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DIFFUSER DESIGN USING HUFFMAN SEQUENCES

JAS Angus JASA Consultancy, York, England.

1. INTRODUCTION

Studios, concert halls, control and dubbing rooms, reverberation chambers, or any place where high quality acoustics are required, need to have a diffuse reverberant sound field. The means of achieving this has vexed acousticians throughout history. Traditionally, plaster mouldings, niches and other decorative surface irregularities have been used to provide diffusion in an "ad hoc" manner. More recently diffusion structures based on patterns of wells whose depths are formally defined by an appropriate mathematical sequence have been proposed and used [1-8].

Integer based sequences are, unfortunately, limited in their frequency response The purpose of this paper is to present an alternative form of diffusion structure based on non integer based phase reflection gratings. This paper will present a new method of generating such diffusers Based on Huffman sequences. The theory, design, advantages and limitations of these structures will be discussed and simulation results of their performance will be presented. The paper will show that it is possible to develop diffusing structures which have a better frequency performance than conventional phase reflecting structures. Finally we will discuss the implication of the results for the design of studios, performance spaces and reverberation chambers.

2. THEORY

There is a well-known theorem in both antenna theory and Fourier optics [9,10] which states:

$$P(\sin\alpha) = \frac{1}{\lambda}\int_{-\infty}^{\infty} E_x(x,0)\exp(-jkx\sin\alpha)dx \tag{1}$$

Which can be written as:

$$P(\sin\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_x(x,0)\exp(-jkx\sin\alpha)dkx \tag{2}$$

This can be compared with the equation:

$$F(\omega) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \exp(-j\omega t) dt$$

where:

$$\sin\alpha = \omega \qquad (3)$$

$$kx = t$$

Equation (3) is the well-known Fourier transform and so equation (1) can be seen to represent a Fourier transform as well. Equation (1) therefore states that the power as a function of sinα (the polar pattern) is related by the Fourier transform to the complex pressure distribution at the object radiating the sound. This means that if we want a uniform diffusion of sound from a reflecting surface we must have a set of complex pressures whose Fourier transform is uniform. Such a requirement is met by phase reflection gratings.

These gratings alter the phase of the wave front by using the different delays of wells of different depths. They work because they perturb the phase of the wave front such that the Fourier transform of the reflected wavefront is uniform, with the possible exception of dc. To make design of such sequences a little clearer, let us consider the two different types of sequences that are used for diffusers.

i) **Quadratic residue sequences**

Well depth = n^2 mod p where n is the set of integers 0,1,2,··· and p is a prime number. If p = 5 this gives a set of well depths of:

0, 1, 4, 4, 1, 0, 1, ..., etc. (4)

So the sequence repeats with a period of 5.

ii) **Primitive root sequences**

Well depth = a^n mod p where p is a prime and a is a suitable constant called a primitive root. For a = 2 and p = 5 we get the sequence:

1, 2, 4, 3, 1, 2, ..., etc. (5)

In this case we have a sequence that has a period of 4 (5-1).

At the lowest design frequency for these examples a well of depth 5 would correspond to $\lambda/2$. Note that when the frequency gets high enough so that $\lambda/2$ becomes equal to the minimum difference in depths (1) then the surface again becomes equivalent to a flat surface.

The effect of these two sequences on the wavefront can be expressed as a set of complex rotations of the phase. This can be expressed mathematically as:

$$r_n = e^{\frac{j2\pi n^2}{p}} \text{ where } n \text{ and } p \text{ are as defined previously} \qquad (6)$$

for the quadratic residue sequence and:

$$r_n = e^{\frac{j2\pi a^n}{p}} \quad \text{where } n,\ a \text{ and } p \text{ are as defined previously} \tag{7}$$

for the primitive root sequence.

Schroeder [1] shows that the Discrete Fourier transform of a single version (no repeats) of the quadratic residue and primitive root sequences are uniform, except for a reduction in the dc component, that is, the specular direction, in the case of the primitive root sequence. This is ideal behaviour for such surfaces but as their complexity of construction, and size, increases with the modulus most practical sequences are based on small prime numbers such as 7, 11, etc. Therefore when a large area needs to be covered the small sequences are concatenated to form larger areas. Indeed some products are designed to facilitate this.

A problem with the quadratic residue and primitive root sequences is that they are based on integer sequences with a finite number of steps. This has two main effects.

- The sequence has a maximum frequency which is determined by the step size of the wells. Because the well depths are all based on small integers the steps are in simple integer ratios. This means that when the step size is equal to $\lambda/2$ of the incident wavelength the wavefronts reflected from the individual well depths are phase shifted by multiples of λ and so are reflected in phase with each other. This means that at this frequency the diffuser behaves like a flat plate. However above this frequency the wavefronts will no longer add in phase and so the diffuser will begin to scatter again, although there will be some reduction in efficacy due to the finite width of the diffusion wells.
- A more subtle effect is that the diffusion performance is only specified at the discrete frequencies determined by the maximum depth of the diffuser. Between these frequencies the diffusion performance is poorer, although still better than a flat plate of equivalent extent. This is due to the fact that between these discrete frequencies the phase shifts are no longer optimum and, due to the integer relationship of the well depths, this deviation from optimum phase occurs across the whole diffuser thus making the effect worse.

These effects are summarised in figure 1 which shows the performance of a length 5 diffuser as a function of frequency compared with a flat plate of the same size, the lowest design frequency is 500Hz. Figure 1 shows the standard deviation, from optimum, of the diffuser as a function of frequency and it clearly shows the effects discussed above. In particular note the fact that at 2500Hz, and its multiples, the performance of the diffuser is identical to a flat plate. At these frequencies the diffuser acts in an identical fashion to a flat plate and thus could be called the flat-plate frequencies of the diffuser. Also note that there are ripples in the diffusion characteristics between these frequencies with the best diffusion occurring at the discrete frequencies determined by the modulus and the depth, in this case at multiples of 500Hz, in between these frequencies the diffusion performance is not as good.

3. DISCUSSION

Thus we have a problem, integer based small diffusers work well and are practical for both construction and installation but we would ideally like the performance at frequencies away from the discrete design frequencies, to be better. Ideally, we would also like the frequency response of the diffuser to be extend beyond the flat-plate frequencies discussed above. This cannot be achieved using integer based sequences and so some other form of sequence which is not based on integers must be used. A class of sequences which can be used are Huffman sequences [11].

4. HUFFMAN SEQUENCES

The integer sequences above achieve their performance because their autocorrelation function is an impulse, or a close approximation. Their algebraic structure, in particular their cyclic property, allows one to work out recipes for generating sequences with the requisite autocorrelation function. Huffman sequences use the algebraic properties of polynomials in the field of complex numbers to design sequences with the desired autocorrelation function. The complex field is used because it is the only non integer number system that has elements with cyclic properties. There are two main ideas behind Huffman sequences:

- The use of the coefficients of complex polynomials, whose roots satisfy certain constraints, as the sequence values.
- Forming the sequence values required from a selection of coefficients from two related polynomials.

The length of the sequence is determined by the order of the polynomial and the autocorrelation properties are determined by the roots of the polynomial. so in order to design a Huffman sequence we need to work out the positions of the roots of the polynomials and then multiply them out to form the coefficients, and hence the sequence amplitudes. It can be shown [] that the roots of the polynomials must lie on a zero centred circle in the complex plane in order to achieve the requisite autocorrelation function. This is analogous to the cyclic properties of the integer based sequences. However the roots must not only lie on a circle in the complex plane, they must also be in specific locations. In order to work out these locations we must examine the polynomials in more detail.

To generate a Huffman sequence, of length *N+1*, we must generate two related polynomials:

$$P = a_0x^N + a_1x^{N-1} + \cdots + a_{N-1}x + a_N = a_0\prod_{n=1}^{N}(x - r_n) \tag{8}$$

The a_1 represent coefficients which will ultimately form the sequence amplitudes and the r_n the roots of the polynomial *P*. These roots are also related to the roots of a second polynomial *Q*:

$$Q = a_0 + a_1x + \cdots + a_{N-1}x^{N-1} + a_Nx^N = a_0(-1)^N\prod_{n=1}^{N}r_n(x - \frac{1}{r_n}) \tag{9}$$

Where a_1 and r_n have the same meaning. In fact to form Huffman sequences we require the complex conjugate of Q, Q^*, given by:

$$Q^* = a_0^* + a_1^*x + \cdots + a_{N-1}^*x^{N-1} + a_N^*x^N = a_N^*(-1)^N\prod_{n=1}^{N}r_n^*(x - \frac{1}{r_n^*}) \tag{10}$$

Equation (10) is important because its roots are reciprocals of the ones in equation (8) this means that if the roots of *P* lie on a circle then the roots of Q^* will also lie on a circle concentric with that of *P*. They will also lie on the radial lines extending from the origin which pass through the roots of *P*. This means that the two polynomials will lie on either side of the unit circle in the complex plane, as shown in figure 2. If the roots are equally spaced, with respect to angle, around round the circle we can also say that:

$$r_n = \zeta e^{\left(\frac{j2\pi}{N}n\right)} \quad \text{and} \quad \frac{1}{r_n^*} = \frac{1}{\zeta} e^{-\left(\frac{j2\pi}{N}n\right)} = \frac{1}{\zeta e^{\left(\frac{j2\pi}{N}n\right)}} \tag{11}$$

This means we can generate the required two polynomials by generating *N* equally spaced roots on the unit circle via:

$$r_n = e^{\left(\frac{j2\pi}{N}n\right)} \qquad 0 \le n \le N-1 \tag{12}$$

And then multiply by ζ to form *P* and $\frac{1}{\zeta}$ to form Q^*. This results in two possible root values at each of N angles round the circle. In order to form the Huffman sequence it is necessary to select one root from each angular position. That is, the final sequence is formed from a selection of roots from *P* and Q^*. There are 2^N possible selections and, in principle, any selection has similar correlation properties. Figure 3 shows the selection the selection process and figure 4 the final root selection. In order to form the final sequence one simply has to multiply out the roots to form the polynomial coefficients. Figure 5 shows the resulting output sequence for the root pattern shown in figure 4. In principle this sequence could have been complex but by choosing the roots to be symmetrical about the x-axis in the complex plane, complex conjugate roots, real coefficients are assured. The aperiodic autocorrelation function of the sequence is shown in figure 6 and one can see that it displays almost ideal properties, having only 2 small sidelobes. These correlation properties will be maintained if the sequence is scaled up or down in amplitude.

To summarise in order to generate a Huffman sequence:

- Choose a value for ζ, a value of about 1.25 seems to work but other values are possible.
- Choose the length of sequence desired and generate *N* equally spaced roots on the unit circle.
- Multiply by ζ to form *P* and by $\frac{1}{\zeta}$ to form Q^*.
- Randomly choose from P and Q^* to form the final sequence's roots. However if real coefficients are required then make sure the roots chosen are symmetric about the x-axis.
- Multiply out the roots to form the final sequence.

5. DIFFUSER DESIGN USING HUFFMAN SEQUENCES

In order to design a diffuser one must assign the final polynomial coefficients to well depths. The 14 bit sequence we have been using is shown below.

$$\begin{aligned} a_n = [&1.00,\ 0.91,\ -0.11,\ 0.61,\ 1.82,\ 0.27,\ -1.15, \\ &-0.24,\ -0.34,\ -0.21,\ 1.28,\ -0.77,\ 0.75,\ -0.82] \end{aligned} \tag{13}$$

As the sequences have both negative and positive values which range to greater than one we need to normalise them into the range 0 to 1 for the purposes of diffuser design. This can be done by first subtracting the most negative number from the sequence.

$$a_n = [2.15,\ 2.06,\ 1.04,\ 1.76,\ 2.97,\ 1.42,\ 0,\ 0.91,\ 0.81,\ 0.94,\ 2.43,\ 0.38,\ 1.90,\ 0.33] \qquad (14)$$

Then one divides the sequence by the maximum value to give:

$$a_n = [0.72,\ 0.69,\ 0.35,\ 0.59,\ 1.00,\ 0.48,\ 0,\ 0.31,\ 0.27,\ 0.32,\ 0.82,\ 0.13,\ 0.64,\ 0.11] \qquad (15)$$

Using this sequence we can calculate the required well depth by multiplying the maximum depth of the sequences, determined by $\frac{\lambda}{2}$ at the lowest design frequency.

6. RESULTS

The simulated response of single and repeated length fourteen Huffman phase reflection gratings was calculated. The width of the wells were kept constant at 4cm and the well depth corresponding to the modulus was 34.4cm, corresponding to a lower design frequency of 500Hz.

6.1 Single Gratings

Figure 7 shows the diffusion performance, as a function of frequency, of a length 14 Huffman diffuser, compared to a flat plate of equivalent length. It shows that the diffuser performs well and does not suffer from either the flat plate frequency or the inter integer variation in diffusion performance. For comparison a length thirteen quadratic residue diffuser is shown in figure 8 and exhibits these faults.

6.2 Periodic Gratings

Figure 9 shows the diffusion performance, as a function of frequency, of a several repeats of a length fourteen Huffman grating, compared to a flat plate of equivalent length. Here the improvement in performance, over the quadratic residue diffuser structure, is again marked with similar characteristics to the single grating case.

6.3 Diffusion Gain

Another way of comparing the performance of these diffusers is to look at the diffusion performance in terms of the improvement over an equivalent length of flat plate. If this is expressed as the ratio of flat plate to diffuser performance in decibels we can define a diffusion 'gain' for these structures.

The diffusion 'gain' for the single grating and the repeated grating is plotted in figure 10. It shows that both single and repeated gratings offer good performance.

7. CONCLUSION

By using Huffman sequences a diffuser can be designed which does not suffer from the flat plate frequency of conventional integer based diffusers. In addition, due to the non-integer relation ship between the diffuser's well depths, an extended frequency range of action and a smoother diffusion response, as a function of frequency, is achieved. These techniques add additional materials to the acoustic designer's armoury, for tackling real acoustic designs that have physical and practical, as well as theoretical, constraints.

8. REFERENCES

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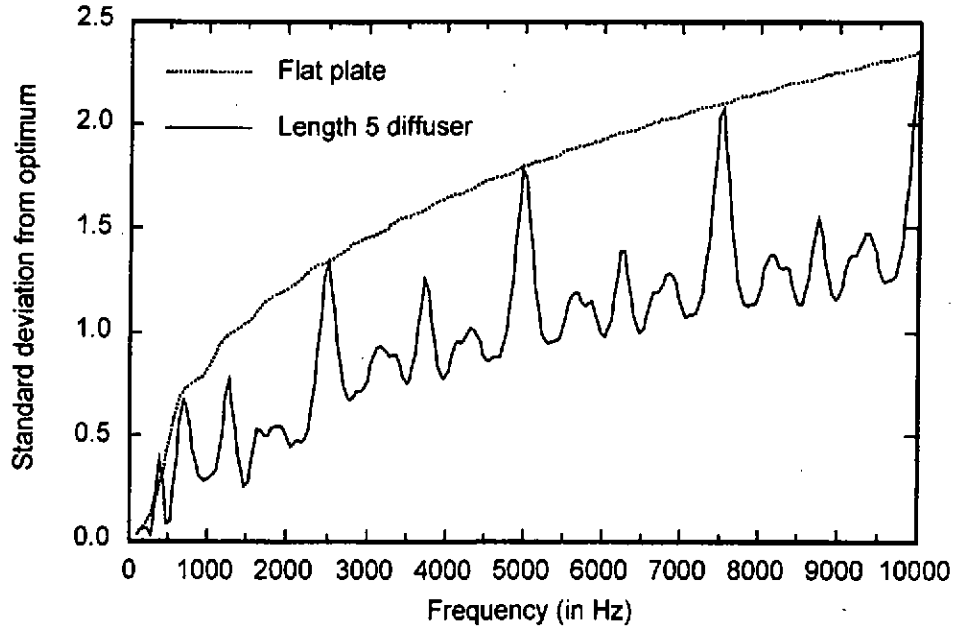

Figure 1 Diffusion performance of a phase reflection grating with frequency.

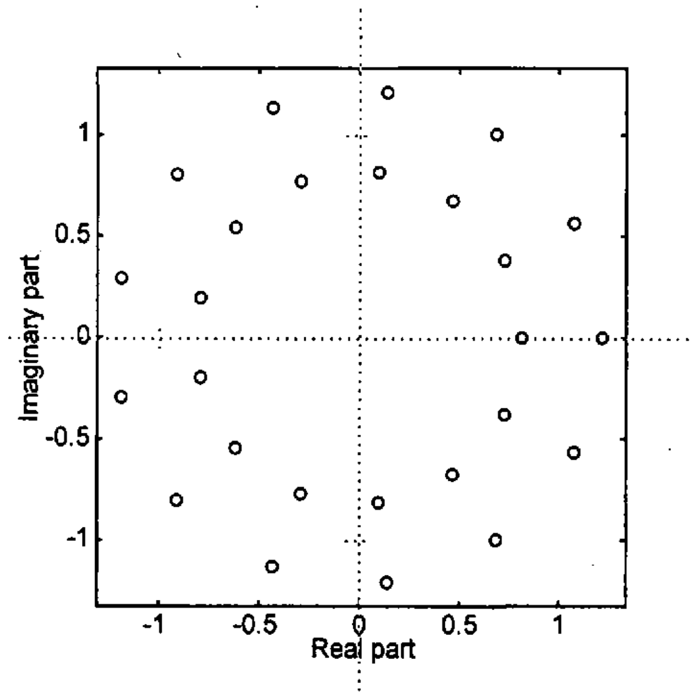

Figure 2 P and Q^* polynomial root locations.

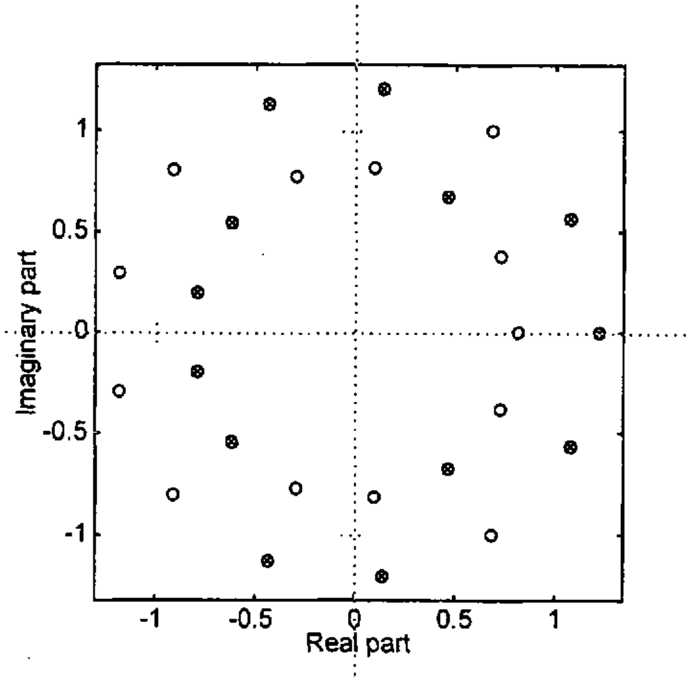

Figure 3 Selecting the root locations for a length 14 Huffman sequence.

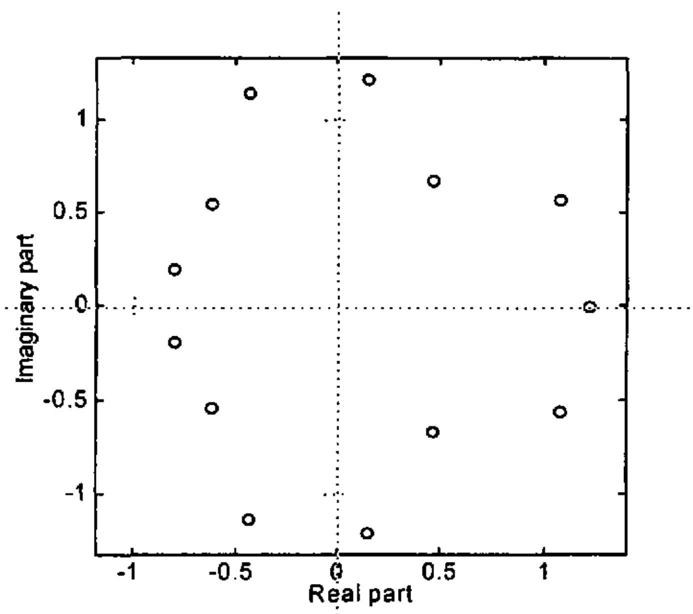

Figure 4 Root locations for a length 14 Huffman sequence.

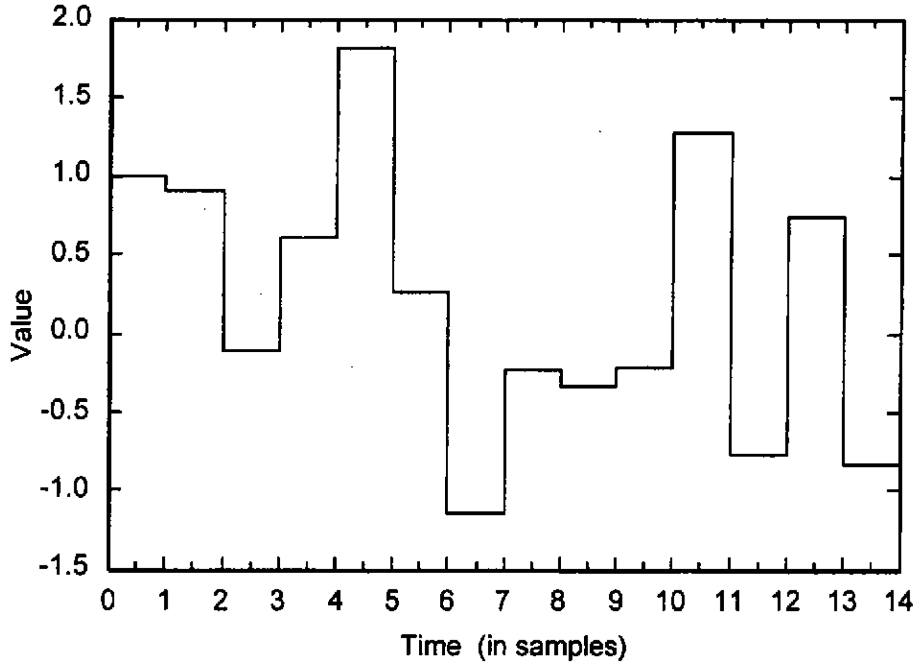

Figure 5 A length 14 Huffman sequence.

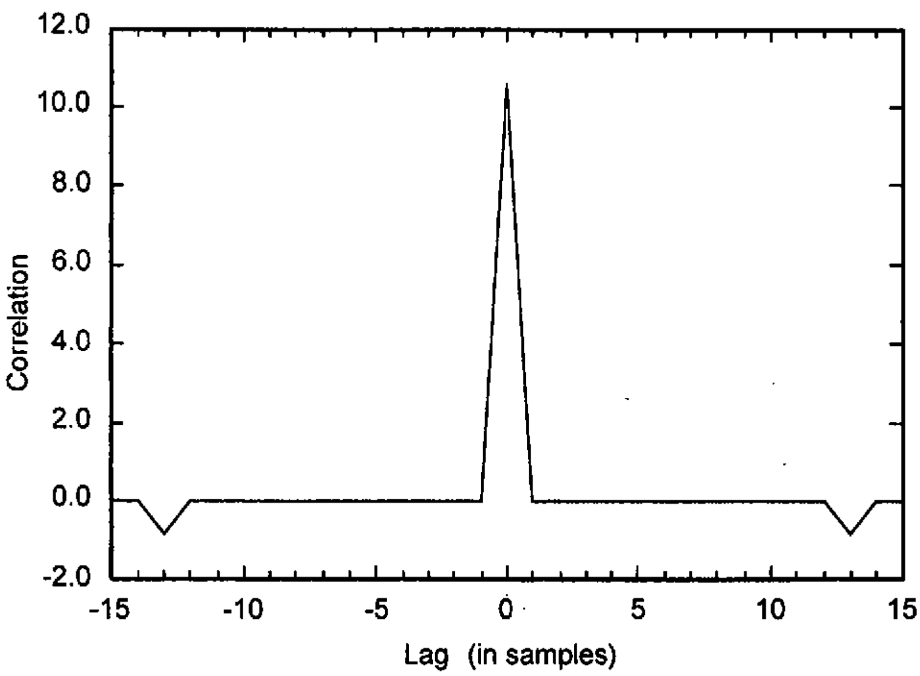

Figure 6 Aperiodic Autocorrelation of a length 14 Huffman sequence.

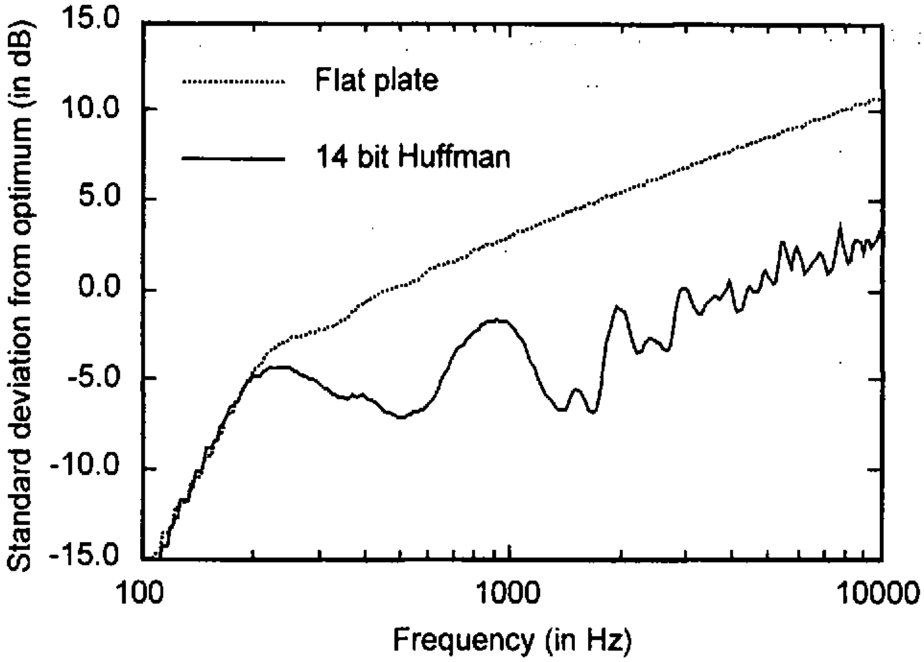

Figure 7 Diffusion performance of a single Huffman diffuser of length 14.

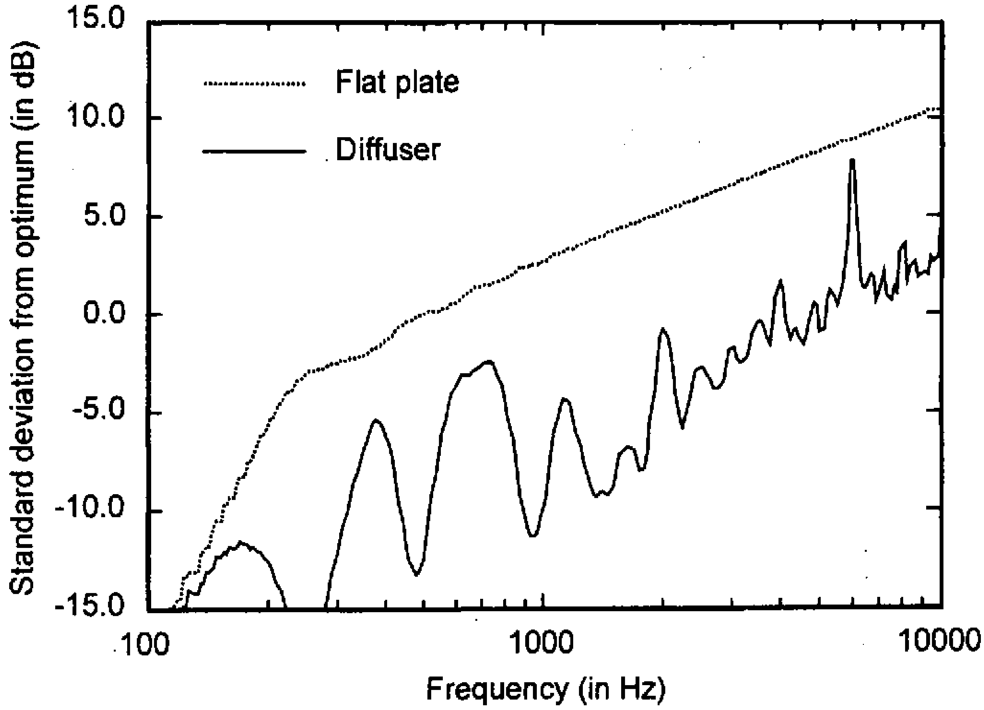

Figure 8 Diffusion performance of a single quadratic diffuser of length 13.

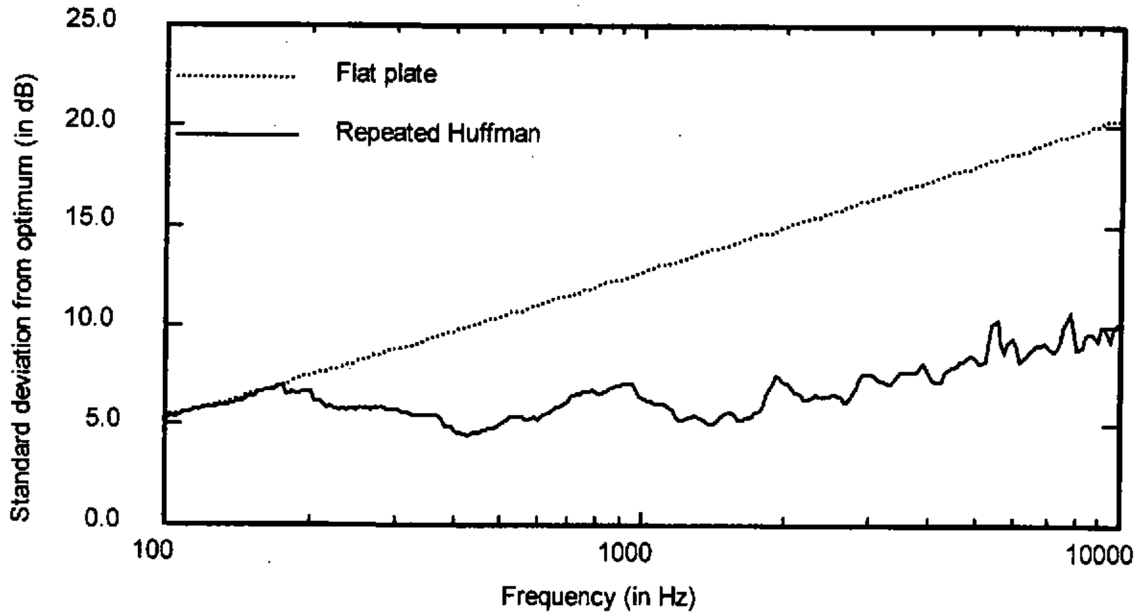

Figure 9 Diffusion performance of a repeated Huffman diffuser of length 14.

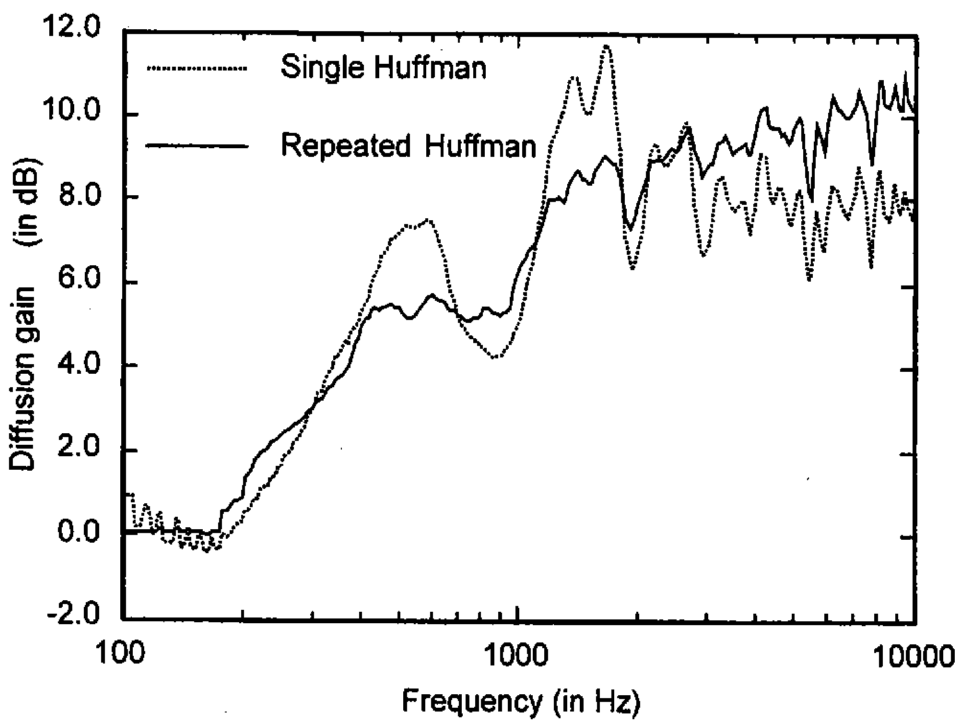

Figure 10 Diffusion 'gain' of a Huffman diffuser of length 14 compared with a repeated version.