THE USE OF CONTINUOUS PHASE FOR INTERPOLATION, SMOOTHING AND FORMING MEAN VALUES OF COMPLEX FREQUENCY RESPONSE CURVES

Joerg Panzer R&D-Team, Salgen, Germany
Lampos Ferekidis R&D-Team, Barsinghausen, Germany

1 INTRODUCTION

The problem at hand is best introduced with the help of an example, namely the problem of interpolating two functions as sketched in figure 1. Say, given two functions, f1 and f2, we want to create a third function f3, somewhere between f1 and f2 depending on the parameter p.

$$f_3 ? p?f_2 ? ?1 ? p??f_1 \qquad (1)$$

Assuming a linear interpolation with p = 0 ... 1. If p = 0 then f3 = f1, and if p = 1 then f3 = f2.

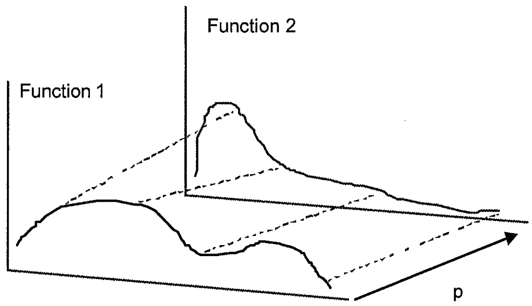

Figure 1: Interpolating two functions

If there is no other constrain, which would render the space between f1 and f2 uncertain, such as aliasing problems, then this interpolation schema yields correctly what we would expect, namely that function f3 morphs continuously from function f1 into function f2.

2 COMPLEX INTERPOLATION

The interpolation formula (1) yields unexpected results if the start and end functions, f1 and f2, are complex valued. A practical example is an acoustic measurement of a loudspeaker on a turntable in order to measure the directivity. In this measurement an impulse response is taken for each angular step, say each 10 degrees.

Figure 2 demonstrates graphically as an example, how directivity data can be displayed. The contour belongs to a software module, which not only displays data in a variety of ways but mainly is also the container of the measured data. The module stores the complex frequency responses in a matrix for each turntable position. In the above example, there are 19 complex spectra covering the range from -90 to +90 degrees, or, each 10 degree.

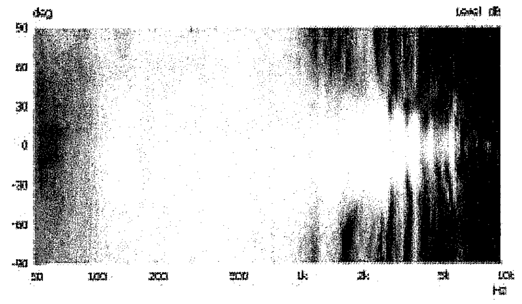

Figure 2: SPL contour of directivity measurement.
x-axis: log-frequency 50 to 10k Hz,
y-axis: linear radiation angles (0 degree = on-axis),
z-axis: log sound pressure level in dB

The aim is now to provide a function, which extracts data from this matrix and displays either a single frequency response at a certain angle, or, a single directivity polar plot at a certain frequency as demonstrated in figure 3. Both parameters, the angle and the frequency, should be arbitrary, i.e. also values in-between the measurement points should be allowed. Further, the total response, i.e. the complex valued data stream, should be mapped and stored in the module of the frequency or the polar plot, respectively. In this way further complex-valued processing can be performed on the mapped data, such as a Fourier transformation for example.

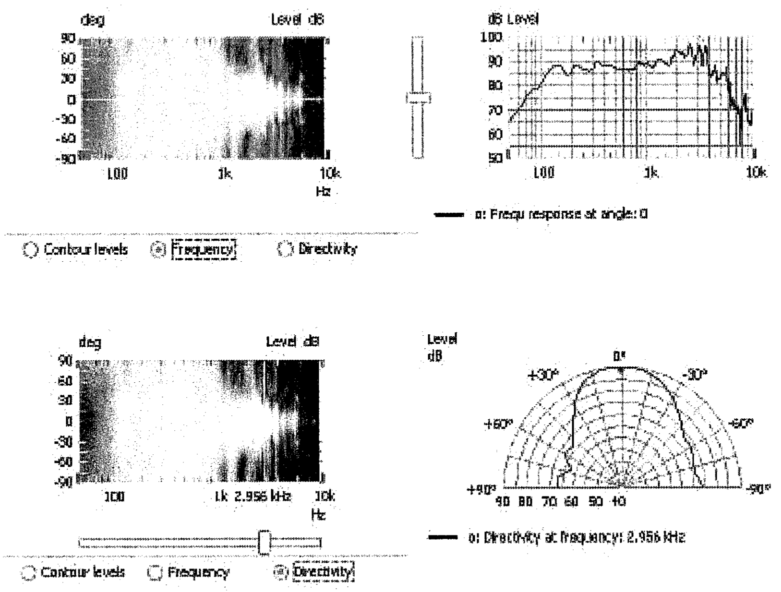

Figure 3: Mapping of a frequency response and a directivity polar plot

The problem is then to obtain a response in-between the angular measurement steps. Following the example in the introduction, function f1 may be the response at zero degree and function f2 the response at the next response curve at 10 degree, as an example. The interpolated function f3 describes the response anywhere between 0 … 10 degree where the parameter, p runs from 0 to 1.
If formula (1) is applied to the impulse response h1(t) at zero degree and h2(t) at 10 degree then the interpolation rule is

$$h_3(t) = p \cdot h_2(t) + (1 - p) \cdot h_1(t) \quad (2a)$$

Or, in spectral form, after a time-frequency Fourier transformation:

$$H_3(j?)\ ?\ p\ ?H_2(j?)\ ?\ ?1\ ?\ p??H_1(j?) \qquad (2b)$$

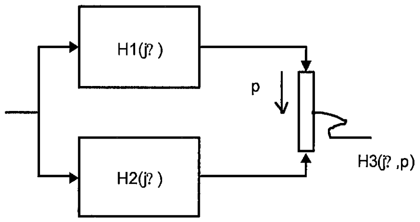

Figure 4: Mixing interpolation between system H1 and system H2

Graphically this mixing algorithm can be sketched as displayed in figure 4. The interpolated response is a mix of response H1 and H2 according the parameter p.

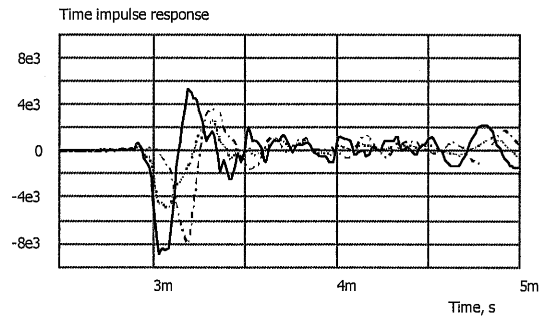

Figure 5: Mixing interpolation between systems h1(t) and h2(t),
Solid: h3 at 0 degree (h3 = h1)
Dotted: h3 at 5 degree (interpolated)
Dashed: h3 at 10 degree (h3 = h2)

The time response version of the data is displayed in figure 5. The solid and dashed curves are the response at 0 and 10 degree as measured. The dotted curve interpolates according to the mixing rule, equations (2a). Because we have selected 5 degree, which is halve way through, the dotted curve shows the mean value of both responses (p = 1/2).

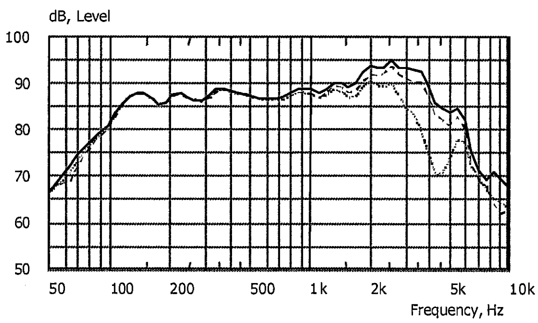

Figure 6: Mixing interpolation between systems H1(j?) and H2(j?),
Level in dB of complex frequency response, 3rd octave smoothed
Solid: H3 at 0 degree (H3 = H1)
Dotted: H3 at 5 degree (interpolated)
Dashed: H3 at 10 degree (H3 = H2)

The frequency response version of the data is displayed in figure 6. The solid and dashed curves are again the response at 0 and 10 degree as measured. The dotted curve interpolates according to the mixing rule, equations (2b). However, in the frequency domain the result, although mathematically correct, does not yield the expected interpolation. We would like to have the interpolated amplitude curve (dotted) to be located somewhere in-between the solid and dashed curves.
The question is, what went wrong, and then, which way to go for a better interpolation strategy, which yields reasonable interpolations, both in frequency and time domain.
Let us first have a closer look to the mixing formula 2b, which can be written in polar form:

$$H_3(j\omega) = p \cdot |H_2(j\omega)| \cdot e^{j\varphi_2(j\omega)} + (1-p) \cdot |H_1(j\omega)| \cdot e^{j\varphi_1(j\omega)} \quad (3)$$

The squared amplitude of H3 is then, after some manipulations:

$$|H_3(j\omega)|^2 = p^2 \cdot |H_2(j\omega)|^2 + (1-p)^2 \cdot |H_1(j\omega)|^2 + 2 \cdot p \cdot (1-p) \cdot |H_2(j\omega)| \cdot |H_1(j\omega)| \cdot \cos(\varphi_2(j\omega) - \varphi_1(j\omega)) \quad (4)$$

Here, the main point to note is that no interpolation parameter, p occurs in the phase difference of the correlation term. That means, first, there is an interference phenomenon and, second, the amount of interference is fixed by the phase difference of H1 and H2. The dip of the interpolated curve in figure 6 (dotted) at high frequencies is hence caused by interference. If we imagine the extreme case, where H1 and H2 are identical but phase inverted, then formula (4) would yield identical zero for H3 for p = 1/2. This would be fine for the interaction of waves but misleading for the purpose of morphing one function into another one.

2.1 The "morphing" approach

An alternative way interpolating complex functions starts in the frequency plane:

$$H_3(j\omega) = (p \cdot |H_2(j\omega)| + (1-p) \cdot |H_1(j\omega)|) \cdot e^{j(p\varphi_2(j\omega) + (1-p)\varphi_1(j\omega))} \quad (5)$$

Equation (5) interpolates the amplitude and phase separately. Immediately it is clear that the squared amplitude yields the expected interpolation behavior:

$$|H_3(j\omega)|^2 = p^2 \cdot |H_2(j\omega)|^2 + (1-p)^2 \cdot |H_1(j\omega)|^2 \quad (6)$$

There is no phase dependence of the amplitude of H3 and, hence, no interference phenomenon. The result can be seen in figure 7. The amplitude (or level in this case) behaves as expected by smoothly morphing from one response to the other. The same is true for all the other components of the complex response, such as the real- and imaginary parts or the phase response.

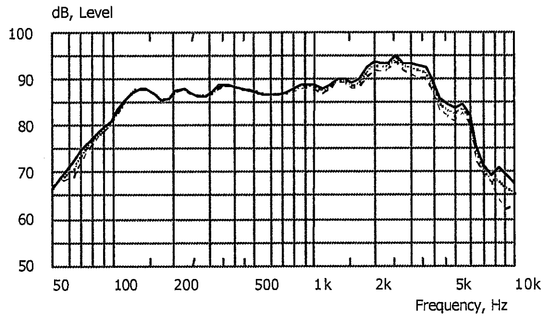

Figure 7: Morphing interpolation between systems H1(j?) and H2(j?),
Level in dB of the complex frequency response, 3rd octave smoothed
Solid: H3 at 0 degree (H3 = H1)
Dotted: H3 at 5 degree (interpolated)
Dashed: H3 at 10 degree (H3 = H2)

Most interesting however is the effect on the time response, h3(t), i.e. the inverse Fourier transform of H3(j?) as demonstrated in figure 8:

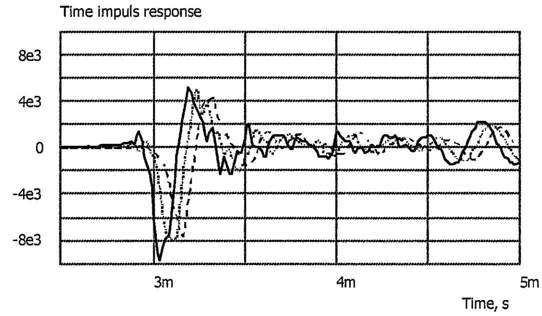

Figure 8: Morphing interpolation between systems h1(t) and h2(t),
Solid: h3 at 0 degree (h3 = h1)
Dotted: h3 at 5 degree (interpolated)
Dashed: h3 at 10 degree (h3 = h2)

The dotted curve in figure 8 includes the time delay in order to morph response h1 into h2. This result is the one we would expect when interpolating response functions.
There is also another explanation one could employ. In polar coordinates a response function can be written

$$H(j?,t)\ ?\ \left|H(j?)\right|?e^{j?\varphi?(j?)\ ?\ ??}$$ (7)

The function exp(j?) is an infinity-to-one mapping because exp(j?) = exp(j? + j2?n). Hence, a continuous phase maps onto a circular repeating function with period 2?. The projections onto the x- and y-axis are the real and imaginary part.

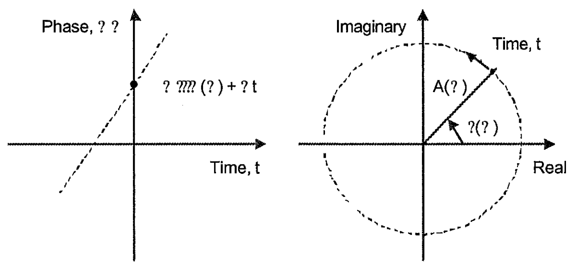

Figure 9: Mapping of exp[j?(? + ? t)]

Because exp(j?) is repeating itself, any interpolation in the exp(j?)-plane would need to count the cycles of revolution. Therefore, the interpolation is easier to perform in the phase plane (left in figure 9), which is continuous.
The latter statement is may be "easier" from the theoretical point of view, but in practice we usually do not have access to a continuous phase function on which to perform the interpolation. In most cases the phase function is calculated from the real and imaginary parts by using the principal value of the arcus-tangens function, which is multi-valued:

$$?(j?)\ ?\ \tan^{?1}\left\{\frac{\mathrm{Im}?H(j?)?}{\mathrm{Re}?H(j?)?}\right\}$$ (8)

However, there is a way to numerically unwrap the phase response into a continuous function, the derivation is given in the Appendix.

3 SMOOTHING

The smoothing procedure levels out ripples of a curve. If the curve is seen as a signal then this is achieved by applying a low-pass filter to the "signals" spectrum, which cuts the high frequency content. Smoothing is frequently used in audio engineering and measurement evaluation, where typically a constant Q window function is applied to the frequency response of sound pressure data.

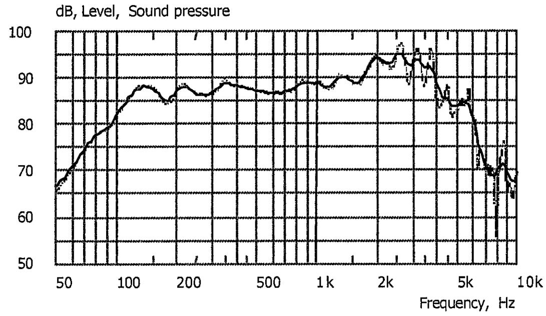

Figure 9: Smoothing example of a sound pressure response curve.
Dotted: Original data
Solid: 1/3 octave log smoothing

Figure 9 displays the effect of smoothing by using a window function of 1/3 of an octave on some sound pressure response data. The smoothed curve typically runs through the original curve displaying the local mean value at a given frequency, hence the term "running mean value". Beyond the scope of this paper is the detailed discussion of the various ways of smoothing and of various forms of weighting functions.

Beside many other advantages of smoothing there is particularly one, which makes the extension to complex-valued functions interesting. A log-smoothed function can easily be sub-sampled, such as on a logarithmic abscissa, yielding a substantial reduction in data size. For example, the original curve in figure 9 (dotted) has 4096 data-points on a linear grid, whereas the smoothed and down-sampled curve (solid) has 200 points on a logarithmic grid, which gives a compression factor of approximately 20.

Traditional power spectrum smoothing and sub-sampling yields scalar results, i.e. all phase information is lost. But often we would like to retain the phase information, either for the sole purpose of qualification, or much more often to be able to use the sub-sampled data as an input for further complex-valued processing. For example, it would be interesting if we could measure, say a woofer and a tweeter, apply the log-smoothing/sub-sampling and then use these data in a network-solver in order to design the cross-over. Because then the data-set, especially of directivity data, is much smaller, the processing is faster, and optimizers can work more efficiently. The question is, how does complex smoothing/sub-sampling work?

Let us first start with an amplitude-smoothing in the frequency domain, which can be written:

$$S(?) \; ? \int |H(j?')| \, ?W(?, ? ? ?') d?' \qquad (9)$$

H(j?) is the frequency response of the original data-set. W(?, ?') is usually a real valued window function of short bandwidth, which can vary with ?, and S(?) is the smoothed and scalar function. The convolution runs in principal from minus to plus infinity, but is usually truncated where W tends to be zero. A power spectrum smoothing is obtained if we sum squared amplitudes instead, and take the square root after the integration.

In order to approach our quest for complex smoothing, we could convolve W with the complex transfer-function, which yields:

$$S(j?) \; ? \int H(j?') \, ?W(?, ? ? ?') d?' \qquad (10)$$

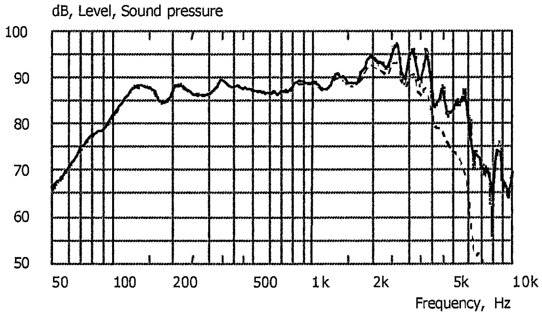

Figure 10: Smoothing example of a sound pressure response curve.
Dotted: Original data
Solid: 1/12 octave log smoothing with formula (9) (amplitude)
Dashed: 1/12 octave log smoothing with formula (10), (complex)

The dashed curve in figure 10 shows the esult of smoothing the complex transfer function. For comparison the amplitude smoothing using formula (9) is added (solid). Here, a smoothing parameter of 1/12 per octave has been chosen for W.
The problem with the smoothing of the complex curve is obvious: the level drops at high frequencies. The reason for this typical behavior [1] becomes clear if we have a look to the real part of this response as shown in figure 11.

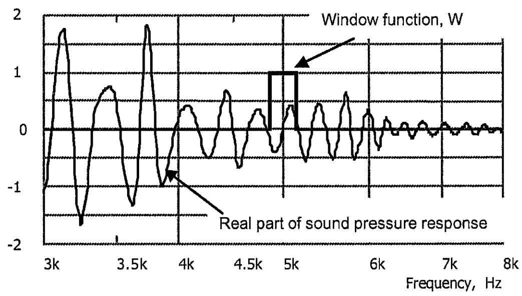

Figure 11: Complex smoothing
Thin line: Real part of sound pressure response,
Thick line: Window function, W at 5kHz

For 1/12 of an octave W has a bandwidth of 288Hz at 5kHz, which covers approximately one phase cycle of the response chosen here as an example. For demonstration, the window function is laid over the real part of the complex response. During the integration according to formula (10) configurations like this will cancel out, thus yielding the loss of energy of the smoothed curve at high frequencies. The amount of canceling depends on the nature of the signal. Here, we chose an example-response with a strong time delay, which causes the real and imaginary parts to cross zero many times.
If the real- and imaginary parts are crossing zero then the phase function (8) may be multi-valued. In this case we should have integrated along a spiral by taking care of the branch cuts. A smoothing which takes care of the multi-valued property of the original data can be written in logarithmic form as:

$$S(?) \; ? \exp \left\{ \int \ln \left| H(j?') \right| ?W(?, ? ? ?)d? ? \; j? \int ?(j?') ?W(?, ? ? ?)d? ? \right\} \qquad (11)$$

The first integral is the convolution of the log-amplitude and similar to formula (9). The second convolves the phase assuming the phase-function ? to be continuous (see Appendix). The continuous phase of the example response, which was already used in figure 11, is displayed

together with W in figure 12. It is clear that now the integration encounters no problems in forming the mean-value for the smoothed phase response at 5kHz.
The steps for complex smoothing based on amplitude and continuous phase are: 1) Calculate the continuous phase. 2) Apply the convolution to the amplitude[1] and phase function independently. 3) Transform back to complex if necessary.

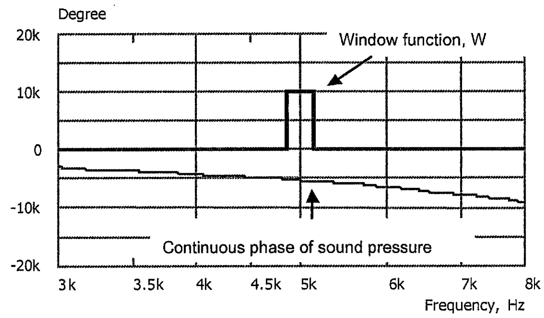

Figure 12 Continuous phase smoothing
Thin line: Phase of sound pressure response in degree,
Thick line: Window function, W at 5kHz

Figure 13 displays the inverse Fourier transform of the result of the continuous phase smoothing, as given by formula (11) together with the transform of the original response data. The time response of the complex smoothing, formula (10), is not shown because it is almost identical to the curve of continuous phase smoothing. The only difference is that it would have a slightly weaker start. This is expected since it is damped at high frequencies.

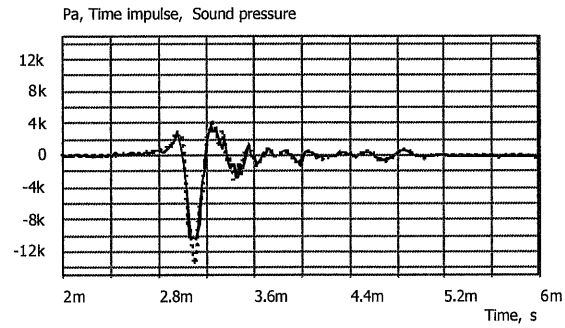

Figure 13 Time response
Solid: Continuous phase smoothing applied
Dotted: Original signal

In the next example the response of a woofer and a tweeter shall be added. For the sake of demonstration we omit the cross-over. The individual measured sound pressure response curves are displayed in figure 14. Both response curves are already smoothed by 1/12 per octave and down sampled to 200 points each (from 4096 and 2048 points).

[1] It works equally well by smoothing the amplitude directly without taking the logarithm first and exponential function later.

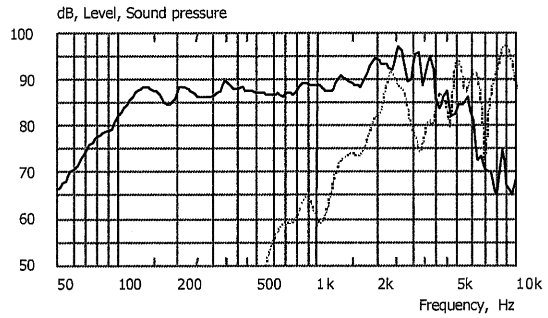

Figure 14 Smoothed with 1/12 per octave by amplitude and continuous phase method.
Solid: Woofer.
Dotted: Tweeter

Figure 15 displays now the sum of the woofer and tweeter-curves. The solid curve is the sum of the smoothed curves from figure 14. The dotted curve is the sum of the original data, which are smoothed after the summation for comparison. The match is quiet good, noting, that in order to produce the solid curve 400 points are involved, whereas the dotted curve is made from approximately 6000 points.

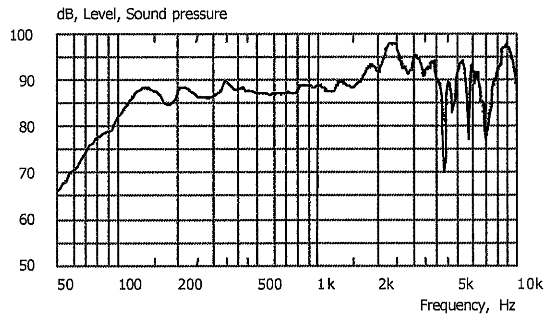

Figure 15 Adding response curves of woofer and tweeter.
Solid: Using smoothed curves from figure 14 (continuous phase method)
Dotted: Using original data and then smoothing the total.

4 FORMING MEAN VALUES

The continuous phase method comes in handy if we have to form the mean value of several measurements, say the mean response of several sound pressure measurements around a certain point in space for example. Here again, we assume that we want to keep the mean response complex valued. The problem is best introduced with the help of an example as given in figure 16. Three similar response functions are added. The difference is in a small time-delay added, which corresponds to +/-5mm spatial offset. The solid curve is the mean value formed by the continuous phase method, whereas the dotted curve dsplays the result of the complex method. The first curve is almost completely identical to the original curves. The dotted curve rolls off at high frequencies, which is typical for the complex method.

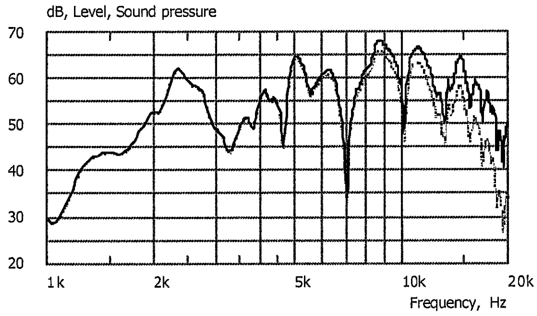

Figure 16: Forming mean value
Solid: Continuous phase method
Dotted: Complex method

The reason for the roll-off can be understood by investigating the phase. Forming the mean-value with the complex method is done by summing all N complex valued response functions $H_i(j\omega)$:

$$H(j\omega) = \frac{1}{N}\sum_i H_i(j\omega) \tag{12}$$

If we have only two such functions then formula (12) yields in polar notation:

$$H(j\omega) = \frac{1}{2}\left(\left|H_1(j\omega)\right| e^{j\varphi_1(j\omega)} + \left|H_2(j\omega)\right| e^{j\varphi_2(j\omega)}\right) \tag{13}$$

The squared amplitude of H is then, after some manipulation:

$$\left|H(j\omega)\right|^2 = \frac{1}{2}\left\{\begin{matrix}\left|H_1(j\omega)\right|^2 + \left|H_2(j\omega)\right|^2 + \\ 2\left|H_1(j\omega)\right|\left|H_2(j\omega)\right|\cos(\varphi_1(j\omega) - \varphi_2(j\omega))\end{matrix}\right\} \tag{14}$$

From the correlation term we can see that the amplitude depends on the phase difference of the two response functions at hand. The behavior is similar to the problem of interpolation in connection with formula (4). Again, the interference effect is welcomed for the interaction of waves, but in the case of forming mean values, we expect the mean amplitude to be independent of the phase.
Better results are obtained by separately summing the amplitude and phase. However, the requirement for this to work is that the phase is available in continuous form (see Appendix).

$$H(j\omega) = \frac{1}{N}\sum_i \left|H_i(j\omega)\right| \cdot e^{j\frac{1}{N}\sum_k \varphi_k(j\omega)} \tag{5}$$

5 CONCLUSION

The extension to complex-valued functions in the procedures of interpolation between, smoothing and forming mean values of response functions may yield wrong results due to interference effects. In the context of these three applications it has been demonstrated that the use of the continuous phase yield the expected results.
In the case of interpolation the continuous phase method morphs from one transfer function into the other one by taking into account the time-delay on the path of interpolation.
The complex smoothing/down-sampling on a log-grid using continuous phase can yield a substantial data reduction and saves the phase information.
Forming mean values of complex response functions succeed only if the phase is continuous.
The authors would like to thank Dr Neil Harris for help and inspiring discussions.

6 APPENDIX

6.1 Continuous phase response

A given complex frequency response H(j?) can be written in polar format

$$H(j?) \; ? \; |H(j?)| ? e^{j?(j?)} \qquad \text{(A.1)}$$

Taking the logarithm of H(j?), which yields

$$\ln H(j?) \; ? \; \ln|H(j?)| \; ? \; j??(j?) \qquad \text{(A.2)}$$

And the derivative with respect to ? gives

$$\frac{?\ln H(j?)}{??} \; ? \; \frac{1}{H(j?)} ? \frac{?H(j?)}{??} \; ? \; \frac{1}{A(j?)} ? \frac{?A(j?)}{??} \; ? \; j?\frac{??(j?)}{??} \qquad \text{(A.3)}$$

We are interested in the very last term, which is the derivative of the phase response with respect to frequency (i.e. the negative of the group delay definition). Hence from formula A.3 follows:

$$\frac{??(j?)}{??} \; ? \; \mathrm{Im} ? \frac{1}{H(j?)} ? \frac{?H(j?)}{??} ? \qquad \text{(A.4)}$$

In order to calculate the continues phase response formula A.4 needs to be integrated:

$$?_c(j?) \; ? \; \mathrm{Im} ? \int_0^{?} \frac{1}{H(j?')} ? \frac{?H(j?')}{???} d?? ? \; ? \; ?_0 \qquad \text{(A.4)}$$

$?_0$ is an integration constant, which can be found by comparing the continuous phase with the principal phase.

7 REFERENCES

[1] P. Hatziantoniou and J. Mourjopoulos: "Generalized fractional-octave smoothing of audio and acoustic responses", J. Audio Eng. Soc., vol. 48, pp. 259-280, (2000 April)