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Sound transmission of finite single walls using a variational technique

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ABSTRACT

The single wall is the simplest element of concern in building acoustics. Even though air borne sound transmission has been investigated since the early 20'th century, there still remain some open questions even for this simple case. Actually, the sound insulation of single constructions is rather complicated. There are mainly two reasons for this: The first being the effects on the excitation and radiation of the wall when it having a finite size: In the most simplified theory the wall is assumed infinite, or an ad hoc correction from the infinite theory is used. The second reason is the fact that the wave field in the wall is consisting of two types of waves, namely forced waves due to the exiting acoustic field, and resonant bending waves due to reflections in the boundary and consists of free bending waves. The aim of the present paper is to derive simple analytical formulas for the airborne sound insulation of a single homogeneous wall of finite size, using a variational technique. Then both simple approximations and more elaborated approximations can be found with the same approach, which then can be compared.

1. INTRODUCTION

The sound insulation of walls and floors in a building is often of major concern by the people using the building – and for society. The airborne noise sources in a building are typically people talking and the sound from, e.g., speech, stereo equipment and music instrument. Thus, the noise source is in the air field inside or outside the building.

The single wall is the simplest element of concern in building acoustics. Even though air borne sound transmission has been investigated since the early 20'th century, there still remain some open questions even for this simple case. Actually, the sound insulation of single constructions is rather complicated. There are mainly two reasons for this: The first being the effects on the excitation and radiation of the wall when it having a finite size: In the most simplified theory the wall is assumed infinite, see e.g. [1, 2], or an *ad hoc* correction from the infinite theory is used, see e.g. [3]. The second reason is the fact that the wave field in the wall is consisting of two types of waves, namely forced waves due to the exiting acoustic field, and resonant free bending waves due to reflections in the boundary, see e.g. [4]. Thus, both of these questions concern the effects of having walls of finite size. For an infinite wall, only the forced field will be present, which simplifies the problem significantly. Moreover, for an infinite wall the radiation into the receiver room is determined by a term $1/\cos(\theta)$, were θ being the incidence angle. For a finite wall a radiation impedance or a radiation efficiency instead have to be introduced, usually the latter.

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The transmission coefficient τ is defined as the ratio between transmitted and incident power. When the incidence angel approaches gracing incidence, $\theta \to \pi$, there will be problems: 1) The incident power is defined as the power of the incident wave projected on the wall. This power is zero for gracing incidence, even if the sound field anyway is producing motions of the plate that will radiate sound (c.f. with diffraction). 2) The transmitted power is the power radiated from the wall, which for an infinite plate is dependent of the factor $1/\cos(\theta)$ which becomes infinite for gracing incidence. A common 'trick' to avoid this is to placing an *ad hoc* upper limit on the angles of incidence in the incident field in the integration associated with diffuse field formulas of the transmission coefficient, see e.g. [4]. This will produce a closer agreement with experimental results. However, in the case of sound absorption of finite patches, Thomasson [5, 6] have shown that the area dependency (and thereby the so called edge-effect) can be described with a radiation (or field) impedance, depending on the geometrical shape of the absorption patch and the incidence angles. Thomasson derives simple formulas for the area correction of the absorption coefficient using an variational approach. This approach will also be use in the present paper.

For finite walls two transmission mechanisms are present: the forced and the resonant transmission. Forced transmission are due to the exciting acoustic field and consist of projected wave numbers, directly created by the sound pressure of incident sound wave. Resonant transmission consists of resonant free bending waves due to reflections in the boundary. In mathematical terms, these two fields are the particular and the homogeneous solution to the governing differential equation. These two fields are in play at the same time. Traditionally, the resonant vibrations are dealt with using SEA, see again [4]. Above the critical frequency transmission of the reverberant field is usually dominating, whereas below this frequency the forced field is dominating.

As will be evident, an exact calculation of the sound transmission of a finite wall is rather difficult, because of the radiation load for a finite plate is a very complicated function of the size of the wall, boundary conditions and mode shape. The radiation load is therefore often neglected. This is a good assumption in many cases, especially for heavy constructions. However, near the coincidence frequency near grazing incidence it is usually important, see e.g. [7]. As this region is the most problematic, it seems important to estimate the effect of the radiation load in the present paper.

Both these problematic aspects of the sound transmission through single walls are related through the sound field caused by the vibrating wall. The problem considered in the present paper is a plane single thin wall of finite size is located in an infinite baffle, having a semi-infinite acoustic field on each side. This can be described as an differential integral equation. This mathematical problem have been studied before by, e.g., Sewell [8], using a spatial Fourier transform approach for the sound field and a modal sum for the wall panel. An exact formulation is then found, but in order to solve it, approximations are introduced and only the forced case below critical frequency is then studied more closely. Related papers to the present one are also Ljunggren [9, 10] and Villot et al. [11], where the radiation efficiency using spatial windowing technique is used, which is closely related to the radiation efficiency.

The aim of the present paper is to derive simple analytical formulas for the airborne sound insulation of a single homogeneous wall of finite size, using a variational technique. Using the variational technique, both simple approximations and more elaborated approximations can be found with the same approach. Thus, the simple result will be compared with more elaborated results derived using the same approach.

2. THEORY

Small amplitudes and linear theory is assumed. The time dependency is of the form $e^{i\omega t}$, which is suppressed throughout.

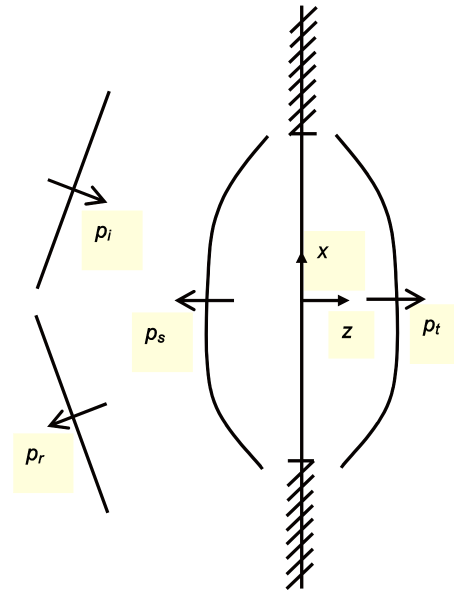

Figure 1: The model considered. A finite wall is located in an infinite rigid baffle in the $x-y$-plane, $z = 0$. The wall is occupying the region $(x, y) \in Sw$. On the left side of the wall is the source side ($z < 0$), the right side is the receiver side ($z > 0$). On the source side the total acoustic field consist of one plane incident wave, one plane geometrically reflected wave and one scattered field due to the motion of the finite wall. On the receiver side only the radiated field will be present, it being the same as the scattered field but with opposite z-direction propagation..

A. Formulation of the problem

Consider figure 1. A plane single thin wall of finite size is located in an infinite baffle, an acoustically hard plane, at $z = 0$. On the source side, the total acoustic field will consist of one plane incident wave p_i, one plane geometrically reflected wave p_r and one scattered field p_s that is due to the motion of the finite wall.

$$p = p_i + p_r + p_s \tag{1}$$

In the receiver side, only the transmitted wave is present, $p = p_t$. With the coordinate system defined as the wall being in the $x-y$-plane and z pointing into the receiver room, see figure 1, the incident and the reflected plane waves are

$$p_i(x, y, z) = \hat{p}_i \exp(-i(k_x x + k_y y + k_z z)) \tag{2}$$

and

$$p_r(x, y, z) = \hat{p}_i \exp(-i(k_x x + k_y y - k_z z)) \tag{3}$$

where $\hat{p}_i$ is the complex amplitude of the incident wave, and $k_x = k \sin\theta \cos\varphi$, $k_y = k \sin\theta \sin\varphi$. The scattered and transmitted wave will be described by means of the Rayleigh integral, for scattered wave

$$p_s(x, y, z) = i\omega\rho \int_{S_w} v(S') G(S, z|S', 0) dS' \tag{4}$$

and for the transmitted wave

$$p_t(x,y,z) = -i\omega\rho \int_{S_w} v(S')G(S,z|S',0)dS' \ , \tag{5}$$

where x', y' and $S' = (x', y')$ refers to the integration point, v is the vibration velocity of the wall and $G(S,z|S',0)$ is the free field Greens function,

$$G(S,z|S',0) = -\frac{e^{-ikr}}{2\pi r} \tag{6}$$

The integral is a surface integral over the wall area S. The distance r between the observation point and the integration point is

$$r = \sqrt{(x - x')^2 + (y - y')^2 + z^2}. \tag{7}$$

It should be noted that due to the symmetry of the transmitted field is the same as the scattered field, but with opposite phase due to the direction of the vibration velocity of the wall v.

The vibrations of the wall can in general terms be describe with a differential operator

$$\mathcal{Z}v(x,y) = p_i(x,y,0) + p_r(x,y,0) + p_s(x,y,0) – p_t(x,y,0) \tag{8}$$

where the wall impedance operator $\mathcal{Z}$ will be described in more detail below. Using the equations above,

$$\mathcal{Z}v(S) = 2\hat{p}_i e^{-i(k_x x + k_y y)} + 2i\omega\rho \int_{S_w} v(S')G(S,0|S',0)dS' \tag{9}$$

valid for $(x,y) \in S_w$. This is an integral-differential equation with the vibration velocity of the wall v as unknown. Equation (9) will generally have constrains in form of boundary conditions at ∂S_w, the boundary of S_w.

The wall impedance operator $\mathcal{Z}$ is used as a general description of wall. If the wall is a thin plate, the Kirchhoff plate equation is the governing equation for the wave motion in the plate,

$$\mathcal{Z} = \frac{B\prime}{i\omega}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 + i\omega m'' \tag{10}$$

which might be combined with boundary conditions at the plate edges. If applying a vibration field in terms of a travelling wave of the form $\exp(-i(k_x x + k_y y))$, k_x and k_y being wave-numbers, the wall impedance operator $\mathcal{Z}$ is reduced to an algebraic wall impedance, $Z = B'\left(k_x^2 + k_{yx}^2\right)^2/i\omega + i\omega m''$.

Generally there will be two boundary conditions at $(x,y) \in \partial S$. The exeption is the case of an infinite plate excited on just the finite area S_w, which is the case studied by Ljunggren [9]. (There will however be a continuity condition in that case.)

C. Variational formulation of the problem

We will formulate a variational principle to solve the problem approximately. The procedure found in Morse and Ingard [12] will be used. In order to make the physical problem symetric, also consider the adjoint situation of a incident wave traveling in the negative $x-y$-direction. Multiply equation (9) with of the vibration velocity v_a of the adjoint problem and integrate over the wall area S_w,

$$\int_{S_w} v_a(S)\mathcal{Z}v(S)\,dS = 2\hat{p}_i \int_{S_w} v_a(S)e^{-i(k_x x+k_y y)}\,dS + 2i\omega \int_{S_w}\int_{S_w} v_a(S)v(S')\,G(S,0|S',0)dS'dS. \quad (11)$$

Tactically, define a functional as

$$V(v) = 2\hat{p}_i \int_{S_w} v(S)e^{i(k_x x+k_y y)}\,dS \quad (12)$$

which thus is related to the first term of the right hand side of (11). Now, move the left side term of (11) to the right hand side and then add equation (11) and (12)

$$V(v,v_a) = 2\hat{p}_i \int_{S_w} v(S)e^{i(k_x x+k_y y)}\,dS + 2\hat{p}_i \int_{S_w} v_a(S)e^{-i(k_x x+k_y y)}\,dS - \int_{S_w} v_a(S)\mathcal{Z}v(S)\,dS + 2i\omega \int_{S_w}\int_{S_w} v_a(S)v(S')\,G(S,0|S',0)dS'dS. \quad (13)$$

This functional is now symmetric in v and v_a, if $\mathcal{Z}$ is a symmetric operator (meaning that $v_a\mathcal{Z}v = v\mathcal{Z}v_a$). The functional $V(v,v_a)$ will be used to find approximate solutions to the problem by means of minimizing it.

D. Radiated power, radiation efficiency and transmission loss

The power transmitted power is the power radiated from the structure into the receiver side,

$$\Pi_t = \frac{1}{2}\Re\left\{\int_{S_w} p_t v^* dS\right\} \quad (14)$$

With the use of equation (5)

$$\Pi_t = -\Re\left\{\frac{i\omega\rho}{2}\int_{S_w}\int_{S_w} v(S')v^*(S)G(S,z|S',0)dS'\,dS\right\} \quad (15)$$

The incident power is usually defined from the fraction of the power in the incident wave that is projected on the wall

$$\Pi_t = \frac{S}{2}\Re\{p_i v_{in}^*\} = \frac{S\cos\theta}{2\varrho c}|\hat{p}_i|^2 \quad (16)$$

where v_{in} is the normal component of the velocity of the plane incident wave. Equation (2) have been used for the last step.

The transmission coefficient is defined as $\tau = \Pi_t/\Pi_i$,which with the present notations becomes

$$\tau = -\frac{\omega\rho^2 c}{S\cos\theta|\hat{p}_i|^2}\Re\left\{i\int_{S_w}\int_{S_w} v(S')v^*(S)G(S,z|S',0)dS'\,dS\right\} \quad (17)$$

For a diffuse field result, an integration over al angels of propagation have to be preformed.

The radiation efficiency is defined as $\sigma = \Pi_t / \rho c S \langle |v|^2 \rangle$, where $\langle |v|^2 \rangle$ is the spatial average square velocity and the transmitted power is found in equation (14).

E. Forced variational solution

As a first use of the variational functional V , assume the vibration velocity of the wall to be of the same form as the incident wave

$$v(S) = \hat{v} e^{-i(k_x x + k_y y)} \tag{18}$$

and for the adjoint field

$$v_a(S) = \hat{v} e^{i(k_x x + k_y y)} \tag{19}$$

Then $\mathcal{Z}v = Z\hat{v}e^{-i(k_x x + k_y y)}$ (the same apply for v_a) where Z is the algebraic expression for the wall impedance. Applying these expressions in equation (13) then yields

$$V(\hat{v}) = 4\hat{p}_i\hat{v}S - Z\hat{v}^2 S + 2i\omega\rho\hat{v}^2 \int_{S_w}\int_{S_w} e^{i\left(k_x(x-x') + k_y(y-y')\right)} G(S,0|S',0)dSdS' \tag{20}$$

Differentiate equation (20) with respect to $\hat{v}$ and equal to zero to find the optimum, and then finally

$$\hat{v} = \frac{2\hat{p}_i}{Z + 2\rho c\, z_f} \tag{21}$$

Where the integral in (20) have be identified as the normalised radiation impedance, as defined by Thomasson [3] (the negative sign being due to the opposite time factor $e^{-i\omega t}$ being used by Thomasson),

$$z_f = -\frac{ik}{S}\int_{S_w}\int_{S_w} e^{i\left(k_x(x-x') + k_y(y-y')\right)} G(S,0|S',0)dSdS' \tag{22}$$

This result can be compared with the result for an infinite wall, as found in
Cremer and Heckl [2]

$$\hat{v} = \frac{2\hat{p}_i}{Z + 2\rho c / \cos\theta}$$

which is equal to (21) for an infinite wall as $z_f \to 1/\cos\theta$ as $S \to \infty$.

F. Forced transmission

Applying the solution (21) in equation (17) yields

$$\tau = \frac{4\rho^2 c \Re\{z_f\}}{\cos\theta \left|Z + 2\rho c\, z_f\right|^2} \tag{23}$$

where again the radiation impedance (32) have been identified. This result can be compared with the result for an infinite wall, as found in Cremer and Heckl [2] (slightly rewritten)

$$\tau = \frac{4\rho^2 c}{\cos^2\theta |Z + 2\rho c/\cos\theta|^2} \quad (24)$$

which is equal to (23) for an infinite wall.

The radiation efficiency becomes very easy for this case, making use of equations (26) in (22)

$$\sigma_{for} = \frac{\Re\{z_f\}}{2} \quad (25)$$

and thus

$$\tau = \frac{2\rho^2 c\sigma_{for}}{\cos\theta |Z + 2\rho c\, z_f|^2} \quad (26)$$

E. Forced and resonant variational solution

If assuming there to be no boundary condition at the boundaries, but rather an infinite plate excited on just the finite area S_w, which is the case studied by Ljunggren [9], an extra resonant term can be added

$$v(S) = \hat{v}e^{-i(k_x x + k_y y)} + \hat{w}e^{-i(k_{bx} x + k_{by} y)} \quad (27)$$

The functional in equation (13) now becomes

$$V(\hat{v},\hat{w}) =$$
$$4\hat{p}_i\hat{v}S + 4\hat{p}_i\hat{w}\int_{S_w}\cos\left((k_x - k_{bx})x + (k_y - k_{by})y\right)dS -$$
$$Z\,\hat{v}\int_{S_w}\left(\hat{v} + \hat{w}e^{-i\left((k_x-k_{bx})x+(k_y-k_{by})y\right)}\right)dS +$$
$$2i\omega\int_{S_w}\int_{S_w}\left(\hat{v}e^{i(k_x x+k_y y)} + \hat{w}e^{i(k_{bx}x+k_{by}y)}\right)\left(\hat{v}e^{-i(k_x x\prime+k_y y\prime)} + \hat{w}e^{-i(k_{bx}x\prime+k_{by}y\prime)}\right)G(S|S')dS'dS. \quad (28)$$

Introducing the following integrals

$$I_c = \frac{1}{S}\int_{S_w}\cos\left((k_x - k_{bx})x + (k_y - k_{by})y\right)dS$$
$$I_e = \frac{1}{S}\int_{S_w} e^{-i\left((k_x-k_{bx})x+(k_y-k_{by})y\right)}\,dS$$
$$z_f = -\frac{ik}{S}\int_{S_w}\int_{S_w}\left(e^{-i(k_x x'+k_y y')}e^{i(k_x x+k_y y)}\right)G\left(S,0\middle|S',0\right)dS'dS$$
$$z_{fb} = -\frac{ik}{S}\int_{S_w}\int_{S_w} e^{-i(k_{bx}x'+k_{by}y')}e^{i(k_{bx}x+k_{by}y)}\,G\left(S,0\middle|S',0\right)dS'dS$$
$$z_{fm} = -\frac{ik}{S}\int_{S_w}\int_{S_w}\left(e^{-i(k_{bx}x'+k_{by}y')}e^{i(k_x x+k_y y)} + e^{-i(k_x x'+k_y y')}e^{i(k_{bx}x+k_{by}y)}\right)G\left(S,0\middle|S',0\right)dS'dS, \quad (29)$$

and optimizing the functional V with respect to the constants $\hat{v}$ and $\hat{w}$ simultaneously, the following matrix equation can be set up

$$\begin{bmatrix} Z2 + c4z_f & Z\,I_e + c2z_{fm} \\ Z\,I_e + c2z_{fm} & c4z_{fb}\end{bmatrix}\begin{bmatrix}\hat{v}\\ \hat{w}\end{bmatrix} = \begin{bmatrix}4\hat{p}_i \\ 4\hat{p}_i I_c\end{bmatrix}$$

3. DISCUSSION AND CONCLUDING REMARKS

Unfortunate no numerical results are yet ready. However, one can clearly see from equations (21), (23) and (26) that the present approach can provide a theoretically solid base for the simplified formulas used in building acoustics. Moreover, the same approach can also be used finding more elaborated results, as indicated in section 2.E. It is also possible introducing boundary conditions and more terms. The present approach also naturally incorporates the radiation impedance z_f (and versions of it) and thereby the size effect in the same way as for the absorption coefficient, c.f. Thomasson [5,6].

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