# NON-UNIFORMLY SAMPLED LOUDSPEAKER ARRAYS

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## 1. INTRODUCTION

An early example of an array loudspeaker was the column loudspeaker. In this arrangement a number of small loudspeakers were arranged in a closely space line. Because of the extended length of the source in one plane directivity control was achieved in that plane. However, the beam pattern would get progressively more directive with frequency. They could also exhibit unwanted side lobes at higher frequencies, which reduced their utility. Techniques were developed to reduce this behaviour, usually by applying the necessary frequency dependent weighting, or tapering, using simple electrical circuits. Methods of steering these line speakers were also developed either, by using simple analogue delay techniques, or by using the inherent phase shifts in the filters used to taper the array. However, the limitations, and cost, of these methods limited their broad application. Other less expensive methods, such as constant directivity horns, were developed to achieve the need for controlled directivity over a broad frequency range.

With the advent of relatively inexpensive digital signal processing array loudspeakers have become more popular. They offer unprecedented control and have been widely used. However, both the new systems, and the old column loudspeakers, suffer from the problem of being undersampled at some point in their frequency range. That is, above some frequency, the spacing between the drivers is greater that half the wavelength of the sound being produced. This causes *spatial aliasing* and results in loss of control of the beam pattern. To avoid spatial aliasing requires a huge number of small loudspeakers, resulting in a prohibitive cost for the array. Some ad-hoc techniques have been developed, such as logarithmic spacing; have been developed to reduce this problem. However, a better understanding of how to subsample the array in order to achieve a desired level of sidelobe performance would be useful. This paper looks at the problem of achieving controlled directivity from array loudspeakers when the density of drivers is less than the minimum required to avoid spatial aliasing. In particular, it looks at using non-uniform sampling techniques as a means of specifying the positions of the drivers within the array. It first examines the basic theory behind array loudspeakers and then goes on to look at the effects of spatial aliasing. Methods of reducing this, including spatial filtering are then discussed. Finally, various strategies for designing non-uniformly sampled arrays using Gaussian quadrature spacing are described.

# 2. THEORY

In order to understand how the spacing of the drivers might affect a non-uniformly sampled loudspeaker array's performance we must first look at some theory behind conventional array loudspeaker performance.

In order to understand sparse loudspeaker arrays we must first look at the theory behind array loudspeaker design.

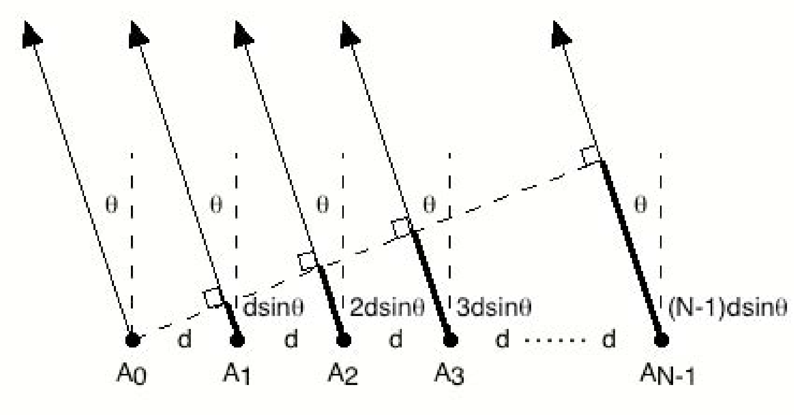


Figure 1 A linear array of N point sources.

## 2.1 Conventional Array Loudspeakers

Consider an, evenly spaced, linear array of perfect point source radiators, as shown in figure 1, with complex amplitudes $A_0 \cdots A_{N-1}$. This corresponds to the reradiated sound from a diffuser when it is illuminated with a plane wave normal to the diffuser surface. If we are an infinite, or at least very large, distance away, we can make the following approximations:

1. The wavefronts are planar, and therefore all the radiators will have the same angle of incidence ($\theta$) to the far off point.
2. The differences in path-lengths are so small that only the initial phase difference, due to, affects the received amplitude.

These approximations are known as the farfield assumptions and, in theory, will be satisfied providing one is a reasonable distance from the array.

Assuming, for the moment, that the far-field assumptions are satisfied we can say the following about our linear array of ideal point sources.

1. The far-field response will be given by the sum of the individual point sources with an additional phase delay/advance due to due to $\theta$, which is the angle from the normal, as shown in figure 1.
2. The phase delay due to will be given by:

$$Phase\ delay = nd\sin\theta \tag{1}$$

Where n is proportional to the point source number, as shown in figure 1.

For the example shown in figure 1, this results in an equation for the far-field polar response, at a frequency who's wavenumber is *k*, which is:

$$P(\theta_k) = A_0 e^{-j(0)kd\sin\theta} + A_1 e^{-j(1)kd\sin\theta} + A_2 e^{-j(2)kd\sin\theta} + \cdots + A_{N-1} e^{-j(N-1)kd\sin\theta} \quad (2)$$

Where the wave number *k* is given by:

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} = \frac{2\pi f}{c}$$

This can be rewritten as:

$$P(\theta_k) = \sum_{n=0}^{N-1} A_n e^{-jnkd\sin\theta} \quad (3)$$

If we make $\psi = kd\sin\theta$ sin then Equation 3 can be rewritten as:

$$P(\theta_k) = \sum_{n=0}^{N-1} A_n e^{-jn\psi} \quad (4)$$

Equation 4 is in fact a Discrete Fourier Transform (DFT) in which $\psi = kd\sin\theta$. This means that the far-field polar pattern of an array of point sources is related to the applied signals by a Fourier Transform relationship and therefore all the theorems that apply to the Discrete Fourier Transform apply to an array of point sources. In particular, these are:

1. **Linearity and Superposition:** Weighted addition in the time, or spatial, domain is equivalent to addition in the transformed domain.

2. **The Convolution Theorem:** This theorem states that convolution in the time, or spatial, domain is equivalent to multiplication in the Fourier domain. The converse is also true.

3. **The Wiener-Khinchin Theorem:** The Wiener-Khinchin theorem states that the squared Fourier transform magnitude of a sequence is equal to the Fourier transform of its autocovariance (or autocorrelation function).

4. **The Shift Theorem:** A shift in the spatial, or time, domain leads to a linear (progressive) phase change in the Fourier domain and vice versa.

As we shall see later these have some important consequences.

## 2.2 The Visible Region

Although, in theory, the variable in equation 4 can range from $-\infty$ to $+\infty$, in reality it cannot. In fact, because $\sin\theta$ cannot exceed ±1, there is only a limited range of that makes any physical sense. This region is known as the “visible region” and, because $\Omega = kd\sin\theta$, the visible region corresponds to $-kd \le \Omega \le +kd$. The visible region corresponds to the angles between ±90º of the normal direction.

This is shown in figure 2 for a 10-element array of points, with the elements spaced 4.3cm apart, at 1khz ( *kd*=0.79). If we double the frequency to 2kHz then *kd* doubles ( *kd*=1.58) and the visible region also doubles, as shown in figure 3.

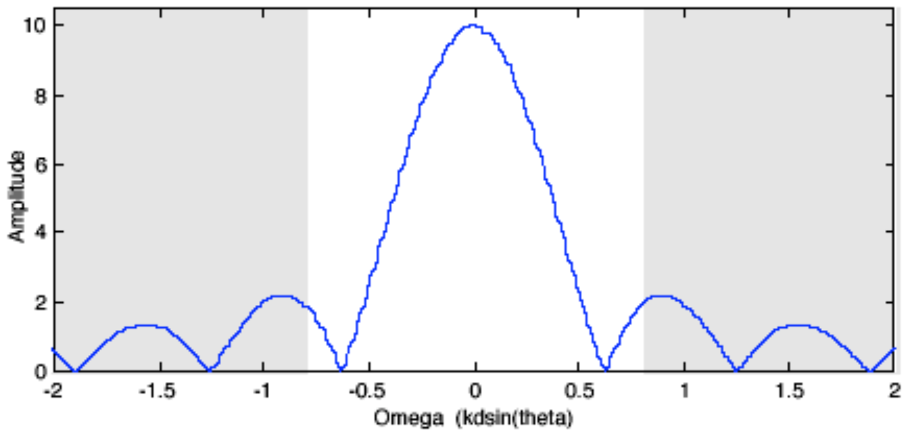


Figure 2 The visible region of an array of points in Ω space ( *kd*=0.79).

As the visible region corresponds to the angles between ±90º of the normal direction the effect of doubling the visible region also implies a narrowing of the main lobe, if its shape does not change as the visible region increases, as in our examples.

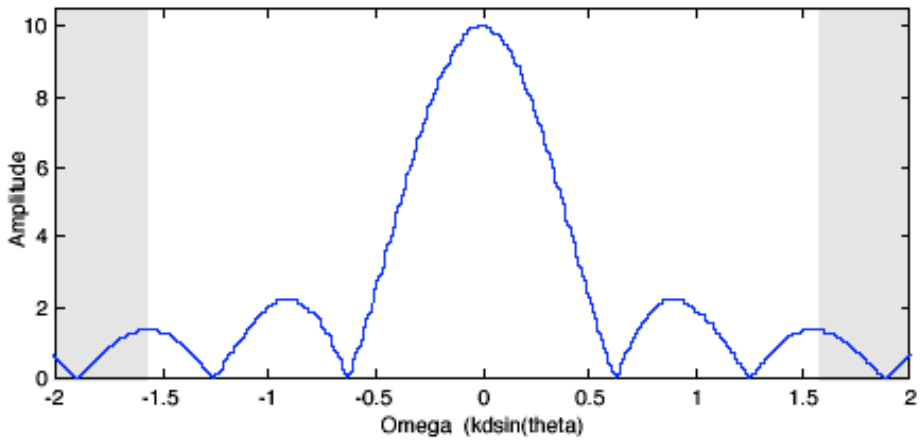


Figure 3 The visible region of an array of points in Ω space ( *kd*=1.58).

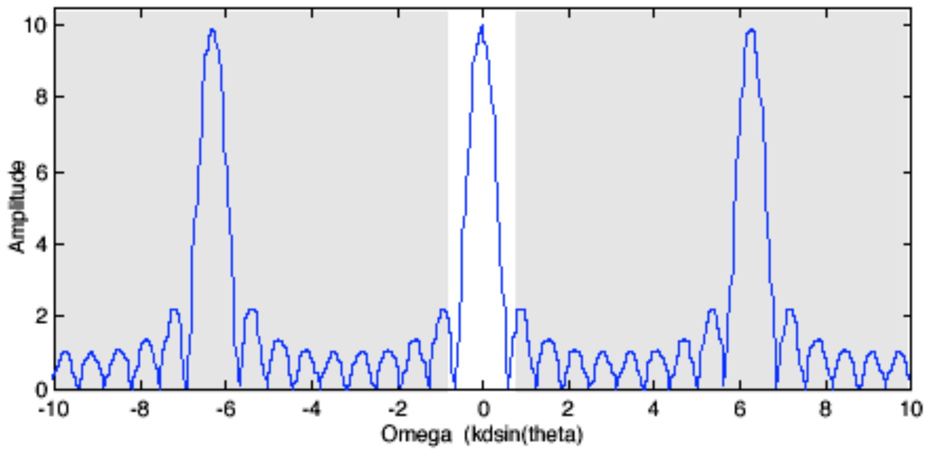


Figure 4 The visible region of an array of points in a larger Ω space (*kd*=0.79).

### 2.3 The Effect of Sampling

When the frequency gets high enough so that the spacing between the point sources becomes greater than half a wavelength the array becomes under-sampled. Under these conditions one gets *spatial aliasing,* which results in multiple main lobes. Figures 4, 5 and 6 illustrate this. Figure 4 shows the 1kHz example with the scale expanded. The first thing to note is that the visible region still covers the same region as that of figure 2. The second thing to note is that the expanded scale reveals the multiple peaks that indicate the possibility of spatial aliasing.

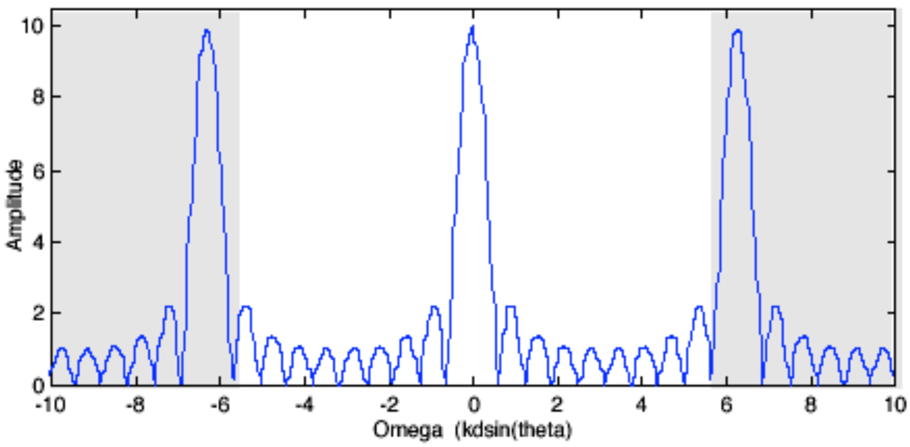


Figure 5 The visible region of an array of points in a larger Ω space (*kd*= 5.5).

Figure 5 shows the visible region when the frequency equals 7kHz ( *kd*=5.5). Here we can see that, although the aliased main lobe is not visible there is an increase in side-lobe levels due to the spatial aliasing. Figure 6 shows the visible region when the frequency equals 10kHz ( *kd*= 7.85). Here we can see that the aliased main lobe is now visible and there is a large increase in the sidelobe levels due to the spatial aliasing.

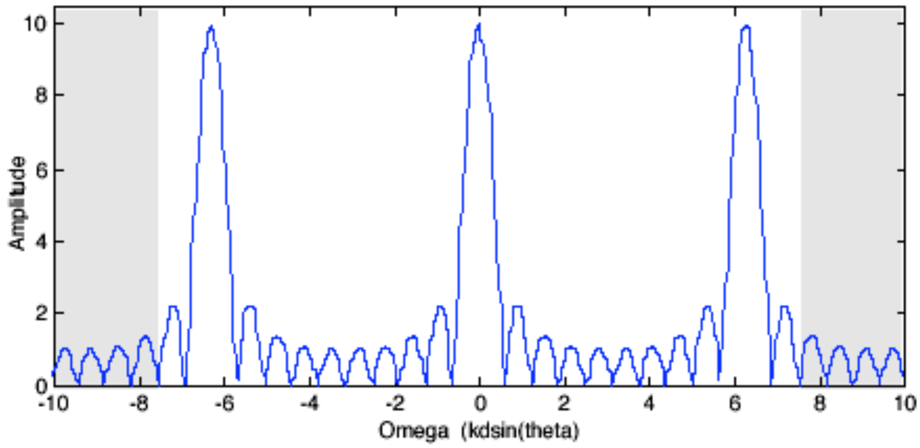


Figure 6 The visible region of an array of points in a larger Ω space ( *kd*= 7.85).

### 2.4 The Effect of a Progressive Phase Shift

From the shift theorem, we know that a shift in the spatial domain leads to a linear (progressive) phase change in the Fourier domain and vice versa. Thus, a progressive phase shift in the spatial domain would result in a linear shift of the function in Ω space. This would result in the main lobe moving to an angle off the central axis. However, the visible region would remain in the same place.

## 3. APPLICATION TO ARRAY LOUDSPEAKERS

An early example of an array loudspeaker was the column loudspeaker. In this arrangement a number of small loudspeakers were arranged in a closely space line. Because of the extended length of the source in one plane directivity control was achieved in that plane. However, the beam pattern would get progressively more directive with frequency, as predicted by the Fourier transform. Techniques were developed to reduce this behaviour, usually by applying the necessary frequency dependent weighting, tapering, or windowing using simple electrical circuits, a direct application of the convolution theorem. Methods of steering these line speakers were also developed either, by using simple analogue delay techniques, or by using the inherent phase shifts in the filters used to taper the array. Again, this is a direct application of the shift theorems of the Fourier transform.

They could also exhibit unwanted side lobes at higher frequencies, due to aliasing, which reduced their utility. That is, above some frequency, the spacing between the drivers is greater that half the wavelength of the sound being produced. This results in spatial aliasing and results in a loss of control of the beam pattern.

To avoid spatial aliasing requires a huge number of small loudspeakers, resulting in a prohibitive cost for the array. For example, ideally we want pattern control over the entire audio frequency range. However, even if we make the speaker spacing 4.3cm, which is unfeasibly small because we would need a large number to achieve low frequency pattern control, we still have significant aliasing at 10kHz.

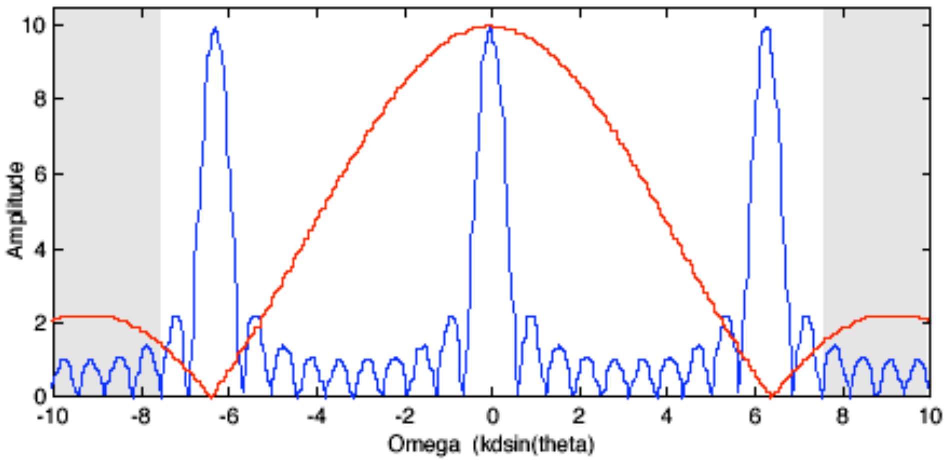


Figure 7 An array speaker and a continuous source equal to the spacing ( *kd*= 7.85).

### 3.1 Acoustic Spatial Filtering

One way of reducing the effect of spatial aliasing is to use directive loudspeakers, instead of point sources, as the array elements. If one uses directive sources then their polar patterns will act as a form of spatial filter. That is the off axis sidelobes will be reduced by the of axis reduction in sound level that a directive source affords. Figure 7 shows an array response at 10kHz ( *kd*= 7.85) with the response of a continuous line source, of length equal to the element spacing, superimposed upon it. Of particular note is that the zeros of the continuous line source fall on the aliased main lobes from the point source array. Because the farfield polar pattern of an array of point sources is related to the applied signals by a Fourier Transform relationship, all the theorems that apply to the Discrete Fourier Transform apply to the array loudspeaker. This means that the theorem that convolution in one domain is equal to multiplication in the other domain applies to this situation. Replacing each of the point sources with a continuous line source is equivalent to convolving it with the point array. Therefore, the effect of replacing the point source with the continuous sources is to multiply their farfield patterns together.

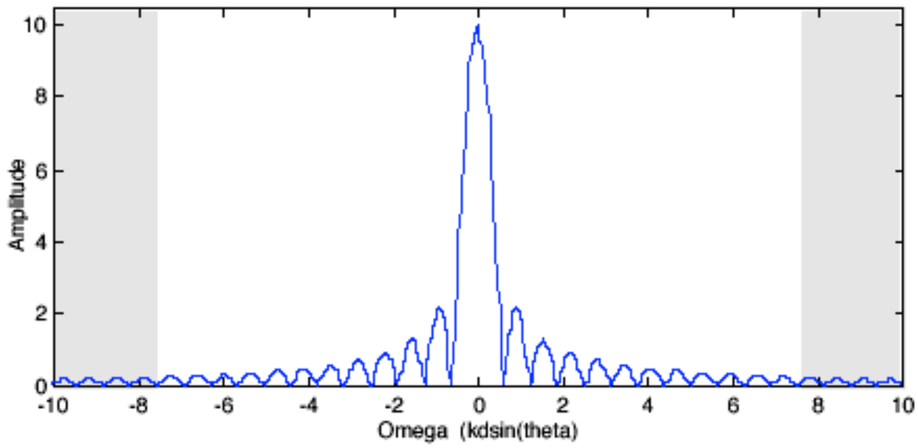


Figure 8 An array speaker made of continuous sources equal to the spacing ( *kd*= 7.85).

This pattern multiplication is well known and the effect is for our example is shown in figure 8. One can see that the aliased main lobes have been eliminated. In fact, the response has become equivalent to a continuous line source of the same extent as the array. Clearly using directional sources, such as constant directivity horns, or the Kef UniQ loudspeaker, can also be used to achieve similar effects. It is this that results in the success of large arrays based on them because, providing they horns have directivity control before spatial aliasing occurs. Once the directivity of the individual elements is considered the need for curved arrays also becomes apparent, as the directivity of the sources must also be taken into account.

## 4. NON UNIFORM SPACING

### 4.1 Logarithmic Spacing

One possible spacing for an array is logarithmic spacing, in which the spacing between the sources exponentially increases away from the centre element. Figure 9 shows the polar response of an array in which the spacing doubles between each source. Two points are of note. Firstly, the centre lobe is very narrow because the extent is equivalent to a length 16 linearly spaced array. Secondly, the main lobe peak is only 5 because there are only 5 active elements in the array, compared to 16 in the linearly spaced array of the same size. In some senses figure 9 is disappointing, because there are still alias lobes and the sidelobe level is not that good. However, the main lobe is very narrow giving good directivity at low frequencies (or low *kd*). In general, such arrays are used where the array is frequency tapered, that is, outer elements are progressively turned off as the frequency increases. In these circumstances, the aliasing is less of a problem, although the sidelobe level can be.

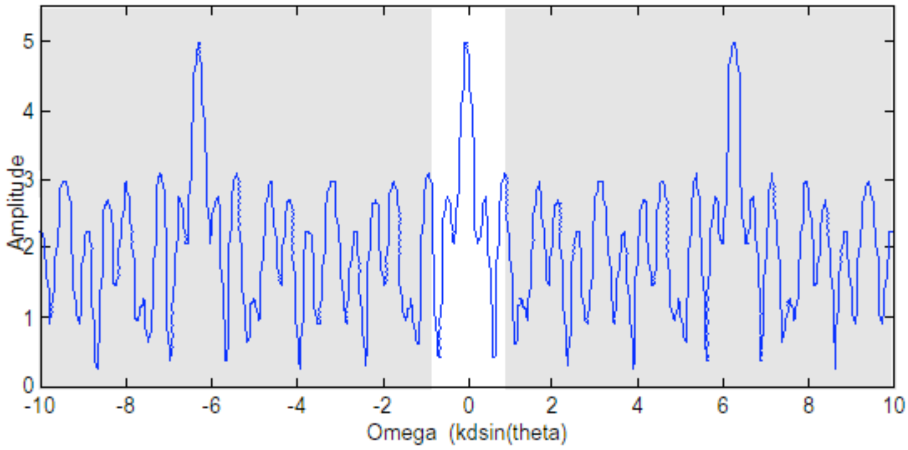


Figure 9 A, 5 element, logarithmically spaced array, spacing doubles between sources (*kd*=0.79).

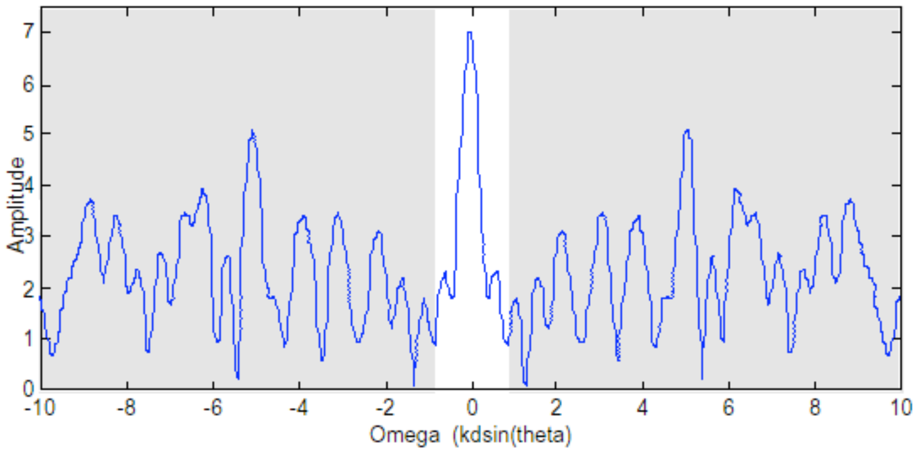


Figure 10 A 7 element logarithmically spaced array, non-integer spacing between sources ( *kd*=0.79).

One of the reasons for the poor performance of the array in figure 9 is the fact that all the spacings are integers. This means that ripples in the polar pattern due to missing elements tend to add constructively, especially at the alias spatial frequencies. Figure 10 shows the effect of using non-integer based spacings (1.0, 1.28, 1.65, 2.12, 2.72, 3.5). The length of the array is equivalent to a 12 element linearly spaced array and contains 7 sources. One can see that the performance has improved a little particularly near the main lobe and that the alias sidelobes have also been reduced a little.

### 4.2 Gaussian Quadrature Spacing

If one looks at the equation (4) for the far-field polar pattern, it resembles a numerical integration, with equally spaced evaluation points. A better form of numerical integration is Gaussian Quadrature, which uses evaluation points that are not equally spaced in order to improve the accuracy of the integration. Perhaps placing the drivers at evaluation points specified by an appropriate Gaussian Quadrature formula would improve the performance of the speaker array. A popular Gaussian Quadrature formula is Gauss Legendre whose weightings and evaluation points are shown in table 1 for a 10-point formula, and plotted in figure 11.

| Position | Weight |
|---|---|
| -0.97391 | 0.066671 |
| -0.86506 | 0.14945 |
| -0.67941 | 0.21909 |
| -0.4334 | 0.26927 |
| -0.14887 | 0.29552 |
| 0.14887 | 0.29552 |
| 0.4334 | 0.26927 |
| 0.67941 | 0.21909 |
| 0.86506 | 0.14945 |
| 0.97391 | 0.066671 |

Table 1 Weights and evaluation points for a 10-point Gauss Legendre formula.

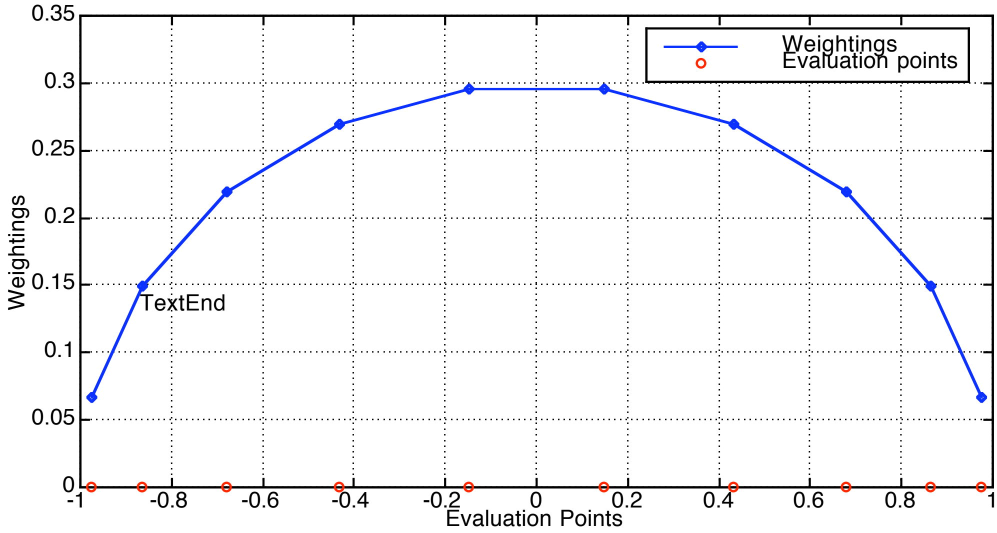


Figure 11 Weights and evaluation points for a 10-point Gauss Legendre formula.

As one can see from figure 11, the points are more closely spaced at the outside edges compared to the middle. This is counterintuitive for array loudspeakers, as one would have thought closer in the centre was better.

A 10-element array was simulated using the Gauss Legendre positions and weightings with the same value of *kd* as the logarithmic arrays in figures 9 and 10. The spacing was adjusted such that the smallest spacing was 4.3cm and the weights were scaled to give the same output as a uniformly sampled 10-element array. The results are presented in figure 12. Comparing figure 12 to figure 10 one can see that the Legendre spacing has a lower *kd* value for the first sidelobe, probably due to the increased spacing of the centre sources. The levels of the sidelobes are about -2.5dB to -3dB, which is comparable to the performance of the nonuniformly sampled log array. However, the close in performance is much better with the worst sidelobe being -14dB compared to the -8dB of the logarithmic array.

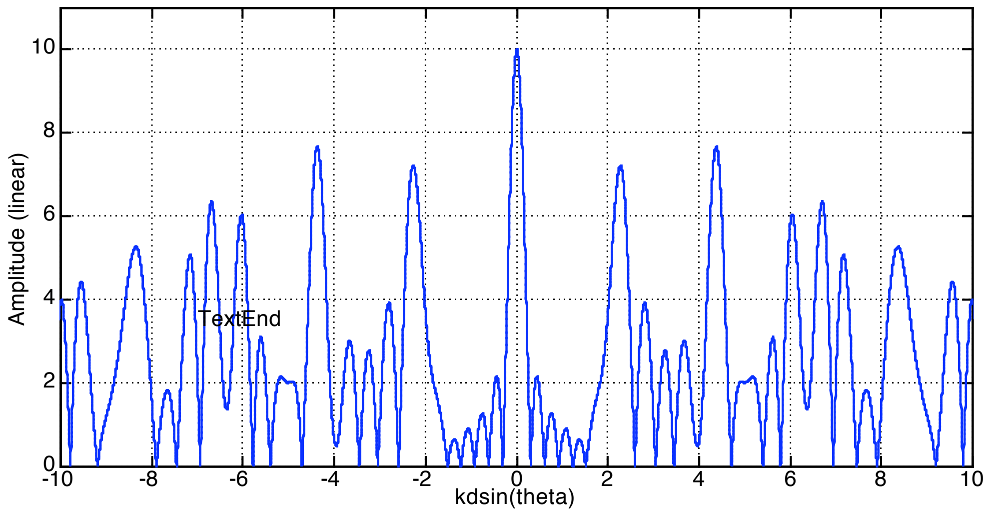


Figure 12 A 10-element with Gauss-Legendre spacing between sources (*kd*=0.79).

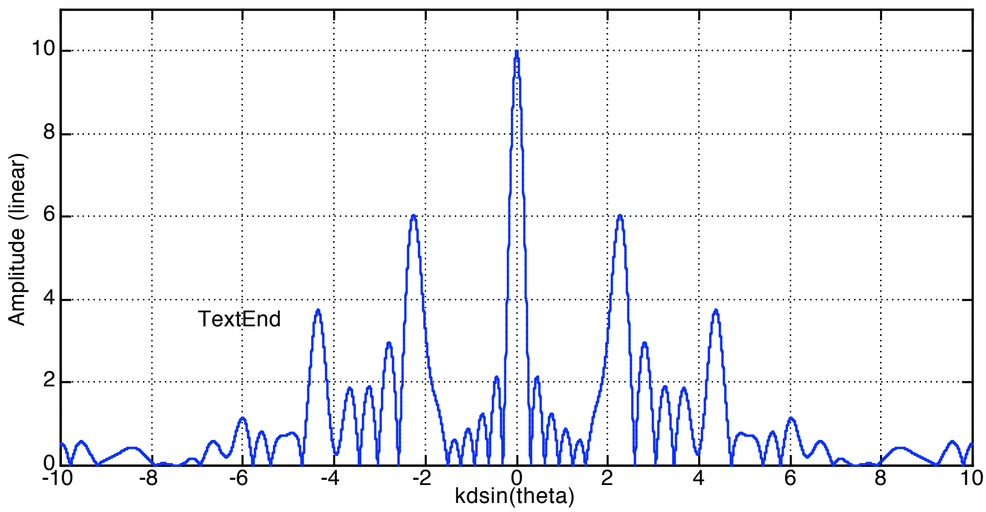


Figure 13 A 10-element with Gauss-Legendre spacing between sources and additional spatial filtering due to finite sized (4cm in diameter) sources (*kd*=0.79).

The results presented in figure 12 are for point sources and so represent the worst case for alias sidelobes. Figure 13 shows the effect of assuming that the array is constructed from 4cm in diameter piston sources and in this case the alias sidelobe levels are reduced to less than -4.5dB.

### 4.3 Discussion on Gaussian Quadrature Spacing

Although the performance of the Gauss-Legendre array is encouraging, it is almost certainly not the correct quadrature formula to apply. Gaussian quadrature formulae assume an underlying structure to the integrand that they are evaluating. In the case of Gauss-Legendre quadrature the integrand approximated is:

$$\int_{-1}^{1} f(x)dx \approx \sum_{n=1}^{N} W_n f(x_n) \qquad (5)$$

This is exact if f(x) is a polynomial of degree 2m-1 or less.

However, we wish to approximate the following, from equation (3):

$$P(\theta) = \int A(x)e^{-jkx\sin\theta}dx \approx \sum_{n=1}^{N} W_n e^{-jkx_n \sin\theta} \qquad (6)$$

Clearly equation 6 is different to equation 5 and thus we could expect better results if we considered other forms of Gaussian Quadrature. However, although Gauss-Laguerre, Gauss-Hermite, or Gauss-Chebyshev may be better they do not really address the problem. More research is required to arrive at a suitable formula.

## 5. CONCLUSIONS

This paper has looked at the problem of achieving controlled directivity from array loudspeakers when the density of drivers is less than the minimum required to avoid spatial aliasing. It first examines the basic theory behind array loudspeakers and then goes on to look at the effects of spatial aliasing. Methods of reducing this, including spatial filtering have been discussed. Various strategies for designing sparse arrays have been described. Although sparse arrays allow larger arrays with fewer sources, a price is paid in sidelobe level. However, using spacing and weightings based on Gaussian Quadrature show potential for improving both the alias, and the close in, sidelobe level.

## 6. REFERENCES

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