SPHERICAL CHICKENS AND SOUND REINFORCEMENT: HOW FOURIER RULES THE ROOST

J A S Angus Acoustics Research Centre, Research Institute for the Built and Human Environment, University of Salford, Salford, Greater Manchester, M5 4WT, England
j.a.s.angus@salford.ac.uk

1. INTRODUCTION

Peter Barnet was always simplifying problems down to the bare essentials, in order to illustrate a particular point. In one of his lectures on sound reinforcement he stated,

> "Assume that we have a loudspeaker which has a trapezoidal coverage area, and so perfectly covers the room."

This paper examines the possibility of creating such a mythical loudspeaker" In particular, it examines the power of Joseph Fourier's theorem to help us envision and create such a beast. The paper first examines the application of Fourier transforms to the problem of acoustic polar patterns. In particular, it shows hoe the same principles are applicable to a variety of electroacoustic radiation problems. It then discusses the outworking of the theorem to constant directivity loudspeakers and optimal diffusers. Areas where the approach breaks down are also discussed. Finally, the possible structures for a trapezoidal coverage area loudspeaker are described.

2. THEORY

In order to understand how the properties of sequences affect diffuser performance we must first look at some theory behind diffuser scattering performance.

2.1 An Array of Point Sources

Consider an, evenly spaced, linear array of perfect point source radiators, as shown in figure 1, with complex amplitudes $A_0 \cdots A_{N-1}$. This corresponds to the reradiated sound from a diffuser when it is illuminated with a plane wave normal to the diffuser surface. If we are an infinite, or at least very large, distance away, we can make the following approximations:

1. The wavefronts are planar, and therefore all the radiators will have the same angle of incidence (θ) to the far off point.
2. The differences in path-lengths are so small that only the initial phase difference, due to, affects the received amplitude.

These approximations are known as the farfield assumptions and, in theory, will be satisfied providing one is a reasonable distance from the diffuser.

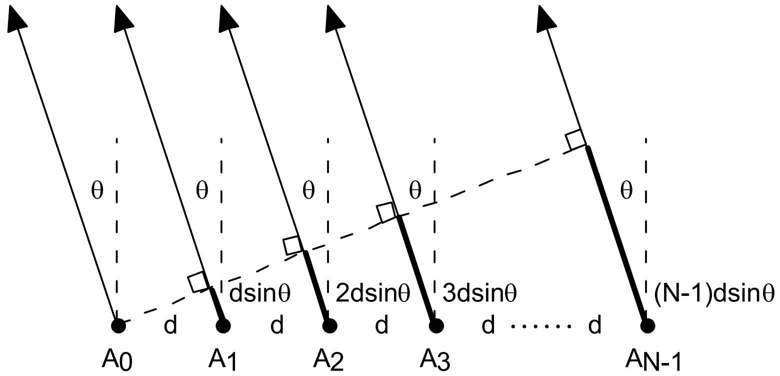

Figure 1 A linear array of N point sources.

Assuming, for the moment, that the far-field assumptions are satisfied we can say the following about our linear array of ideal point sources.

1. The far-field response will be given by the sum of the individual point sources with an additional phase delay/advance due to due to θ, which is the angle from the normal, as shown in figure 1.
2. The phase delay due to will be given by:

$$Phase\ delay = nd\sin\theta \qquad (1)$$

Where n is proportional to the point source number, as shown in figure 1.

For the example shown in figure 1, this results in an equation for the far-field polar response, at a frequency who's wavenumber is *k*, which is:

$$P(\theta_k) = A_0 e^{-j(0)kd\sin\theta} + A_1 e^{-j(1)kd\sin\theta} + A_2 e^{-j(2)kd\sin\theta} + \cdots + A_{N-1} e^{-j(N-1)kd\sin\theta} \qquad (2)$$

Where the wave number *k* is given by:

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} = \frac{2\pi f}{c}$$

This can be rewritten as:

$$P(\theta_k) = \sum_{n=0}^{N-1} A_n e^{-jnkd\sin\theta} \qquad (3)$$

If we make $\Omega = kd\sin\theta$ sin then Equation 3 can be rewritten as:

$$P(\theta_k) = \sum_{n=0}^{N-1} A_n e^{-jn\Omega} \quad (4)$$

Equation 4 is in fact a Discrete Fourier Transform (DFT) in which $\Omega = kd\sin\theta$. This means that the far-field polar pattern of an array of point sources is related to the applied signals by a Fourier Transform relationship and therefore all the theorems that apply to the Discrete Fourier Transform apply to an array of point sources. In particular, these are:

1. **Linearity and Superposition:** Weighted addition in the time, or spatial, domain is equivalent to addition in the transformed domain.

2. **The Convolution Theorem:** This theorem states that convolution in the time, or spatial, domain is equivalent to multiplication in the Fourier domain. The converse is also true.

3. **The Wiener-Khinchin Theorem:** The Wiener-Khinchin theorem states that the squared Fourier transform magnitude of a sequence is equal to the Fourier transform of its autocovariance (or autocorrelation function).

4. **The Shift Theorem:** A shift in the spatial, or time, domain leads to a linear (progressive) phase change in the Fourier domain and vice versa.

As we shall see later these have some important consequences.

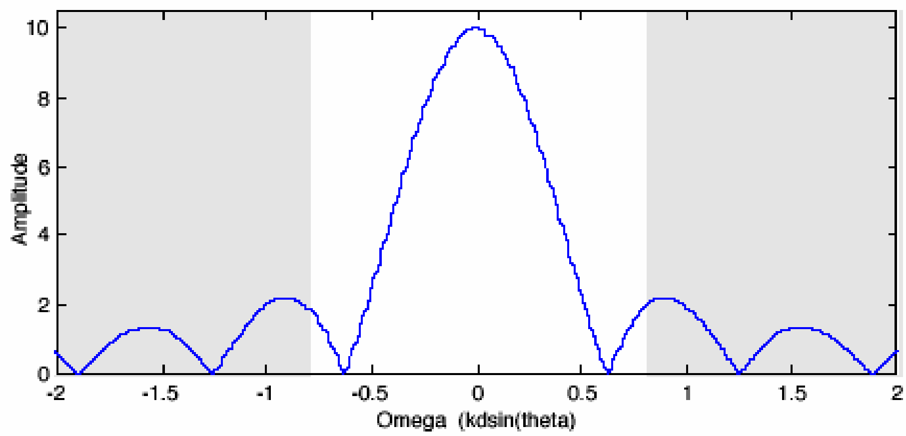

Figure 2 The visible region of an array of points in Ω space (kd=0.79).

2.2 The Visible Region

Although, in theory, the variable in equation 4 can range from $-\infty$ to $+\infty$, in reality it cannot. In fact, because $\sin\theta$ cannot exceed ±1, there is only a limited range of that makes any physical sense. This region is known as the "visible region" and, because $\Omega = kd\sin\theta$, the visible region corresponds to $-kd \le \Omega \le +kd$. The visible region corresponds to the angles between ±90° of the normal direction.

This is shown in figure 2 for a 10-element array of points, with the elements spaced 4.3cm apart, at 1khz (kd=0.79). If we double the frequency to 2kHz then kd doubles (kd=1.58) and the visible region also doubles, as shown in figure 3.

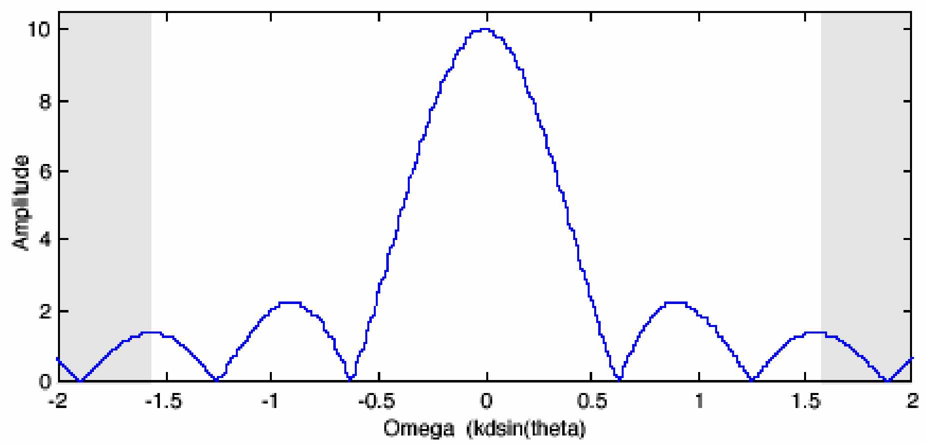

Figure 3 The visible region of an array of points in Ω space (kd=1.58).

As the visible region corresponds to the angles between ±90° of the normal direction the effect of doubling the visible region also implies a narrowing of the main lobe, if its shape does not change as the visible region increases, as in our examples.

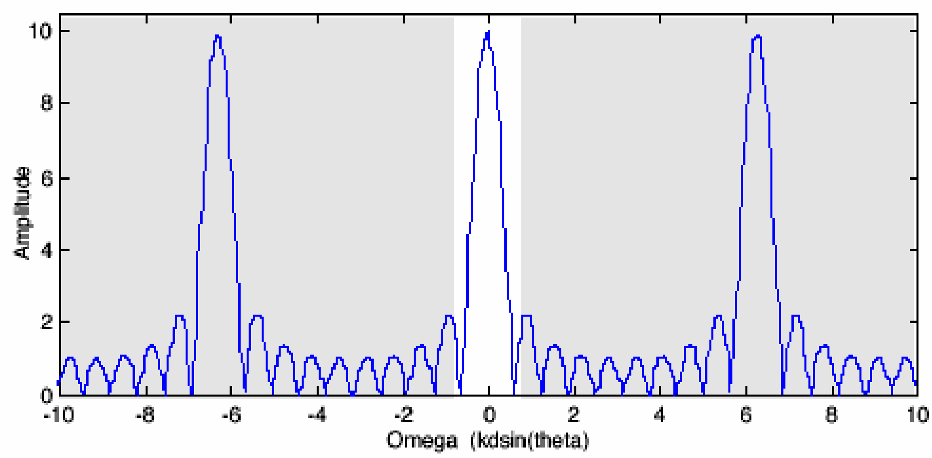

Figure 4 The visible region of an array of points in a larger Ω space (*kd*=0.79).

2.3 The Effect of Sampling

When the frequency gets high enough so that the spacing between the point sources becomes greater than half a wavelength the array becomes under-sampled. Under these conditions one gets *spatial aliasing,* which results in multiple main lobes. Figures 4, 5 and 6 illustrate this. Figure 4 shows the 1kHz example with the scale expanded. The first thing to note is that the visible region still covers the same region as that of figure 2. The second thing to note is that the expanded scale reveals the multiple peaks that indicate the possibility of spatial aliasing.

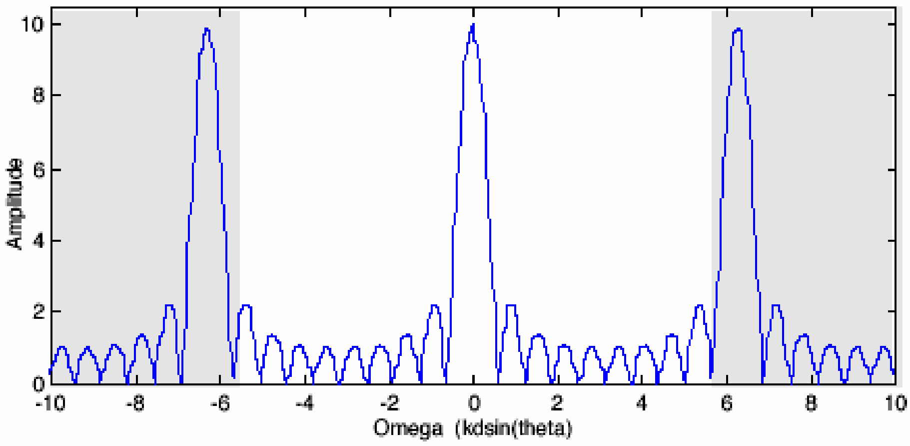

Figure 5 The visible region of an array of points in a larger Ω space (*kd*= 5.5).

Figure 5 shows the visible region when the frequency equals 7kHz (kd=5.5). Here we can see that, although the aliased main lobe is not visible there is an increase in side-lobe levels due to the spatial aliasing. Figure 6 shows the visible region when the frequency equals 10kHz (kd= 7.85). Here we can see that the aliased main lobe is now visible and there is a large increase in the sidelobe levels due to the spatial aliasing.

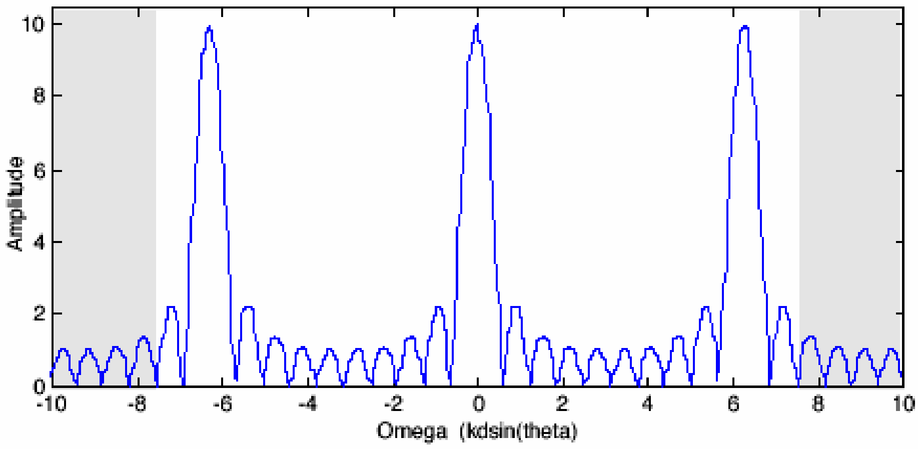

Figure 6 The visible region of an array of points in a larger Ω space (kd= 7.85).

2.4 The Effect of a Progressive Phase Shift

From the shift theorem, we know that a shift in the spatial domain leads to a linear (progressive) phase change in the Fourier domain and vicar versa. Thus, a progressive phase shift in the spatial domain would result in a linear shift of the function in Ω space. This would result in the main lobe moving to an angle off the central axis. However, the visible region would remain in the same place.

3. APPLICATION TO DIFFUSER DESIGN

For a diffuser we wish to have a $P(\theta_k)$ that is uniform with respect to angle. This corresponds to a pattern of coefficients, which corresponds to the diffuser structure that has a constant Fourier transform magnitude over the visible region. The effect of an obliquely incident wave-front is to add an additional progressive phase shift across the diffuser's reradiated sound. This causes the visible region to be shifted in angle. Therefore, in addition we would like the Fourier transform magnitude to be uniform outside the visible region as well to cover oblique incidence. Therefore, we need to find diffusion structures that have uniform magnitude Fourier transforms, such as Schroeder [1] diffusers. We can use the Fourier Transform relationship between the far-field polar pattern and the pattern of amplitudes at the diffuser's surface to help us choose appropriate sequences for diffusion structures.

The Wiener-Khinchin theorem states that the squared Fourier transform magnitude of a sequence is equal to the Fourier transform of its autocovariance (or autocorrelation function). Therefore sequences, whose autocovariance either is a delta function, or close to a delta function, will form good diffusers, because the Fourier transform magnitude of a delta function is uniform.

The convolution theorem states that convolution in the spatial domain is equivalent to multiplication in the Fourier domain and that the converse is true. This means that multiplication, or modulation, in the spatial domain is corresponds to convolution in the polar pattern domain. This allows us to use a variety of modulation techniques on short diffusers to achieve good diffuser [2-4], without the lobe narrowing that results from a repeated set of short diffusers. In fact, the Fourier relationship allows one to develop new diffusion structure, such as "Binary Amplitude" [4,5] and "Ternary" [6] diffusers.

4. APPLICATION TO ARRAY LOUDSPEAKERS

An early example of an array loudspeaker was the column loudspeaker. In this arrangement a number of small loudspeakers were arranged in a closely space line. Because of the extended length of the source in one plane directivity control was achieved in that plane. However, the beam pattern would get progressively more directive with frequency, as predicted by the Fourier transform. Techniques were developed to reduce this behaviour, usually by applying the necessary frequency dependent weighting, tapering, or windowing using simple electrical circuits, a direct application of the convolution theorem. Methods of steering these line speakers were also developed either, by using simple analogue delay techniques, or by using the inherent phase shifts in the filters used to taper the array. Again, this is a direct application of the shift theorems of the Fourier transform.

They could also exhibit unwanted side lobes at higher frequencies, due to aliasing, which reduced their utility. That is, above some frequency, the spacing between the drivers is greater that half the wavelength of the sound being produced. This results in spatial aliasing and results in a loss of control of the beam pattern.

To avoid spatial aliasing requires a huge number of small loudspeakers, resulting in a prohibitive cost for the array. For example, ideally we want pattern control over the entire audio frequency range. However, even if we make the speaker spacing 4.3cm, which is unfeasibly small because we would need a large number to achieve low frequency pattern control, we still have significant aliasing at 10kHz.

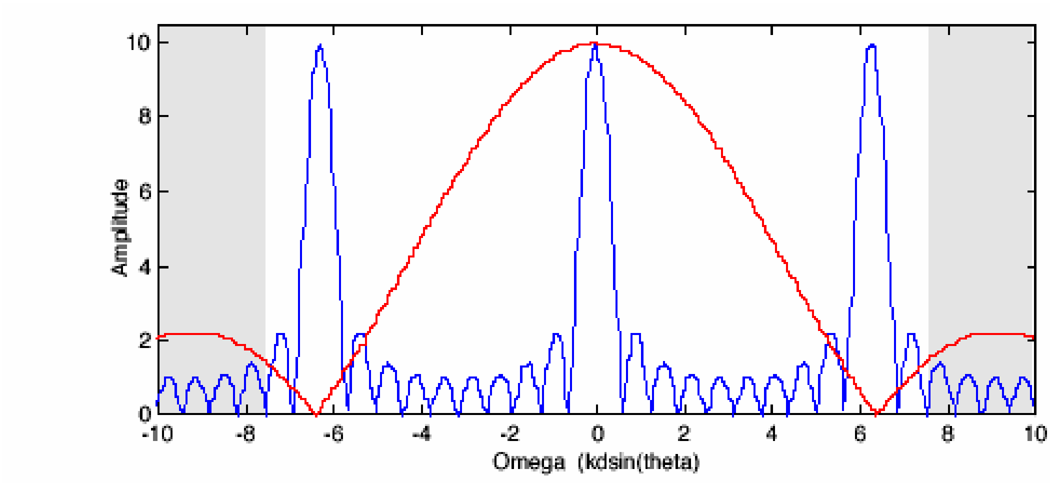

Figure 7 An array speaker and a continuous source equal to the spacing (kd= 7.85).

4.1 Acoustic Spatial Filtering

One way of reducing the effect of spatial aliasing is to use directive loudspeakers, instead of point sources, as the array elements. If one uses directive sources then their polar patterns will act as a form of spatial filter. That is the off axis sidelobes will be reduced by the of axis reduction in sound level that a directive source affords. Figure 7 shows an array response at 10kHz (kd= 7.85) with the response of a continuous line source, of length equal to the element spacing, superimposed upon it. Of particular note is that the zeros of the continuous line source fall on the aliased main lobes from the point source array. Because the farfield polar pattern of an array of point sources is related to the applied signals by a Fourier Transform relationship, all the theorems that apply to the Discrete Fourier Transform apply to the array loudspeaker. This means that the theorem that convolution in one domain is equal to multiplication in the other domain applies to this situation. Replacing each of the point sources with a continuous line source is equivalent to convolving it with the point array. Therefore, the effect of replacing the point source with the continuous sources is to multiply their farfield patterns together.

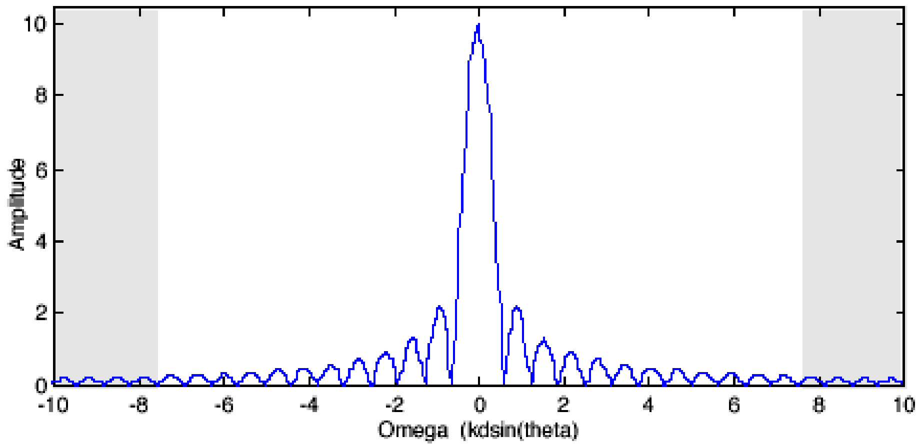

Figure 8 An array speaker made of continuous sources equal to the spacing (kd= 7.85).

This pattern multiplication is well known and the effect is for our example is shown in figure 8. One can see that the aliased main lobes have been eliminated. In fact, the response has become equivalent to a continuous line source of the same extent as the array. Clearly using directional sources, such as constant directivity horns, can also be used to achieve similar effects. It is this that results in the success of large arrays based on them because, providing they horns have directivity control before spatial aliasing occurs. Once the directivity of the individual elements is considered the need for curved arrays also becomes apparent, as the spatial filtering effect of the sources must also be factored in.

5. APPLICATION TO CONSTANT DIRECTIVITY HORNS

Keele [7-11] extended the work of Buren et al. [12,13] to develop the Constant Beamwidth Theory (CBT) arrays. In his papers, the transducer is a circular spherical cap of arbitrary half-angle with Legendre function shading. It provides a constant beam pattern and directivity with extremely low side lobes for all frequencies above a certain cutoff frequency.

To maintain constant beamwidth behaviour, CBT circular-arc loudspeaker line arrays require that the individual transducer drive levels be set according to a continuous Legendre shading function. This shading gradually tapers the drive levels from maximum at the centre of the array to zero at the outside edges of the array. Keele developed approximations to the Legendre shading that both discretise the levels and truncate the extent of the shading so that practical CBT arrays can be implemented. He determined by simulation that a 3-dB stepped approximation to the shading maintained out to -12 dB did not significantly alter the excellent pattern control of a CBT line array.

Conventional CBT arrays require a driver configuration that conforms to either a spherical-cap curved surface or a circular arc. Keele also showed how CBT arrays can also be implemented in flat-panel or straight-line array configurations using signal delays and Legendre function shading of the driver amplitudes. CBT arrays do not require any signal processing except for simple frequency-independent shifts in loudspeaker level. This is in contrast with conventional constant-beamwidth flat-panel and straight-line designs, which require strongly frequency-dependent signal processing.

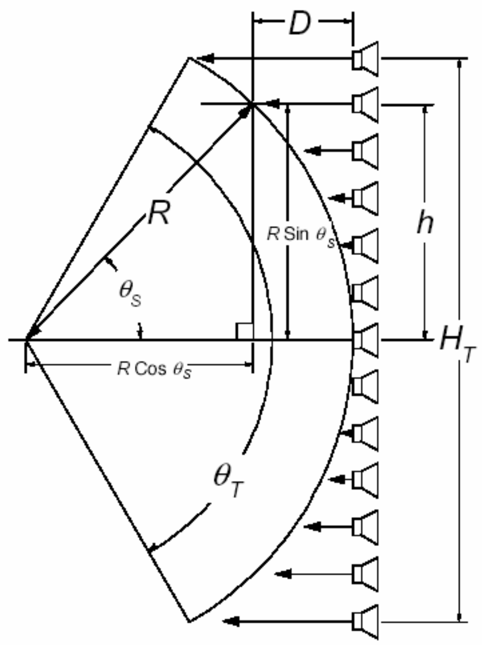

Figure 9 Relationships required to calculate the delays for a planar CBT array. from [9].

These results are important because they also provide a link between array loudspeakers and constant directivity horns. Figure 8, reproduced from [9], shows how the delays for a planar CBT array The delay effectively moves the driver from its position on a straight-line or flat surface to a point on a circular arc or on the surface of a sphere. That is it provides a delay that makes the wavefront at the planar array that is spherical. According to the Fourier theory we have described earlier, the Fourier transform of the combination of phase shifts, due to a spherical wavefront, and Legendre weighting results in a frequency independent constant beamwidth above a certain cutoff frequency. Furthermore, the cutoff frequency is a function of both the required directivity angle and the length of the array.

If we compare this to a constant directivity horn we observe many similarities.

1. Firstly, the conical flare of such horns results in a spherical wavefront at the horn's mouth.
2. Secondly the projection of the spherical wavefront's intensity onto the planar front of the horn results in an intensity that has a cosine rolloff from the centre of the horn. This approximates a Legendre weighting at the

centre of the planar front of the horn, as shown in Figure 10. Unfortunately, the weighting is not enough at the edges of the horn aperture. However, practical constant directivity horns have an additional more extreme flare at the mouth. This would have the effect of more rapidly reducing the amplitude at the edge of the horn's mouth and so it would more closely approach Legendre weighting.

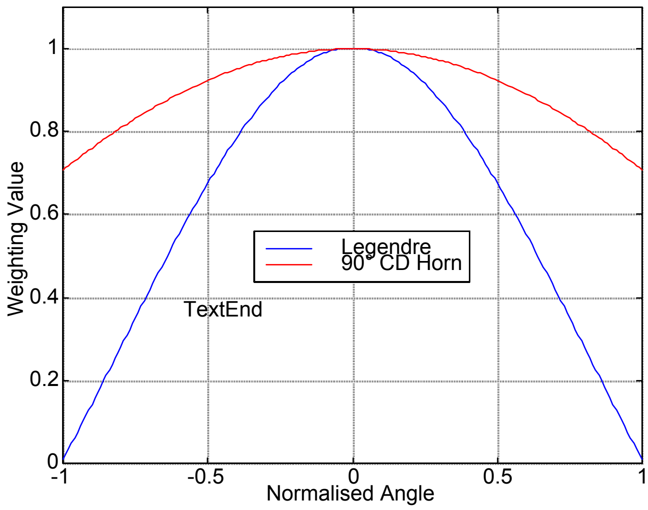

Figure 10 Legendre versus cosine weighting

Thus, constant directivity horns can be seen as a simple approximation to a CBT array!

6. WHERE THE METHOD FAILS

If one has an accurate representation of the near-field pressures then the far-field polar response can be calculated from it via the Fourier transform. If a velocity source, or a mixed velocity and pressure source, then the far-field polar pattern can still be calculated, however one has to allow for the dipole polar pattern of the velocity component. This can be done via the use of the convolution and superposition theorems.

The method breaks down when one is in the near-field. In this case one must explicitly add up the potentials that arise by explicitly including the effect of the varying distances from the various parts of the source.

The other area where the method breaks down is when one is trying to predict the nearfield, as one does in diffuser design for example. In these cases one is assuming that there is no interaction between the individual sources, and whilst this works for welled diffusers, it does not for stepped ones. For these situations, more complex calculations, such as the "Boundary Element Method" [14], must be used.

Finally, the Fourier method assumes a planar surface from which the transform is calculated, If the radiator is not planar then the amplitude and phases of the pressure components must be calculated for a “virtual” planar surface that is close to the actual radiator. These calculations must also take account of any lateral interaction that may take place between the actual and the “virtual” surface.

7. HOW TO ACHIEVE A TRAPEZOIDAL COVERAGE AREA

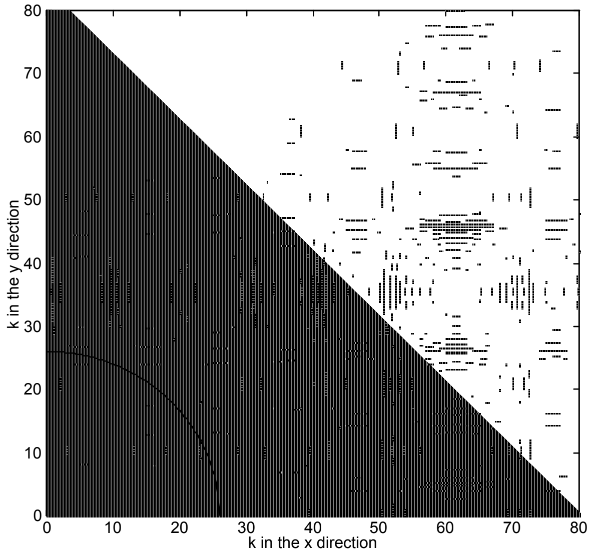

Figure 11 Section of a 2D inverse Fourier transform of a rectangular coverage area

There are two different ways of achieving a trapezoidal coverage area. Both require doing an inverse Fourier transform of the coverage area. Doing the inverse of a rectangularly illuminated area results in a cross shape, as shown in figure 11. This makes sense because the polar pattern must extend further at the corners, and there is a reciprocal relationship between mouth size and coverage angle. In fact the coverage patterns of rectangular constant directivity horns tend to show a narrowing of the coverage angle in the diagonal directions. Thus, the required coverage could be obtain using a CBT Array with a cross shaped arrangement of speakers. Alternatively a cross-shaped constant directivity horn (the “Bizarre-Horn”) could be developed.

8. CONCLUSIONS

This paper has presented the basic Fourier relationship between the farfield polar response and the near field illumination of the aperture. It has demonstrated its utility in a variety of electroacoustic applications. Given its power it can be truly said that in as a tool for electroacoustic design Fourier theory really does ”rule the roost”!

9. REFERENCES

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