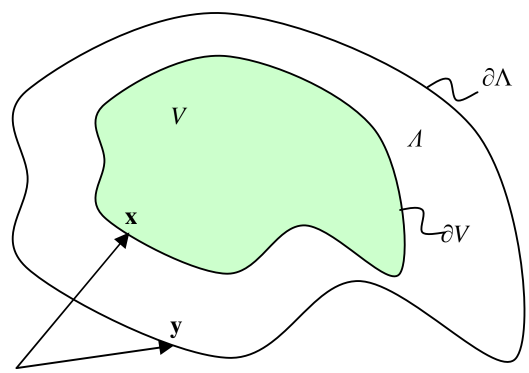

Figure 1: Two dimensional diagram of the reference geometry

Assume that a sound field $p(\mathbf{x})$ is defined within V, and that this field satisfies the homogeneous Helmholtz equation

$$\nabla^2 p(\mathbf{x}) + k^2 p(\mathbf{x}) = 0 \qquad \mathbf{x} \in \overline{V} \tag{1}$$

where $k = \omega/c$ is the wave number, ω is the frequency of the sound and c is the speed of sound propagation, which is supposed to be uniform in all the space. The time dependence $e^{-i\omega t}$ of the sound field is implicitly assumed, where $i = \sqrt{-1}$. The single layer potential S is defined as

$$(Sa)(\mathbf{x}) = \int_{\partial\Lambda} G(\mathbf{x},\mathbf{y}) a(\mathbf{y}) dS(\mathbf{y}) \tag{2}$$

The kernel of this integral is the Green function, also referred to as the fundamental solution of the Helmholtz equation. In what follows, the assumption is made that $G(\mathbf{x},\mathbf{y})$ is the free field Green function, that is

$$G(\mathbf{x},\mathbf{y}) = \frac{e^{ik\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|} \tag{3}$$

Nevertheless, the arguments presented below also hold, in principle, for a different choice of the Green function. The single layer potential S is the mathematical representation of the sound field generated by a continuous distribution of monopole sources (secondary sources), arranged on $\partial\Lambda$. The function $a(\mathbf{y})$ is referred to as the density of the potential and represents the strengths of the secondary sources.

The problem of sound field reconstruction addressed in this paper consists of defining the density $a(\mathbf{y})$ such that the field generated by the single layer potential with that density is the best approximation, in the sense of an L^2 norm, of a the target field $p(\mathbf{x})$ in V. In other words, this means that the target is to define the density $a(\mathbf{y})$ that minimises the reconstruction error

$$E = \int_V \left| p(\mathbf{x}) - (Sa)(\mathbf{x}) \right|^2 dV \tag{4}$$

It has been shown in [11] and [12] that this problem corresponds mathematically to an integral equation of the first kind, and represents therefore an ill-posed problem. In physical terms, this implies that it is theoretically possible to compute a density $a(\mathbf{y})$ such that the potential $(Sa)(\mathbf{x})$ can approximate the target sound field $p(\mathbf{x})$ in V with an arbitrarily high level of accuracy, but on the other hand, the higher is the accuracy with which the reconstruction is attempted, the larger is the chance that errors in the data are amplified and degrade the reconstruction, often resulting in catastrophic effects. In order to compute a stable solution, it is possible to apply a regularisation scheme, as described for example in [13].

A method to compute the density $a(\mathbf{y})$ from the knowledge of the target field $p(\mathbf{x})$ on the boundary ∂V is discussed in [11] and [12]. This method is based on the singular value decomposition of the integral operator S and is briefly explained in what follows. In order to do that, it is useful to define the scalar product between two square integrable functions f and g defined on D as

$$\langle g | f \rangle_D = \int_D f(\mathbf{x}) g(\mathbf{x})^* dS(\mathbf{x}) \tag{5}$$

the symbol $[\cdot]^*$ indicates the complex conjugate. The adjoint operator S^+ of S has the property

$$\langle Sa | p \rangle_V = \langle a | S^+ p \rangle_\Lambda \tag{6}$$

Given the integral operator S defined in (2), it is possible to define a singular system (σ_n, a_n, p_n), constituted by a set of functions $a_n(\mathbf{y})$ and $p_n(\mathbf{x})$, referred to as singular functions, and by a set of real and positive singular values σ_n, satisfying the following relations:

$$\begin{aligned} (S^+ S a_n)(\mathbf{y}) &= \sigma_n^2 a_n(\mathbf{y}) \\ (S a_n)(\mathbf{y}) &= \sigma_n p_n(\mathbf{x}) \\ (S^+ p_n)(\mathbf{x}) &= \sigma_n a_n(\mathbf{y}) \end{aligned} \tag{7}$$

As discussed in [11] and [13], the action of the compact operator S on the function $a(\mathbf{y})$ can be expressed as

$$(Sa)(\mathbf{x}) = \sum_{n=1}^{\infty} \sigma_n p_n(\mathbf{x}) \langle a_n | a \rangle_\Lambda \tag{8}$$

Given the target sound field $p(\mathbf{x})$ on ∂V, it is possible to compute the density S as

$$a(\mathbf{y}) = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} a_n(\mathbf{y}) \langle p_n | p \rangle_V \tag{9}$$

The ill-posedness of the inverse problem is demonstrated by the fact that the singular values accumulate in general at zero, and so the series above diverges. Regularised solutions can be computed, for example, by truncating the order of the series (9) to a given order N, or by applying a Tikonhov regularization, which results in

$$a_R(\mathbf{y}) = \sum_{n=1}^{\infty} \frac{\sigma_n}{\sigma_n^2 + \beta} a_n(\mathbf{y}) \langle p_n | p \rangle_V \tag{10}$$

where β is the regularization parameter.

3 TWO CONCENTRIC SPHERES

The singular system (σ_n, a_n, p_n) for the integral operator S defined by (3) depends on the geometry of the domains V and Λ. For arbitrary geometries, an explicit calculation of the singular functions and of the singular values is in general very difficult, if not impossible. In what follows the analytical expression of the singular system (σ_n, a_n, p_n) is derived, for the case when V and Λ are two concentric spheres of radius R_V and R_Λ, respectively.

As a first step, it is useful to introduce the following expression for the free field Green function [13], [14]:

$$\frac{e^{jk\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|} = \sum_{n=0}^{\infty} ikj_n(kx)h_n(ky) \sum_{m=-n}^{n} Y_n^m(\hat{\mathbf{x}})Y_n^m(\hat{\mathbf{y}})^* \tag{11}$$

where the assumption has been made that $y > x$. The functions $j_n(\cdot)$ and $h_n(\cdot)$ are the *n*–th order spherical Bessel function and *n*-th order spherical Hankel function of the first kind, respectively. Given that the Cartesian components of the unit vector $\hat{\mathbf{x}}$ are $[\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta)]$ the spherical harmonic $Y_n^m(\hat{\mathbf{x}})$ is defined as

$$Y_n^m(\hat{\mathbf{x}}) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_n^m(\cos\theta) e^{im\phi} \tag{12}$$

where $P_n^m(\cdot)$ are associated Legendre functions. The following orthogonality relations follow from the orthogonality of the spherical harmonics [14]

$$\begin{aligned} &\int_{\partial V} Y_n^m(\mathbf{x}/R_V) Y_n^m(\mathbf{x}/R_V)^* \, dS(\mathbf{x}) = \delta_{nn'}\delta_{mm'} {R_V}^2 \\ &\int_{\partial \Lambda} Y_n^m(\mathbf{y}/R_\Lambda) Y_n^m(\mathbf{y}/R_\Lambda)^* \, dS(\mathbf{y}) = \delta_{nn'}\delta_{mm'} {R_\Lambda}^2 \end{aligned} \tag{13}$$

It can be simply verified that the adjoint operator S^+ of the single layer potential defined by (3) is

$$(S^+ p)(\mathbf{y}) = \int_{\partial V} G(\mathbf{x}, \mathbf{y})^* \, p(\mathbf{x}) dS(\mathbf{x}) \tag{14}$$

It is possible to substitute equation (11) into equations (3) and (14) and to rearrange the order of summation and integration in order to obtain

$$(Sa)(\mathbf{x}) = \sum_{n=0}^{\infty} ikR_V R_\Lambda j_n(kR_V) h_n(kR_\Lambda) \sum_{m=-n}^{n} \frac{Y_n^m(\mathbf{x}/R_V)}{R_V} \int_{\partial\Lambda} \frac{Y_n^m(\mathbf{y}/R_\Lambda)^*}{R_\Lambda} a(\mathbf{y}) dS(\mathbf{y})$$
$$(S^+ p)(\mathbf{y}) = \sum_{n=0}^{\infty} -ikR_V R_\Lambda j_n(kR_V) h_n(kR_\Lambda)^* \sum_{m=-n}^{n} \frac{Y_n^m(\mathbf{y}/R_\Lambda)}{R_\Lambda} \int_{\partial V} \frac{Y_n^m(\mathbf{x}/R_V)^*}{R_V} p(\mathbf{x}) dS(\mathbf{x}) \quad (15)$$

Following a similar procedure and applying the first orthogonality relation in (13) it is possible to show also that

$$(S^+ Sa)(\mathbf{y}) = \sum_{n=0}^{\infty} k^2 R_V{}^2 R_\Lambda{}^2 \left| j_n(kR_V) h_n(kR_\Lambda) \right|^2 \sum_{m=-n}^{n} \frac{Y_n^m(\mathbf{y}/R_\Lambda)}{R_\Lambda} \int_{\partial\Lambda} \frac{Y_n^m(\tilde{\mathbf{y}}/R_\Lambda)^*}{R_\Lambda} a(\tilde{\mathbf{y}}) dS(\tilde{\mathbf{y}}) \quad (16)$$

In view of these results and of the orthogonality relations (13), it is possible to make the following choice of the singular system (σ_n, a_n, p_n)

$$\sigma_n = kR_V R_\Lambda \left| j_{n'}(kR_V) h_{n'}(kR_\Lambda) \right|$$
$$a_n(\mathbf{y}) = \frac{Y_{n'}^{m'}(\mathbf{y}/R_\Lambda)}{R_\Lambda}$$
$$p_n(\mathbf{x}) = \gamma_{n'} \frac{Y_{n'}^{m'}(\mathbf{x}/R_V)}{R_V} \quad (17)$$
$$n' = \left\lceil \sqrt{n} - 1 \right\rceil \qquad m' = n - 1 - n' - n'^2 \qquad \gamma_{n'} = i \frac{j_{n'}(kR_V) h_{n'}(kR_\Lambda)}{\left| j_{n'}(kR_V) h_{n'}(kR_\Lambda) \right|}$$

where $\lceil \cdot \rceil$ denotes rounding up to the next integer. From equations (15) and (16) it follows that this singular system satisfies the relations (7). The two indexes n' and m', replacing the single index n, are due to the degeneracy of the singular values, often arising in eigenvalues problems in which symmetrical geometries such as cylinders or spheres are involved. In fact, a singular value related to a given coefficient n' has a $2n'+1$ multiplicity, and the relative eigenspace has dimension $2n'+1$.

As a consequence of these results and in view of equation (9), given a target sound field $p(\mathbf{x})$ on the sphere ∂V, it is possible to calculate the density $a(\mathbf{y})$ as

$$a(\mathbf{y}) = \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \frac{Y_{n'}^{m'}(\mathbf{y}/R_\Lambda)}{ikR_V{}^2 R_\Lambda{}^2 j_{n'}(kR_V) h_{n'}(kR_\Lambda)} \int_{\partial V} Y_{n'}^{m'}(\mathbf{x}/R_V)^* p(\mathbf{x}) dS(\mathbf{x}) \quad (18)$$

In order to analyse the roll-off of the singular values σ_n, it is helpful to report the high order approximations of the spherical Bessel and Hankel functions [13]

$$j_n(t) = \frac{t^n}{1 \cdot 3 \cdots (2n+1)} \left(1 + O\left(\frac{1}{n}\right) \right) \qquad n \to \infty$$
$$h_n(t) = \frac{1 \cdot 3 \cdots (2n-1)}{it^{n+1}} \left(1 + O\left(\frac{1}{n}\right) \right) \qquad n \to \infty \quad (19)$$

As a consequence of this relation, the asymptotic behaviour of the singular values is

$$\sigma_n = kR_V R_\Lambda \left| j_{n'}(kR_V) h_{n'}(kR_\Lambda) \right| = \frac{R_V}{(2n+1)} \left(\frac{R_V}{R_\Lambda} \right)^n \qquad n \to \infty \tag{20}$$

This relation shows that the decay of the singular values is affected by the linear term $(2n+1)^{-1}$ and by the exponential term involving the ratio between the radii of the two spheres. The logical and maybe not surprising consequence of what has been shown is that the ill-conditioning of the inverse problem considered is directly related to the distance between the two boundaries ∂V and $\partial\Lambda$. In the special case when these two boundaries coincide, the roll-off of the singular values is linear and the problem is said to be mildly ill-conditioned.

4 ANALOGY WITH SCATTERING THEORY

It is of interest to consider now the problem of sound field reconstruction with a single layer potential for the special case when the two subsets V and Λ coincide. In the previous section it has been proved that, in the case when the two volumes are spheres, the problem of the ill-conditioning is minimized when the two volumes coincide. The argument could be extended to other geometries too. In what follows the analogy between the sound field reconstruction problem and the scattering theory is described. The argument presented is analogous to that used in [14] to derive the simple source formulation, and holds for any geometry of Λ, respecting the conditions described above.

Consider the target sound field $p(\mathbf{x})$ introduced above, and consider its analytical extension to a larger domain D which strictly contains Λ. Consider now a sound soft object (an object with pressure release boundaries) with the shape of Λ that is introduced in the sound field. The interaction between this object and the sound field $p(\mathbf{x})$ originates a scattered sound field $p_S(\mathbf{x})$ defined in $R^3 \setminus \Lambda$, that is, in the exterior of Λ and on its boundary. The scattered sound field satisfies the Sommerfeld radiation condition [14]

$$\lim_{x\to\infty} x\left(\frac{\partial p_S(\mathbf{x})}{\partial x} - ikp_S(\mathbf{x}) \right) = 0 \tag{21}$$

The sum of the incident sound field $p(\mathbf{x})$ and the scattered field $p_S(\mathbf{x})$ in the domain $D \setminus \Lambda$ defines the total sound field

$$p_T(\mathbf{x}) = p(\mathbf{x}) + p_S(\mathbf{x}) \qquad \mathbf{x} \in R^3 \setminus \Lambda \tag{22}$$

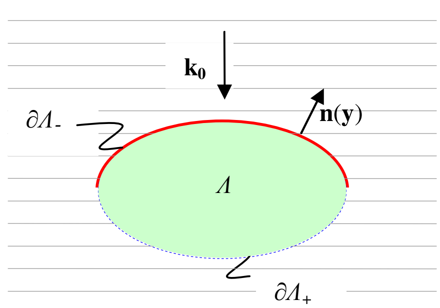

Figure 2: Example of subdivision of the boundary $\partial\Lambda$ of the sound soft object Λ.

The pressures release boundary condition on Λ imposes that

$$p_T(\mathbf{x}) = 0 \qquad \mathbf{x} \in \partial\Lambda \tag{23}$$

With reference to the geometry represented in Figure 2, it is possible to represent the target sound field using the Kirchhoff-Helmholtz integral [13], [14]

$$\alpha p(\mathbf{x}) = \int_{\partial\Lambda} G(\mathbf{x},\mathbf{y})\nabla_{\mathbf{n}} p(\mathbf{y}) - \nabla_{\mathbf{n}} G(\mathbf{x},\mathbf{y}) p(\mathbf{y}) dS(\mathbf{y}) \qquad \mathbf{x} \in R^3$$
$$\alpha = \begin{cases} 1 & \mathbf{x} \in \Lambda \\ 1/2 & \mathbf{x} \in \partial\Lambda \\ 0 & \mathbf{x} \in R^3 \setminus \overline{\Lambda} \end{cases} \tag{24}$$

where the normal derivative $\nabla_{\mathbf{n}}$ is defined as

$$\nabla_{\mathbf{n}} f(\mathbf{y}) = \mathbf{n}(\mathbf{y}) \cdot \lim_{h \to +0} \nabla f(\mathbf{y} - h\mathbf{n}(\mathbf{y})) \tag{25}$$

The vector $\mathbf{n}(\mathbf{y})$ is the unit vector perpendicular to the boundary $\partial\Lambda$ on $\mathbf{y}$ and pointing toward the exterior of Λ. The scattered field $p_S(\mathbf{x})$ can be represented analogously as

$$(1-\alpha) p_S(\mathbf{x}) = -\int_{\partial\Lambda} G(\mathbf{x},\mathbf{y})\nabla_{\mathbf{n}} p_S(\mathbf{y}) - \nabla_{\mathbf{n}} G(\mathbf{x},\mathbf{y}) p_S(\mathbf{y}) dS(\mathbf{y}) \qquad \mathbf{x} \in R^3 \tag{26}$$

with α defined as for equation (24). Subtracting equation (26) from equation (24) one obtains

$$\alpha p(\mathbf{x}) + (\alpha - 1) p_S(\mathbf{x}) = \int_{\partial\Lambda} G(\mathbf{x},\mathbf{y})\nabla_{\mathbf{n}} \big(p(\mathbf{y}) + p_S(\mathbf{y})\big) - \nabla_{\mathbf{n}} G(\mathbf{x},\mathbf{y}) \big(p(\mathbf{y}) + p_S(\mathbf{y})\big) dS(\mathbf{y}) \qquad \mathbf{x} \in R^3 \tag{27}$$

From this formula, from the definition (22) of the total sound field and from the boundary condition (23) follows that

$$p(\mathbf{x}) = \int_{\partial\Lambda} G(\mathbf{x},\mathbf{y})\nabla_{\mathbf{n}} p_T(\mathbf{y}) dS(\mathbf{y}) \qquad \mathbf{x} \in \Lambda \tag{28}$$

Comparing this result with equation (2) implies that the density $a(\mathbf{y})$ for the potential S, which allows the perfect reconstruction of the target sound field in Λ, corresponds to the normal derivative of the total field $\nabla_{\mathbf{n}} p_T(\mathbf{y})$ evaluated on the boundary $\partial\Lambda$. This powerful result defines a direct analogy between the theory of sound field reconstruction and scattering theory.

The boundary $\partial\Lambda$ of a convex scattering object can be divided, as shown in Figure 2, into the *illuminated region* $\partial\Lambda_-$ and the *shadow region* $\partial\Lambda_+$. As an example, if the target sound field is a plane wave $p(\mathbf{x}) = e^{i\mathbf{x}\cdot\mathbf{k}_o}$, then the two regions of $\partial\Lambda$ are $\partial\Lambda_+ : \{\mathbf{y} \in \Lambda : \mathbf{n}(\mathbf{y}) \cdot \mathbf{k_0} \geq 0\}$ and $\partial\Lambda_- = \partial\Lambda \setminus \partial\Lambda_+$. At high frequencies, that is when the wavelength of the sound to be reproduced is much smaller than the physical dimension of Λ, it is possible to approximate the total sound field on the boundary $\partial\Lambda$ by the Kirchhoff approximation, as described in [13]. Firstly, it is possible to consider the effect of acoustic shadowing by the scattering object, thus imposing that the total sound field vanishes in the shadow region $\partial\Lambda_+$. Secondly, in the illuminated region, the scattering object locally may be considered at each point $\mathbf{y}$ as a plane with normal $\mathbf{n}(\mathbf{y})$. This leads to

$$\nabla_{\mathbf{n}} p_T(\mathbf{y}) = a(\mathbf{y}) = \begin{cases} 2\nabla_{\mathbf{n}} p(\mathbf{y}) & \mathbf{y} \in \partial\Lambda_- \\ 0 & \mathbf{y} \in \partial\Lambda_+ \end{cases} \tag{29}$$

Substituting this result into equation (29) leads to

$$p(\mathbf{x}) = \int_{\partial\Lambda_-} G(\mathbf{x},\mathbf{y})\nabla_{\mathbf{n}} p_T(\mathbf{y})dS(\mathbf{y}) + \int_{\partial\Lambda_+} G(\mathbf{x},\mathbf{y})\nabla_{\mathbf{n}} p_T(\mathbf{y})dS(\mathbf{y}) = \int_{\partial\Lambda} G(\mathbf{x},\mathbf{y})2\nabla_{\mathbf{n}} p(\mathbf{y})dS(\mathbf{y}) \qquad (30)$$

$$\mathbf{x} \in \Lambda$$

Again, this approximation is accurate only at high frequencies.

It is interesting to notice that, for the case discussed, the determination of the density $a(\mathbf{y})$ does not involve the solution of an ill-posed inverse problem. In the previous section it has been shown that the severe ill-posedness of the integral equation involved reduces to a mild ill-posedness if the two volumes V and Λ coincide. However, not even this mild ill-posedness appears in equation (28). This is due to the fact that in equation (28) the normal derivative $\nabla_{\mathbf{n}} p_T(\mathbf{y})$ of the total field is assumed to be known on $\partial\Lambda$. The mild ill-posedness arises when the attempt is made to determine the normal derivative of the scattered field, or even the normal derivative of the incident field, from the knowledge of the sound field on $\partial\Lambda$. This can be illustrated from the exemplified case when $\partial\Lambda$ is a sphere of radius R_Λ. Following analysis analogous to that shown in [14], it can be shown that the normal derivative of the incident and of the total field can be computed respectively from

$$\nabla_{\mathbf{n}} p(\mathbf{y}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{kj_n{}'(kR_\Lambda)Y_n^m(\mathbf{y}/R_\Lambda)}{j_n(kR_\Lambda)} \int_{\delta\Lambda} \frac{Y_n^m(\mathbf{y}/R_\Lambda)^*}{R_\Lambda{}^2} p(\mathbf{y})dS(\mathbf{y})$$

$$\nabla_{\mathbf{n}} p_T(\mathbf{y}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{Y_n^m(\mathbf{y}/R_\Lambda)}{ikR_\Lambda{}^2 j_n(kR_\Lambda)h_n(kR_\Lambda)} \int_{\delta\Lambda} \frac{Y_n^m(\mathbf{y}/R_\Lambda)^*}{R_\Lambda{}^2} p(\mathbf{y})dS(\mathbf{y}) \qquad (31)$$

From (19) follows that

$$\frac{kj_n{}'(kR_\Lambda)}{j_n(kR_\Lambda)} = \frac{n}{R_\Lambda} \qquad n \to \infty$$

$$\frac{1}{ikR_\Lambda{}^2 j_n(kR_\Lambda)h_n(kR_\Lambda)} = \frac{(2n+1)}{R_\Lambda} \qquad n \to \infty \qquad (32)$$

These results are perfectly consistent with the inverse $1/\sigma_n$ of the asymptotic approximation of the singular values reported in equation (20), in the case when $R_V = R_\Lambda$.

5 CONCLUSIONS AND FURTHER STUDIES

The fundamentals of a theory of sound field reproduction previously presented by the authors in [11], [12] have been briefly recalled and the method has been extended. Firstly, an analytical expression for the singular system of the single layer potential introduced in equation (2) has been derived for the case when the boundary of the control volume and the secondary source layer are two concentric spheres. It has been shown that the singular functions are directly related to spherical harmonics while the singular functions are related to spherical Hankel and Bessel functions. This allows an analytical expression to be written for the density $a(\mathbf{y})$ for the reproduction of a given target sound field which is known on the boundary of the control volume. These important results shed some light on the analogy between this special case of the method proposed and the theory of High Order Ambisonics. It can be shown that, always for the case of spherical geometries, the singular value decomposition of the integral operator can be rigorously interpreted as the encoding and decoding process for an infinite order Ambisonic system (which includes near field correction [5]). This subject will be discussed in a future paper, which the authors are currently preparing.

Secondly, the special case when the boundaries of the control volume and the secondary source layer coincide has been studied. This has lead to a meaningful analogy between the theory of sound field reproduction and scattering theory. It has been shown that for any smooth and convex secondary source layer $\partial\Lambda$ with arbitrary geometry, the potential $a(\mathbf{y})$ allowing the reproduction of a given target sound field equals the normal derivative, calculated on $\partial\Lambda$, of the total sound field which originates when the target sound field impinges on and is scattered by a sound soft object with the shape of Λ. It has also been shown that at high frequencies the potential $a(\mathbf{y})$ equals twice the normal derivative of the target (incident) sound field on the region $\partial\Lambda_-$ illuminated by the incident field, and vanishes on the shadow region $\partial\Lambda_+$. These results shed some light on the relation between this special case of the theory proposed and the theory of Wave Field Synthesis. It can be shown that the result represented by equation (29) is formally analogous to the driving function of the secondary sources derived with a Wave Field Synthesis approach, as shown for example by [3]. Again, this subject will be discussed in more detail in a future paper in preparation by the authors.

It has been shown that the roll-off of the singular values of the integral operator S, and therefore the ill-conditioning of the inverse problem involved, are strictly related to the distance between the boundary of the control volume and the layer of secondary sources. In more detail, for the case of the concentric spheres it has been shown that the roll-off of the singular values is controlled by the exponential of the ratio between the radii of the two spheres. In the special case when the two surfaces involved coincide, the ill-conditioning of the problem is reduced to its minimum, and the roll-off of the singular values is controlled by a linear factor. The latter has been shown to arise from the computation of the normal derivative of the target field from the knowledge of the field on the boundary. The relation of the ill-conditioning of the inverse problem to the distance between the source layer and the control surface could be extended to other geometries and can be physically related to the presence of evanescent waves in the sound field generated by the single layer potential. This could also be a topic of further studies.

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