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THE BALLOON DANCE IN ELECTROACOUSTICS

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1. INTRODUCTION

An electroacoustic transducer's variation in sensitivity with direction is usually represented via a polar plot. This represents a slice of the three dimensional variation with respect to angle. Unfortunately many transducer's sensitivities are not rotationally symmetric and therefore the polar pattern must be plotted at more than more than one angle. The two orthogonal x and y directions are popular but even this level of specification may not cover variations on the diagonal. Furthermore in many loudspeakers the polar pattern is also a function of frequency. Modern CAD packages require a full specification of the transducer's polar pattern as a function of frequency in order to provide an accurate prediction of its likely performance in a given space.

Unfortunately providing this level of accuracy requires that the polar pattern be measured at a large number of angles, both azimuth and elevation, and over a large number of frequencies. A recent proposal [1] suggests that a large number of measurements be taken over a full sphere at sixth octave frequency spacing. Clearly this level of specification is costly in both measurement time and computer storage. One of the reasons behind the large number of directional measurements is the need to interpolate amplitude values between measured data points. This is exacerbated if the measurement angles are coarse.

This problem of restricted directional resolution can be alleviated by forming a continuous, functional representation of a polar pattern, expressing the polar patterns mathematically, as a continuous function of direction. This paper shows how to form a continuous, orthogonal, and three dimensional, representation by expressing the polar pattern as a weighted sum of Surface Spherical Harmonics - a hierarchical set of basis functions which are orthogonal upon the surface of a sphere. The Surface Spherical Harmonic weights can be calculated from a limited set of experimental measurements by means of a discrete Fourier analysis. The resulting spherical harmonic representation is continuous, yielding a modelled polar pattern for any arbitrary direction. It is also hierarchical, in that the more harmonics that are included the greater the accuracy of the model, and has a meaningful spatial structure, with particular Surface Spherical Harmonic weights expressing particular patterns of directional variation in the polar pattern. This representation may provide not only a more efficient way of expressing directional variation but also may allow new insights into the analysis of the spatial variation of transducers.

The paper firstly explains Surface Spherical Harmonics and presents a means of efficiently deriving them from measured data. The implication and applications are then discussed and finally some results of the analysis applied to polar pattern measurements are presented.

2. SPHERICAL HARMONIC ANALYSIS

In order to perform a Surface Spherical Harmonic analysis of a speaker polar pattern we need to answer two questions:

- What are surface spherical harmonics and why are they useful?
- How can we efficiently calculate surface spherical harmonic coefficients from polar pattern measurements?

2.1 What are Surface Spherical Harmonics?

Conventional, one-dimensional signal processing makes use of sinusoidal harmonics as the basis functions of Fourier analysis. Surface Spherical Harmonics may be similarly applied to the analysis of functions on the surface of a sphere - $f(\theta,\phi)$ where $0°\le\phi<180°$ and $0°\le\theta<360°$. This is possible because Surface Spherical Harmonics form a complete, orthogonal set over this surface Kaplan [2]. Spherical harmonics arise as the solution to the Laplace equation expressed in spherical polar co-ordinates. Surface Spherical Harmonics are the special-case spherical harmonics in which the distance coordinate is constant. We may define the Surface Spherical Harmonics (normalised so that their integrals over the surface of the sphere are unity) as:

$$u_{0n}(\theta,\phi)=\frac{(2n+1)}{4\pi}\cos m\theta\, P_n^m(\cos\phi), \quad m=0$$
$$u_{mn}(\theta,\phi)=\frac{(n-m)!(2n+1)}{2\pi(n+m)!}\cos m\theta\, P_n^m(\cos\phi), \quad m=1,\dots,n \tag{1}$$

$$v_{mn}(\theta,\phi)=\frac{(n-m)!(2n+1)}{2\pi(n+m)!}\sin m\theta\, P_n^m(\cos\phi), \quad m=1,\dots,n \tag{2}$$

Where P_n^m are the *Associated Legendre Functions* defined by:

$$P_n^m(x)=-1^m\left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{dx^m}\left[\frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n, \quad n=1,2,\dots\right] \tag{3}$$

The parameter n represents the degree of each particular harmonic: u_{00} is the fundamental Surface Spherical Harmonic. The three first-degree harmonics are denoted by u_{01}, u_{11} and v_{11}. There are five second-degree harmonics, seven third-degree harmonics and so on. The first few Surface Spherical Harmonics, listed below, will have a structure familiar to chemists as they form the electron shell patterns of 's' (fundamental), 'p' (first-degree), 'd' (second-degree) and 'f' (third-degree) orbitals.

$$u_{00}=\sqrt{\frac{1}{4\pi}},\quad u_{01}=\sqrt{\frac{3}{4\pi}}\cos\phi,\quad u_{11}=-\sqrt{\frac{3}{4\pi}}\cos\theta\sin\phi,\quad v_{11}=-\sqrt{\frac{3}{4\pi}}\sin\theta\sin\phi,\ \dots$$

Because these harmonics form a complete, orthogonal set any function of (θ,ϕ) can be approximated by a weighted sum of Surface Spherical Harmonic components, up to the desired degree:

$$f(\theta,\phi)\approx U_{00}u_{00}+U_{01}u_{01}+U_{11}u_{11}+V_{11}v_{11}+U_{02}u_{02}+U_{12}u_{12}+V_{12}v_{12}+U_{22}u_{22}+\dots$$

The weights required to express a function $f(\theta,\phi)$ in this way can be calculated by Fourier analysis:

$$U_{mn} = \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} f(\theta,\phi)u_{mn}(\theta,\phi)\sin\phi\, d\phi d\theta,\ \ 0 \le m \le n \tag{4}$$

$$V_{mn} = \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} f(\theta,\phi)v_{mn}(\theta,\phi)\sin\phi\, d\phi d\theta,\ \ 1 \le m \le n \tag{5}$$

2.2 Deriving Surface Spherical Harmonic Weights from Measured Data

Equations 4 and 5 allow the calculation of the Surface Spherical Harmonic weights given a continuous function $f(\theta,\phi)$. However, in this investigation we do not have such a function, but instead a set of polar pattern impulse responses. Spherical Harmonic Analysis (SHA) can be applied to the responses for each individual time sample. Thus the representation will be of the form of a new set of weights for each instant of time. Equations 4 and 5 must also be adapted so that the weights U_{mn} and V_{mn} may be calculated given that only discrete values of $f(\theta,\phi)$ are available, effectively sampled at each direction in which a physical polar pattern was measured. Thus, it is necessary for the integration denoted in Equations 4 and 5 to be approximated by means of a summation over N discrete points:

$$\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} f(\theta,\phi)u_{mn}(\theta,\phi)\sin\phi\, d\phi d\theta \approx \sum_{i=1}^{N} w_i f(\theta_i,\phi_i)u_{mn}(\theta_i,\phi_i) \tag{6}$$

Equation 6 is of the form:

$$\int_S F(\theta,\phi)\, dS \approx \sum_{i=1}^{N} a_i F(\theta_i \phi_i) \tag{7}$$

This operation is, therefore, one of numerical integration across the region formed by the surface of a sphere. Ideally, we wish the approximation to the continuous integral to be exact.

McLaren [3], and Stroud [4] describe efficient methods for exact numerical integration of this form, up a given degree of Surface Spherical Harmonic, in terms of the discrete directions to use and the corresponding weights. Although such optimal sets of directions and weights minimise the number of directions that need be considered, given a particular Surface Spherical Harmonic degree, in terms of *experimental* efficiency they are not, in general, suitable for polar pattern analysis. Generally, the optimal directions for numerical integration on the sphere are not distributed in a pattern along which experimental measurements can conveniently be made. In Atkinson [5] an alternative set of directions and weights for exactly this integration is developed. This set is less mathematically efficient, since it requires a greater number of individual directions to integrate exactly functions up to a given degree. However, the choice of discrete directions is more experimentally practical. The approximation is based on following relationship:

$$\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} F(\theta,\phi)\sin\phi\, d\theta d\phi \approx \frac{\pi}{n}\sum_{j=1}^{n}\sum_{i=1}^{2n} w_j F(\theta_i,\phi_j) \tag{8}$$

Under this approximation, measurements are made at the same set of angles of azimuth, θ_i, for each of a set of n angles of elevation, ϕ_j. The $2n$ angles of azimuth are equally spaced, thus forming a 'rectangular' mesh of directions which permits more systematic experimental measurements. In order to gain some mathematical efficiency, and maintain the exactness of the approximate integration, *product Gaussian Quadrature* is used to select the n angles of elevation. The *Gauss-Legendre* quadrature formula uses weighting and an unequal spacing of sampling points to exactly integrate functions varying between -1 and +1. This approach can be applied to the spherical integration region of Equation 8 by selecting the angles of elevation ϕ_j so that $\cos(\phi_j)$ and w_j are the Gauss-Legendre nodes and weights for degree n., These values can be found in tables such as those of Stroud and Secrest [6]. Using this choice of directions Equation 8 permits the exact numerical integration of all functions $F(\theta,\phi)$ less than degree $2n$, Stroud [4]. In Surface Spherical Harmonic analysis, since $F(\theta,\phi)$ is the product of the function under analysis and a particular Surface Spherical Harmonic, it is apparent that this approximation can be used to calculate exactly the weights for Surface Spherical Harmonic components up to and including degree n-1.

Maximum Surface Spherical Harmonic Degree	Number of Measurements
0	2
1	8
2	18
3	32
4	50
5	72
6	98
7	128

Table 1 The total number of measurements required to exactly calculate the Surface Spherical Harmonic weights up to a given spherical harmonic degree.

The total number of measurements required to exactly calculate the Surface Spherical Harmonic weights up to a given spherical harmonic degree is given by the following equation.

$$Number\ of\ measurements = 2(degree + 1)^2 \qquad (9)$$

The results of equation 9 are shown in Table 1 and from it we can see that spherical harmonics up to degree 6 can be achieved using less than one hundred measurements.

The method of Atkinson [9] is efficient if a manual azimuth and elevation apparatus is used, or if the movement time in the elevation position is significantly greater than the azimuth direction. However if one has computer controlled apparatus which can be programmed to measure at arbitrary angle then the methods of McLaren [7] and Stroud [8] may offer significant additional savings providing the mechanical behaviour of the measurement equipment does not cause an additional time penalty for two axis movements.

However both methods have the problem that if one wishes to increase the order of spherical harmonic representation then a complete set of new measurements must be made. This is because, in general, the Gauss-Legendre nodes which determine the elevation positions are different for different values of n. Ideally, we would like to refine the measurement by making the minimum number of new measurements by reusing the old ones. Fortunately it is possible to extend Atkinson's method to achieve this. A method due to Konrod [8] and refined by Patterson [9] allows one to add $n+1$ additional points between the existing ones whilst still keeping the old points. This allows us to increase the order, although in a limited fashion, without having to re-measure all the points.

For example, one could increase the order from 3, requiring 32 measurements, to order 8, which would normally require 162 measurements, but in fact would only require 130 additional measurements if Patterson's [9] technique is used.

3. DISCUSSION

So Surface Spherical Harmonics can be used to approximate a polar pattern but what are the implications of this representation? Firstly The angular variability of spherical harmonics is determined by their degree: As degree increases the spherical harmonic components become more variable as a function of angle. The effect of higher order spherical harmonics is to sharpen the polar pattern in a manner similar to the way higher order Fourier components sharpen waveforms. This means that we would expect to need higher order Surface Spherical Harmonics to represent narrow main lobes. Conversely wider main lobes should require less harmonics for their specification. This means that Surface Spherical Harmonics can be used as a more efficient means of representing polar patterns. Note, that the normal means of representing will require the same number of points to represent it irrespective of the width of the main lobe. Surface Spherical Harmonics also specify the polar pattern in three dimensions automatically and this can provide further savings in storage. Finally Surface Spherical Harmonics can be measured efficiently. The method of Atkinson [5] is efficient if a manual azimuth and elevation apparatus is used, or if the movement time in the elevation position is significantly greater than the azimuth direction. However if one has computer controlled apparatus which can be programmed to measure at arbitrary angle then the methods of McLaren [3] and Stroud [4] offer significant additional savings.

4. RESULTS

Surface Spherical Harmonic can be applied to microphone as well as loudspeaker polar patterns. Here we present results based on measurements of the polar pattern of the entrance to the ear canal. This is a particularly challenging problem because the polar pattern is not aligned along a given axis and is strongly frequency dependent in both pattern shape and direction. A set of 648 frequency response measurements were analysed by means of an Surface Spherical Harmonic analysis. This corresponds to setting n=18 in Equation 8, thus requiring measurements with a 10° azimuthal spacing at each of 18 angles of elevation; ±5.0°, ±14.5°, ±24.0°, ±34.0°, ±44.0°, ±53.5°, ±63.0°, ±73.0° and ±82.5°, where 0° elevation refers to the horizontal plane. Such a set of polar pattern measurements allows the calculation of Surface Spherical Harmonic component weights of up to and including degree 17.

The frequency responses were measured by means of Maximum-Length Sequence (MLS) analysis, using a MLSSA system (DRA Laboratories). The measurements were made in an anechoic chamber (measuring approximately 3m × 3m × 3m) at BT Laboratories, Martlesham Heath, using a Bruel and Kjaer 4127 Head-and-Torso Simulator (HATS) mounted on a turntable, and a Auratone loudspeaker in a movable bracket on an arched frame. It is estimated that the directional error in positioning the loudspeaker and HATS ears relative to each other was rarely more than 0.5° and never more than 1°. The frequency response data collected in the anechoic chamber was processed using the MATLAB software package to provide the Surface Spherical Harmonic coefficients up to order 17 at 48 equally spaced frequencies up to 10kHz.

Figures 4 and 5 compare the measured results and the Surface Spherical Harmonic representation of the same data at two frequencies, 4.305kHz and 9.135kHz. At 4.305kHz the measured and Surface Spherical Harmonic representation are in close agreement. At 9.135kHz the agreement is not as good but is still reasonable especially considering the variability in the polar pattern. Note that the Surface Spherical Harmonic representation is also able to accommodate a shift in angle of the main lobe as a function of frequency as well.

Figures 6 and 7 show the rms. amplitudes of the Surface Spherical Harmonics as a function of both degree and frequency, a sort of Surface Spherical Harmonic Spectrum, for these measurements in linear and logarithmic scales respectively. It is clear from these results that, even for this situation, the amplitude of the Surface Spherical Harmonic weights reduce rapidly with degree. In fact one could argue that above about degree six most of the polar pattern information has been captured. This results augur well for the ability of this representation to capture and represent polar pattern information efficiently.

5. CONCLUSION

Surface spherical harmonics can be used to represent polar patterns. They have the advantage of requiring potentially less measurements for a given level of specification. In addition, because they are an orthogonal basis set, detail can be increased simply by adding in additional harmonics. This offers the potential for optimising the storage of polar patterns within acoustic CAD programs and may allow other forms of analysis of polar and scattering patterns.

6. ACKNOWLEDGEMENTS

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7. REFERENCES

[1] Seidel, F and Staffeldt H, "Frequency and Angular Resolution for Measuring, Presenting, and Predicting Loudspeaker Polar Data", *Journal of the Audio Engineering Society*, **44**, 7/8, July/August 1996, pp 555-568.

[2] Kaplan W, *"Advanced Mathematics for Engineers"*, Addison-Wesley, Reading MA. 1981.

[3] McLaren A D, "Optimal numerical integration on a sphere", *Mathematics of Computation*, **17**, 1963, pp. 361-383.

[4] Stroud A H, "*Approximate Calculation of Multiple Integrals*", Prentice-Hall, Englewood Cliffs NJ, 1971.

[5] Atkinson K, "Numerical integration on the sphere", *Journal of the Australian Mathematical Society (Series B)* , **23**, 1982, pp. 332-347.

[6] Stroud A H, Secrest D, *Gaussian Quadrature Formulas*, Prentice-Hall, Englewood Cliffs NJ, 1966.

[7] Press, D.B. et al, *Numerical recipes in C: second edition,* Cambridge: Cambridge University Press, 1992, pp150-160

[8] Konrod, A S, "Numerical integration on the sphere", *Numerical recipes in C: second edition,* Cambridge: Cambridge University Press, 1992, pp150-160

[9] Patterson T N L, "Numerical integration on the sphere", *Mathematics of Computation,* **23**, 1968, pp. 847-856.

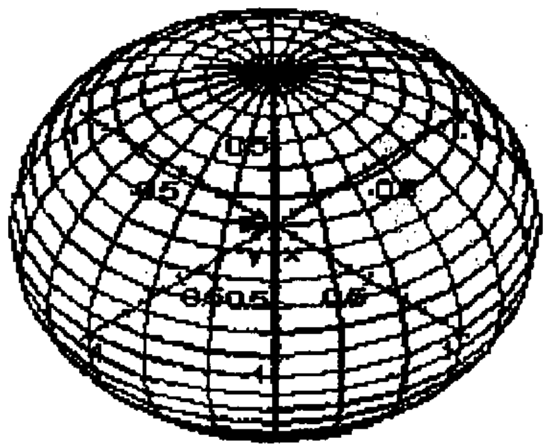

Figure 1 Degree zero spherical harmonic.

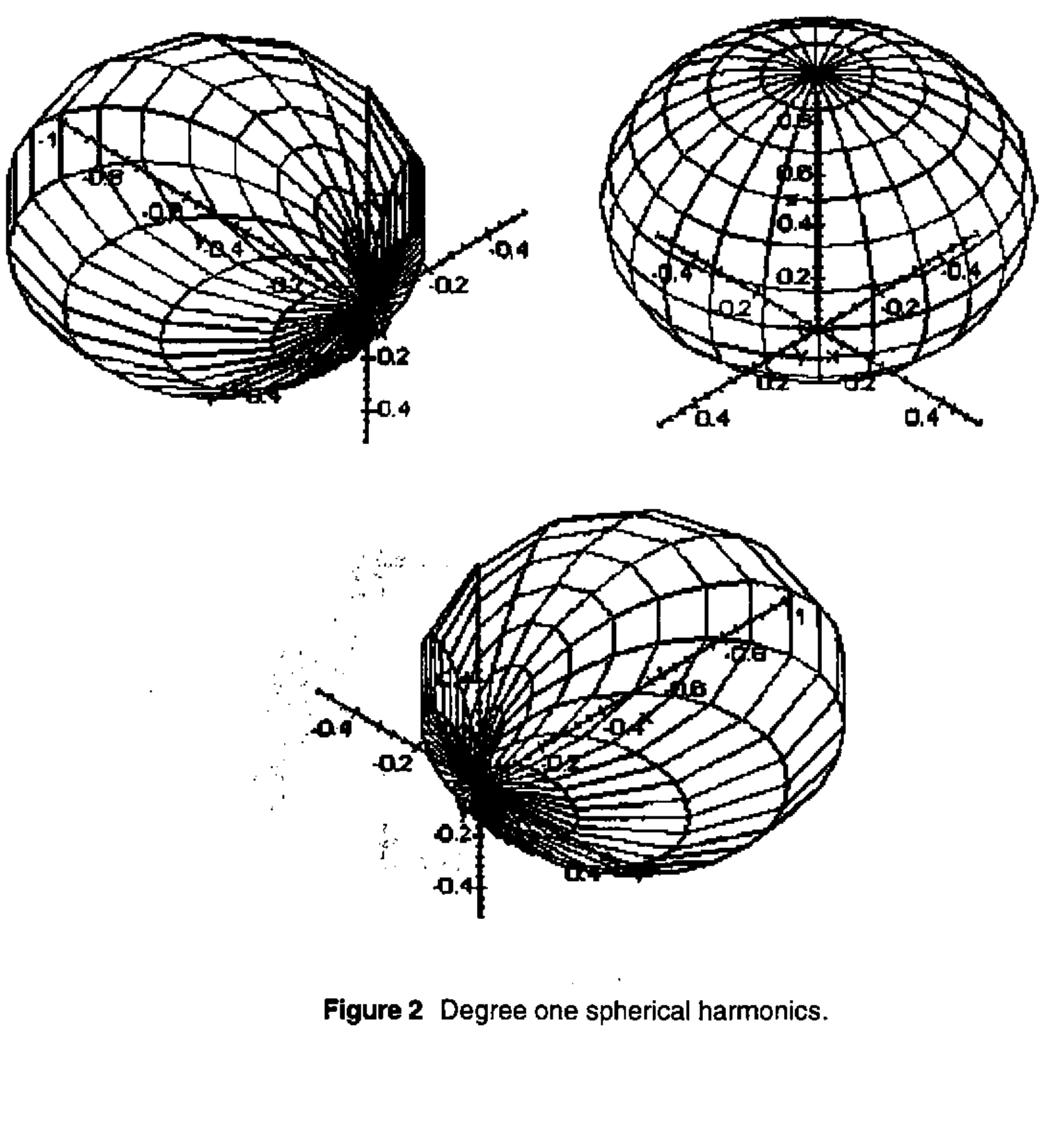

Figure 2 Degree one spherical harmonics.

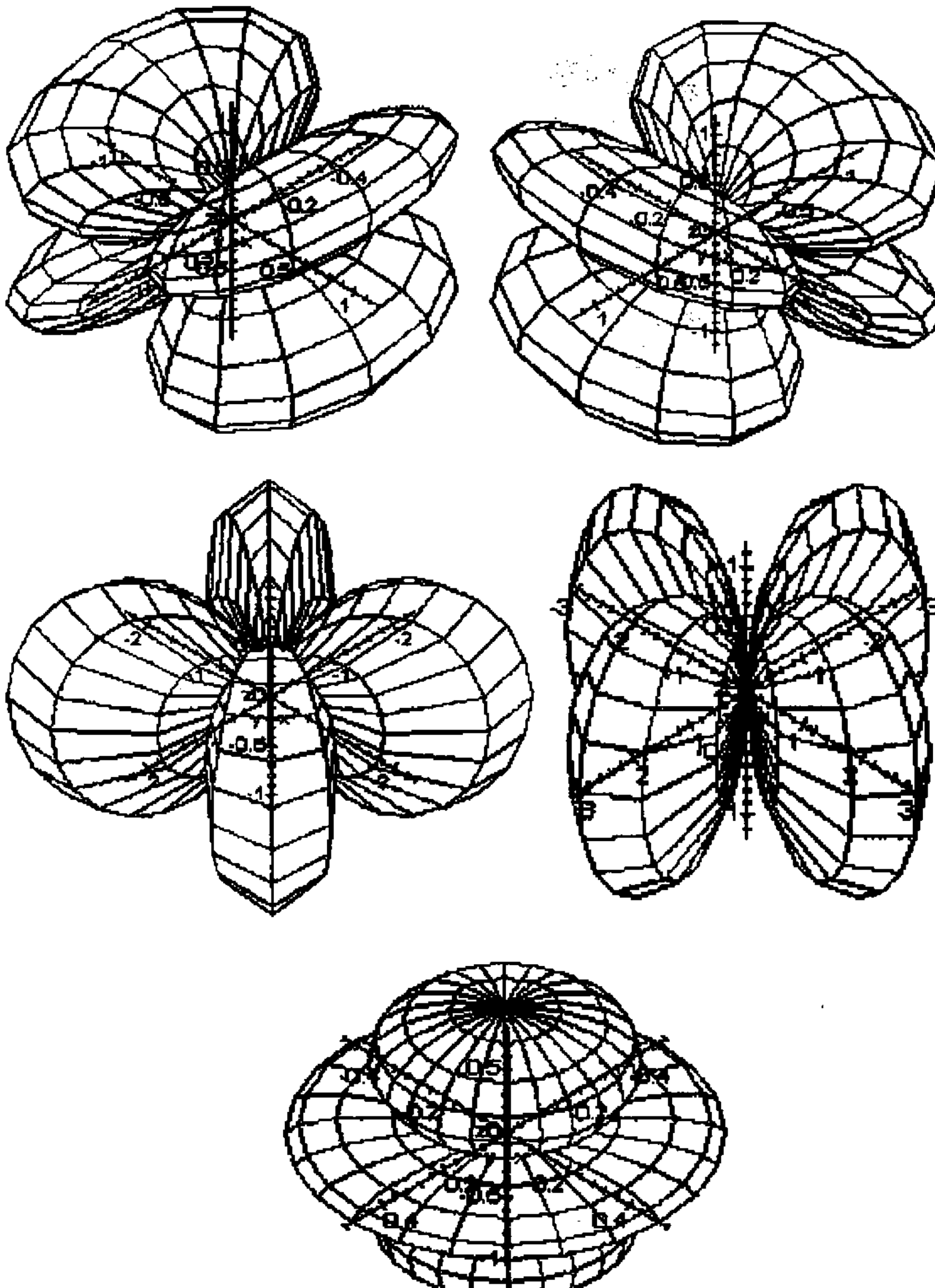

Figure 3 Degree two spherical harmonics.

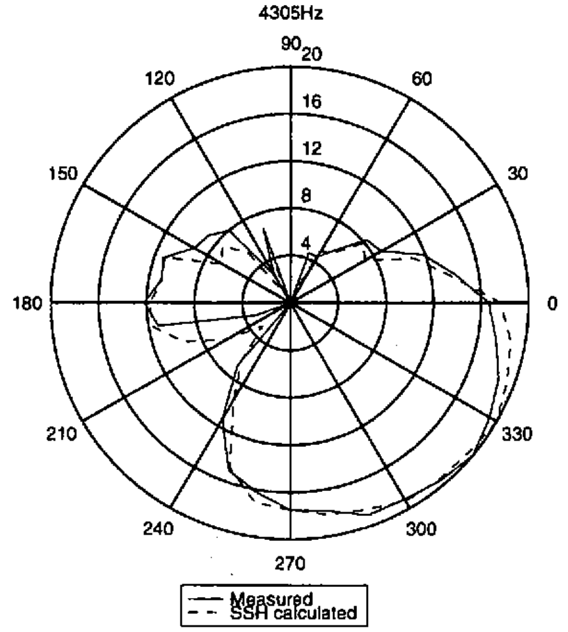

Figure 4 Measured and Spherical Harmonic generated polar plot at 4305Hz.

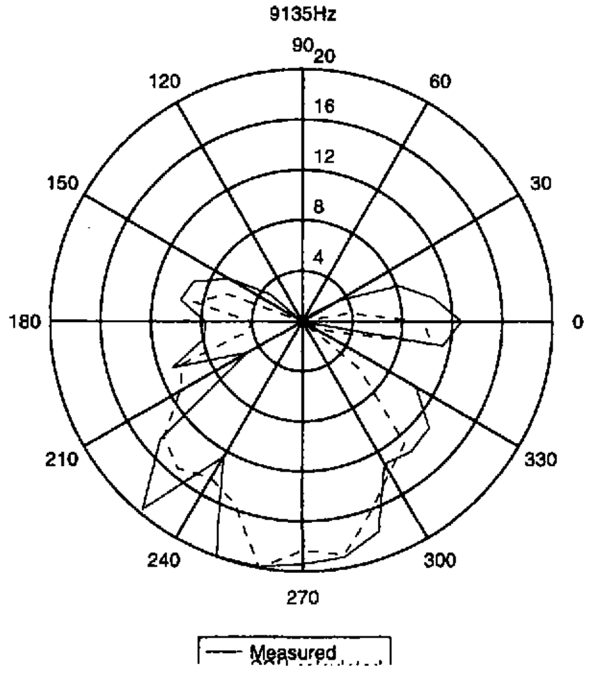

Figure 5 Measured and Spherical Harmonic generated polar plot at 9135Hz.

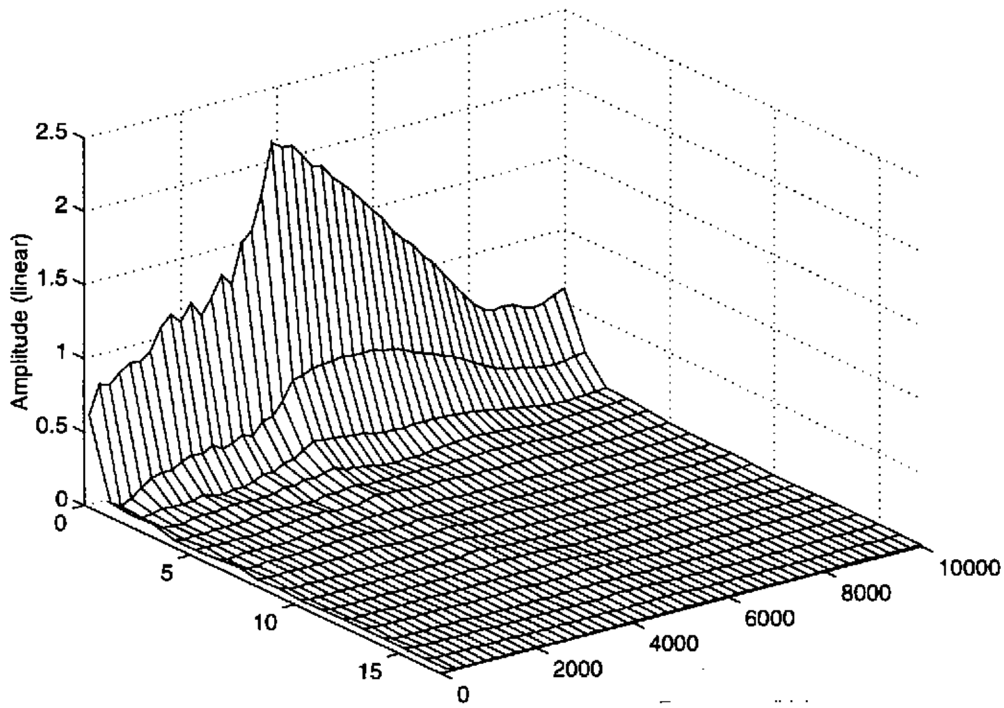

Figure 6 Rms Spherical Harmonic Spectra for each frequency bin (linear scale)

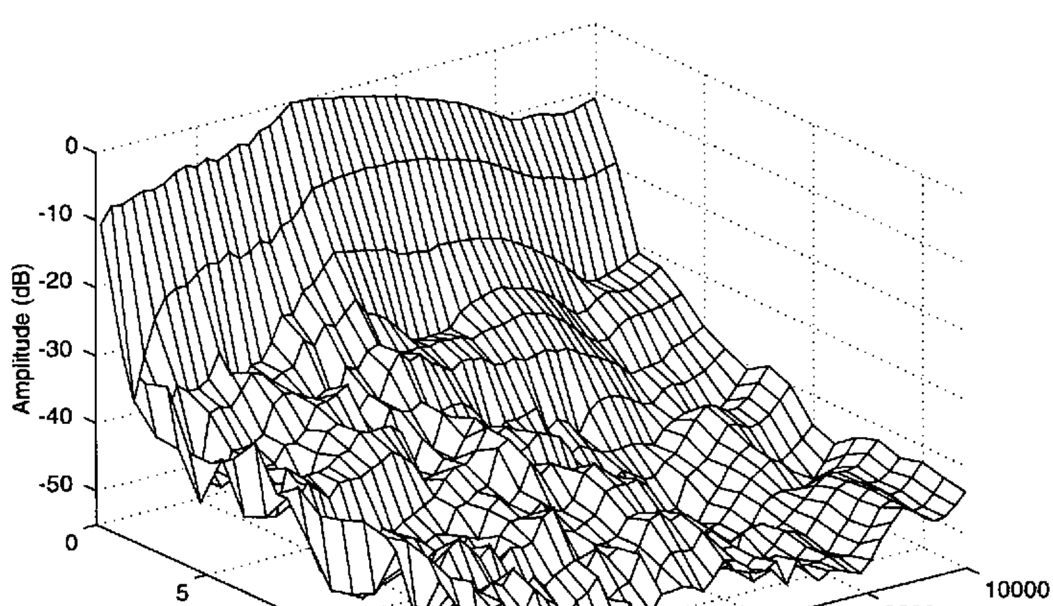

Figure 7 Rms Spherical Harmonic Spectra for each frequency bin (dB scale)