APPLICATION OF THE BOUNDARY ELEMENT METHOD TO THE DESIGN OF A MICROPHONE ARRAY BEAMFORMER

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1 INTRODUCTION

Microphone arrays are widely used in many applications such as sound field analysis, spatial sound recording and beamforming. In general the theoretical study and design of microphone arrays conform to one of the geometries for which the wave equation can be solved by separation of variables, for example, linear, planar, cylindrical and spherical geometry. Many studies have been carried out recently on 3-D microphone arrays, and especially spherical microphone arrays. The main advantage of the spherical arrangement is the 3-D spatial isotropy, which is useful for sound field analysis[11]. A spatial sound recording technique that relies on the use of spherical arrays was firstly introduced in a recent works by Meyer and Elko[8]. The design of this kind of arrays and their limitations are studied by Abhayapala and Ward[1], and Rafaely[10], among others.

A sound field can be represented by the superposition of plane waves arriving from all possible directions. This concept is described mathematically by an integral operator. If the kernel of such integral operator are plane waves that propagate in the free field, the integral operator is the so-called the Herglotz Wave Function (HWF)[4, 5]. This approach can be adopted as a theoretical model for the analysis of the sound field.

In this paper, the plane wave superposition approach is extended to a more general idea. The kernel of the integral operator is the acoustic transfer functions between a plane wave propagating in a given direction and a sensor mounted on the surface of a scattering object. We also present the use of BEM to calculate this acoustic transfer functions. Finally, this approach is applied to the design of a beamformer for a rigid array with unconventional shape, e.g. an array of microphones arranged on a rigid structure with ellipsoidal shape.

The Boundary Element Methods (BEM) is a well-known technique widely used for solving acoustic problem numerically[3]. BEM is a powerful tool to study scattering problems especially for the sound field scattered by objects with arbitrary shape. The first use of this method is reported by Chen and Schweikert[2] to predict sound radiation from a vibrating object with arbitrary shape. The attractiveness of this method has now significantly been increased by the decreasing cost of computing devices. The advantage of BEM over another common method, the Finite Element Method (FEM), is that a smaller number of nodes and meshes is required. A suitable boundary condition, either the acoustic pressure or the particle velocity, is required for the calculation. Some examples of the use of the BEM in the design of the microphone array are reported in the scientific literature. For example, Moquine et al[9] designed a beamformer for a microphone array embedded in an asymmetrically shaped object. A circular microphone array mounted on asymmetrically shape object was studied by Wei Jiang et al [6].

This paper is structured as follows: the theoretical background on the spherical harmonics expansion of sound field is reviewed and the method of designing a beamformer by analytical inversion of an integral operator is analysed in section 2. The numerical implementations are discussed in section 3 and the acoustic transfer functions calculated by BEM are studied in the same section. The beam patterns of the analytical and numerical beamformers applied to a rigid spherical microphone array are discussed in section 4. Finally, the BEM is applied to the design of a beamformer for a rigid ellipsoidal array.

2 BEAMFORMER BASED ON THE ANALYTICAL SOLUTION

2.1 Sound Field Model and Array Processing

We assume that the acoustic pressure of the sound field $p(\mathbf{x})$ to be measured by the microphone array at $\mathbf{x}$ can be represented, in the absence of the array, by a linear superposition of plane waves. This is mathematically expressed by the following integral formula, often referred to as Herglotz wave function

$$p(\mathbf{x}) = \int_{\Omega_S} \exp(ik\mathbf{x}\cdot\hat{\mathbf{y}})a(\hat{\mathbf{y}})\,d\Omega_S(\hat{\mathbf{y}}) \tag{1}$$

where $\hat{\mathbf{y}}$ is the arrival direction of the plane wave $\exp(ik\mathbf{x}\cdot\hat{\mathbf{y}})$, $a(\hat{\mathbf{y}})$ is the density of the plane wave in the direction $\hat{\mathbf{y}}$, Ω_S is the unitary sphere that represents all possible direction of arrival. If a rigid array is introduced in the sound field, the total sound field $p_T(\mathbf{x})$, given by the scattered field plus the incident field, in the exterior region of the array is given by

$$p_T(\mathbf{x}) = \int_{\Omega_s} H(k,\mathbf{x},\hat{\mathbf{y}})a(\hat{\mathbf{y}})\,d\Omega_s(\hat{\mathbf{y}}) \tag{2}$$

where $H(k,\mathbf{x},\hat{\mathbf{y}})$ can be regarded as the acoustic transfer function between a plane wave coming from the direction $\hat{\mathbf{y}}$ and an omnidirectional, ideal microphone arranged in $\mathbf{x}$. The most important role is played here by the density function $a(\hat{\mathbf{y}})$ because it indicates the strength of each plane wave from the given direction $\hat{\mathbf{y}}$.

The ideal objective is to reconstruct the density function $a(\hat{\mathbf{y}})$, and therefore the entire sound field, from the knowledge of the total pressure field $p_T(\mathbf{x})$ measured on the boundary S. A beamformer may be regarded as a signal processing operation that attempts to recover the value of the density function at a given direction $\hat{\mathbf{y}}$. The beamformer output can be determined by the integration of the product of the pressure $p_T(\mathbf{x})$ and the complex conjugate of the so-called aperture function $w(\mathbf{x},\hat{\mathbf{y}}_{look})$ where $\mathbf{x}$ now indicates the position on the boundary Ω_a. In the case that the sound pressures is measured on the surface of a rigid sphere, Ω_a represents the surface of that sphere. The beamformer output $z(\hat{\mathbf{y}}_{look})$ is given by

$$z(\hat{\mathbf{y}}_{look}) = \int_{\Omega_a} p_T(\mathbf{x}) w(\mathbf{x}, \hat{\mathbf{y}}_{look}) d\Omega_a(\mathbf{x}) \tag{3}$$

Substituting (8) into (7) yields

$$z(\hat{\mathbf{y}}_{look}) = \int_{\Omega_a} \int_{\Omega_s} H(\mathbf{x}, \hat{\mathbf{y}}) a(\hat{\mathbf{y}}) w(\mathbf{x}, \hat{\mathbf{y}}_{look}) d\Omega_s(\hat{\mathbf{y}}) d\Omega_a(\mathbf{x}) \tag{4}$$

The wave number k in H has been dropped to simplify the notation. We want the aperture function $w(\mathbf{x}, \hat{\mathbf{y}}_{look})$ to be such that

$$\int_{\Omega_a} H(\mathbf{x}, \hat{\mathbf{y}}) w(\mathbf{x}, \hat{\mathbf{y}}_{look}) d\Omega_a(\mathbf{x}) = \delta(\hat{\mathbf{y}} - \hat{\mathbf{y}}_{look}) \tag{5}$$

If equation (5) holds we have that

$$z(\hat{\mathbf{y}}_{look}) = a(\hat{\mathbf{y}}_{look}) \tag{6}$$

In practical cases the sound pressure is measured at a finite set of locations in Ω_a. For this reason, equation (6) does not hold true but the output $z(\hat{\mathbf{y}}_{look})$ of the beamformer is an approximation of $a(\hat{\mathbf{y}}_{look})$.

The results above assume the knowledge of the transfer functions $H(k, \mathbf{x}, \hat{\mathbf{y}})$. In practice, these should be computed using an analytical method or a numerical method. In this section and in the following one, beamformers are computed using firstly an analytical solution for the scattering problem and then a numerical solution.

2.2 Spherical Wave

We consider now the special case of a spherical microphone array. In this case, the sound field is spatially sampled on a spherical surface and can be described by means of a spherical harmonic expansion. The notation (r, θ, ϕ) represents radius, elevation angle and azimuthal angle in the standard spherical coordinate that is adopted in this work[12].

The spherical spectrum P_ν^μ of a function $p(r, \theta, \phi)$ defined on the surface of a sphere is given by

$$P_{\nu\mu} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} p(r, \theta, \phi) Y_\nu^\mu(\theta, \phi)^* \sin\theta d\phi d\theta \tag{7}$$

$$p(r, \theta, \phi) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} P_{\nu\mu} Y_\nu^\mu(\theta, \phi) \tag{8}$$

where $Y_\nu^\mu(\theta, \phi)$ is the spherical harmonic of order ν and degree μ defined as

$$Y_\nu^\mu(\theta, \phi) \equiv \sqrt{\frac{(2\nu+1)}{4\pi} \frac{(\nu-\mu)!}{(\nu+\mu)!}} P_\nu^\mu(\cos\theta) e^{i\mu\phi}$$

where P_ν^μ is a Legendre function of order ν and degree μ. The completeness and orthogonality properties of the spherical harmonics are given respectively by

$$\sum_{\mu} Y_{\nu}^{\mu}(\theta,\phi)Y_{\nu}^{\mu}(\theta_y,\phi_y)^* = \delta(\cos\theta - \cos\theta_y)\delta(\phi-\phi_y) \tag{9}$$

$$\int_{\Omega} Y_{\nu'}^{\mu'}(\theta,\phi)Y_{\nu}^{\mu\,*}(\theta,\phi)d\Omega = \delta_{\nu-\nu'}\delta_{\mu-\mu'} \tag{10}$$

where (θ_y,ϕ_y) are the elevation angle and azimuthal angle that identify the direction $\hat{\mathbf{y}}$. A plane wave in a free field (in the absence of scatterers) can be expanded in a series of spherical harmonics. This expansion is known also as Jacobi-Angle expansion

$$e^{ik\hat{\mathbf{y}}\cdot\mathbf{r}} = 4\pi\sum_{\nu=0}^{\infty} i^{\nu} j_{\nu}(kr)\sum_{m=-\nu}^{\nu} Y_{\nu}^{\mu}(\theta,\phi)Y_{\nu}^{\mu}(\theta_y,\phi_y)^* \tag{11}$$

where j_ν is the spherical Bessel function of order ν. The expansion of a plane wave scattered by a rigid sphere is given by

$$H(k,\mathbf{x},\hat{\mathbf{y}}) = 4\pi\sum_{\nu=0}^{\infty} i^{\nu}\left(j_{\nu}(kr) - \frac{j_{\nu}'(kr_0)}{h_{\nu}'(kr_0)}h_{\nu}(kr)\right)\sum_{m=-\nu}^{\nu} Y_{\nu}^{\mu}(\theta,\phi)Y_{\nu}^{\mu}(\theta_y,\phi_y)^* \tag{12}$$

Note that j_ν' is the derivative of the spherical Bessel function, h_ν and h_ν' are the spherical Hankel function and its derivative, respectively, and r_0 is the radius of the rigid sphere. The additional term $\frac{j_{\nu}'(kr_0)}{h_{\nu}'(kr_0)}h_{\nu}(kr)$ represents the scattered field. Equation (12) can be regarded as an acoustic transfer function.

2.3 Calculation of the density of the integral operator

In this work we focus on the beamforming for the rigid sphere microphone array of radius r_0. The density $a(\hat{\mathbf{y}})$ is computed analytically by the equation (2). The transfer function matrix H, the sound field p, and the density $a(\hat{\mathbf{y}})$ can be expanded in a spherical harmonic series as given by

$$\begin{aligned} p_T(\mathbf{r}) &= \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} P_{\nu\mu}Y_{\nu}^{\mu}(\theta,\phi) \\ a(\hat{\mathbf{y}}) &= \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} A_{\nu\mu}Y_{n}^{\mu}(\theta_y,\phi_y) \end{aligned} \tag{13}$$

After some mathematical manipulation, the density of the integral operator can be computed from the following formula

$$a(\hat{\mathbf{y}}) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\frac{P_{\nu\mu}}{4\pi i^{\nu}\left[j_{\nu}(kx) - \frac{j_{\nu}'(kr_0)}{h_{\nu}'(kr_0)}h_{\nu}(kx)\right]}Y_{\nu}^{\mu}(\theta_y,\phi_y) \tag{14}$$

In case that we can measure sound pressure on the sphere continuously, $P_{\nu\mu}$ can be computed analytically from (7). However in real applications the array is constituted by a finite number of sensors. This means that $P_{\nu\mu}$ is computed from a discretized version of equation (12). The equation (14) is also reported by Rafaely[10].

3 BEAMFORMER BASED ON THE NUMERICAL SOLUTION

The acoustic transfer function can be computed analytically if the array conforms to one of the geometry for which the wave equation can be solved by separation of variables. For arbitrary shaped arrays, analytical solutions are usually not available. However, the calculation can performed using the BEM. The analytical acoustic transfer function matrix $\mathbf{H}$ and numerical acoustic transfer function matrix $\mathbf{H}_{BEM}$ originate from the discretization of equation (2) and are computed from equation (12) and using the BEM respectively. In this work, we used the open source code MATLAB based software *OpenBEM* which is based on the PhD thesis by Peter Juhl[7] and is available for download at *http://www.openbem.dk*.

3.1 Discrete version of the plane wave superposition

The number of directions of arrival $\hat{\mathbf{y}}$ is truncated to L directions. Therefore the discrete version of (2) is given by

$$p_T(\mathbf{x}_m) = \sum_{l=1}^{L} H(k, \mathbf{x}_m, \hat{\mathbf{y}}_l) a(\hat{\mathbf{y}}_l) \Delta\Omega(\hat{\mathbf{y}}_l) \tag{15}$$

Assuming that we have M transducers we can write the following system of linear equations

$$\begin{Bmatrix} p_T(\mathbf{x}_1) \\ p_T(\mathbf{x}_2) \\ \vdots \\ p_T(\mathbf{x}_M) \end{Bmatrix} = \begin{bmatrix} H(ik\mathbf{x}_1 \cdot \hat{\mathbf{y}}_1) & H(ik\mathbf{x}_1 \cdot \hat{\mathbf{y}}_2) & \cdots & H(ik\mathbf{x}_1 \cdot \hat{\mathbf{y}}_L) \\ H(ik\mathbf{x}_2 \cdot \hat{\mathbf{y}}_1) & H(ik\mathbf{x}_2 \cdot \hat{\mathbf{y}}_2) & \cdots & H(ik\mathbf{x}_2 \cdot \hat{\mathbf{y}}_L) \\ \vdots & \vdots & \ddots & \vdots \\ H(ik\mathbf{x}_M \cdot \hat{\mathbf{y}}_1) & H(ik\mathbf{x}_M \cdot \hat{\mathbf{y}}_2) & \cdots & H(ik\mathbf{x}_M \cdot \hat{\mathbf{y}}_L) \end{bmatrix} \times \begin{Bmatrix} \Delta\Omega(\hat{\mathbf{y}}_1) a(\hat{\mathbf{y}}_1) \\ \Delta\Omega(\hat{\mathbf{y}}_2) a(\hat{\mathbf{y}}_2) \\ \vdots \\ \Delta\Omega(\hat{\mathbf{y}}_L) a(\hat{\mathbf{y}}_L) \end{Bmatrix}$$

which in matrix form is

$$\mathbf{p} = \mathbf{Ha} \tag{16}$$

Note that the coefficients $\Delta\Omega_S(\hat{\mathbf{y}}_s)$ that arise from the discretization of the integration domain Ω_S, are included in the vector $\mathbf{a}$. If the number of the sensors equals the number of directions of arrival of plane waves, $\mathbf{H}$ is a square matrix. This means the exact solution can be obtained if $\mathbf{H}$ has full rank and $\mathbf{a}$ can be computed by

$$\mathbf{a} = \mathbf{H}^{-1}\mathbf{p} \tag{17}$$

However if the number of the sensors does not equal the number of directions of arrival of plane waves, the estimated density vector denoted by $\tilde{\mathbf{a}}$ can be computed from

$$\tilde{\mathbf{a}} = \mathbf{H}^{\dagger}\mathbf{p} \tag{18}$$

where $\dagger$ indicates the pseudo-inverse matrix of $\mathbf{H}$. In order to compute the matrix $\mathbf{H}$ numerically from equation (12), we need to truncate the order of the spherical harmonic expansion to the order $N \leq \sqrt{M} - 1$ according to a study by Rafaely[10]. This approximation will be used throughout this work.

3.2 Inversion of the matrix $\mathbf{H}$

As discussed in section 2.1 it is not feasible to obtain a discretized aperture function or weighting vector $\mathbf{w}$ that completely satisfies equation (5). In this work we propose to calculate the weighting vector $\mathbf{w}$ as one of the rows of the pseudo-inverse (or inverse) matrix $\mathbf{H}^{\dagger}$ (or $\mathbf{H}^{-1}$). It is useful at this stage to study the orthogonality between the matrix $\mathbf{H}$ and its pseudo-inverse $\mathbf{H}^{\dagger}$. The orthogonality matrix is computed by

$$\mathbf{H}_{ortho} = \mathbf{H}^{\dagger} \times \mathbf{H} \tag{19}$$

and is plotted in fig 2, for different values of M (number of sensors) and L (number of directions of arrival).

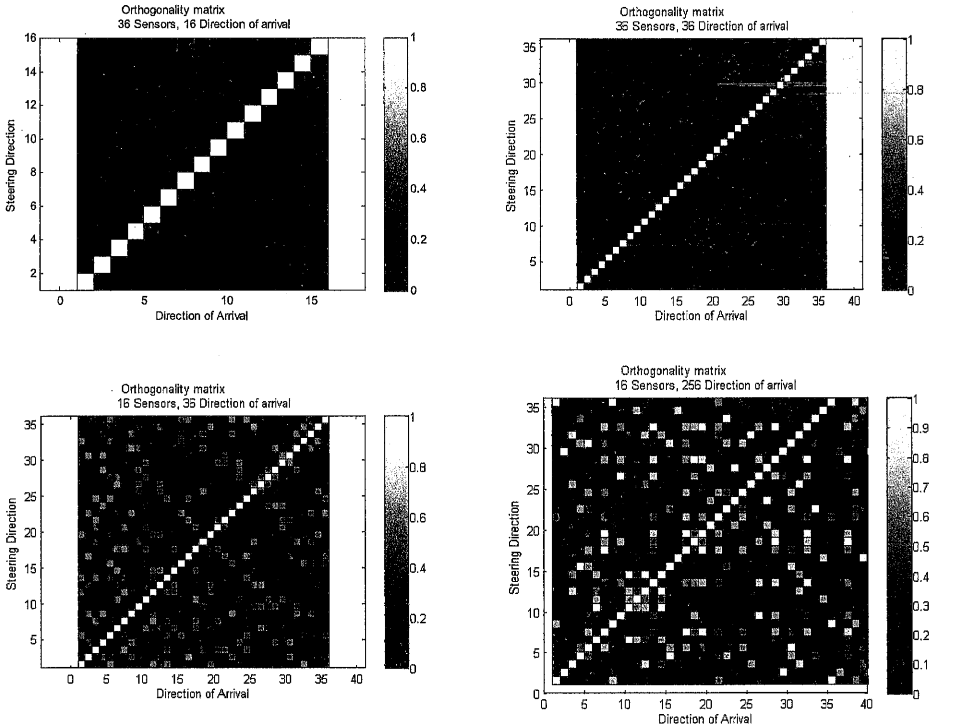

Fig 2. Orthogonality matrix (a) 36 sensors x16 directions (b) 36 sensors x 36 directions (c) 16 sensors x 36 directions and (d) 16 sensors x 256 directions (first 36x40 values)

A black cell in the figure above indicates that the corresponding row of matrix $\mathbf{H}^{\dagger}$ and column of matrix $\mathbf{H}$ are orthogonal. Conversely, a white cell indicates that the elements corresponding to such row and column are parallel. When the number of the acoustic transducers is greater or equal than the number of source directions, the orthogonality matrix is the diagonal matrix as shown in fig 1(a) and 1(b). However, as expected, if the number of the transducers is smaller than the number of the directions, the system of linear equations (16) is underdetermined and the orthogonality matrix is not diagonal as displayed in fig 1(c) and 1(d). This means that equation (5) does not hold in this case and the beamformer will be similar to a sinc function on the sphere.

3.3 Error between analytical and Numerical calculation of the matrix $\mathbf{H}$

As mentioned above, the acoustic transfer function matrix for a rigid sphere was computed numerically and by means of the BEM. 386 nodes were used in the calculation of this matrix, which is hereafter referred to as the matrix $\mathbf{H}_{BEM}$. The error matrix $\mathbf{H}_{err}$ that shows the error between $\mathbf{H}$ and $\mathbf{H}_{BEM}$ is computed by

$$\mathbf{H}_{err} = 20\log_{10}\left|\mathbf{H} - \mathbf{H}_{BEM}\right| \quad (20)$$

The error matrices computed are shown in fig 3 to 8. The number of nodes in the calculation is 98, 386, or 1538 whilst the number of sensors is 25 or 36. It can be seen that, as expected, when the number of nodes increases the error decreases.

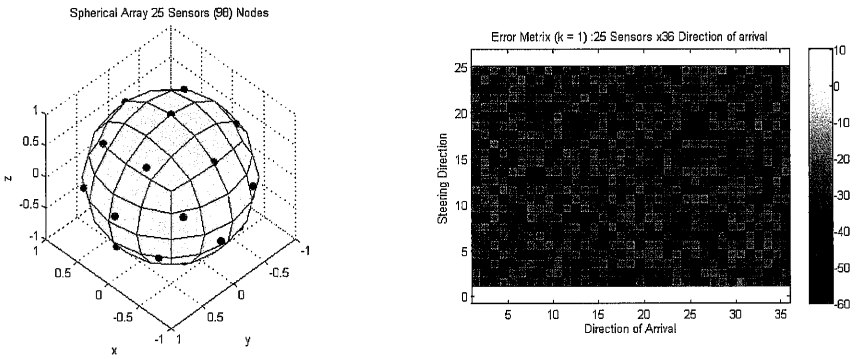

Fig 3. The error matrix of 25 sensors and 36 directions (98 nodes is used in the BEM calculation)

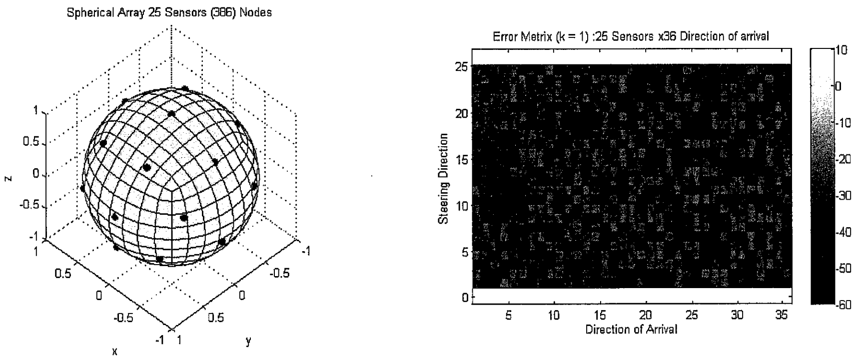

Fig 4. The error matrix of 25 sensors and 36 directions (386 nodes is used in the BEM calculation)

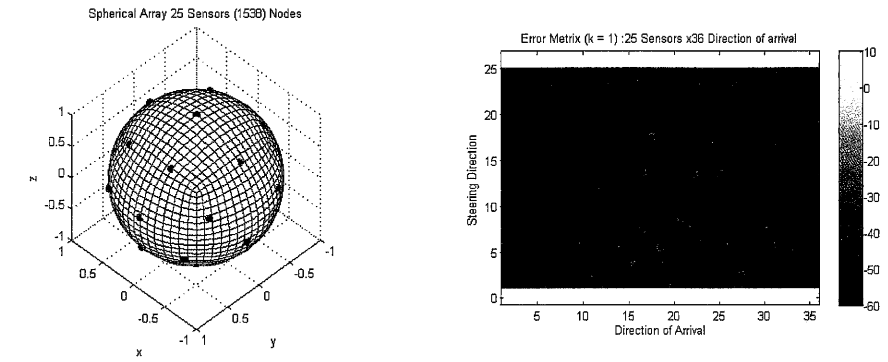

Fig 5. The error matrix of 25 sensors and 36 directions (1538 nodes is used in the BEM calculation)

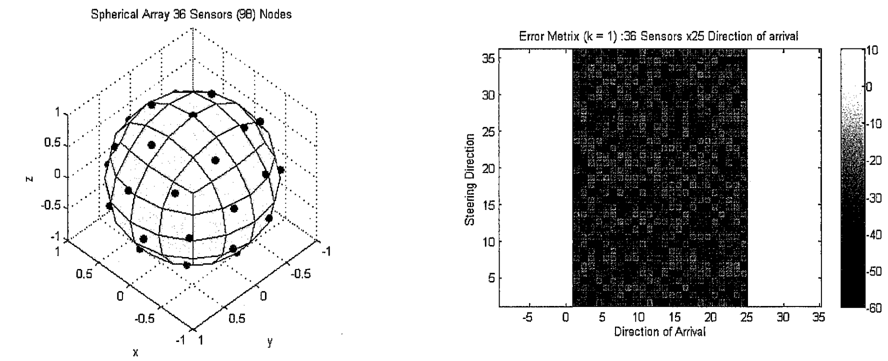

Fig 6. The error matrix of 36 sensors and 25 directions (98 nodes is used in the BEM calculation)

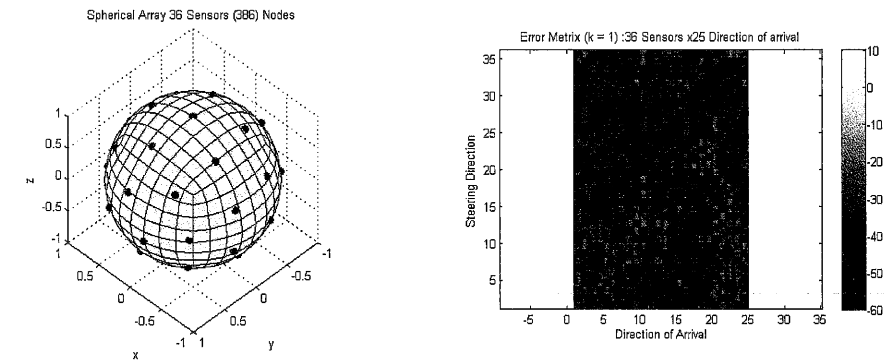

Fig 7. The error matrix of 36 sensors and 25 directions (386 nodes is used in the BEM calculation)

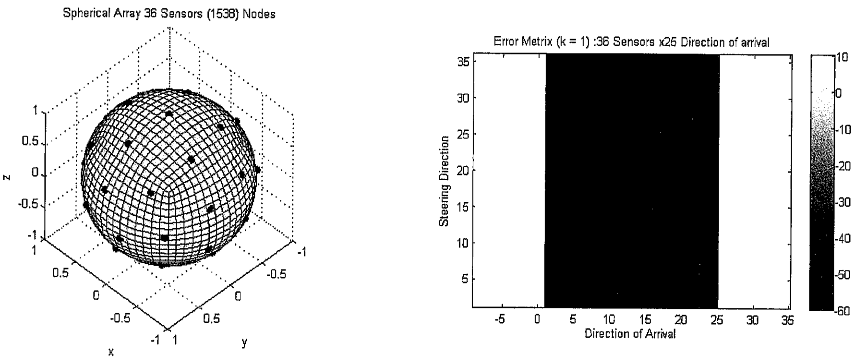

Fig 8. The error matrix of 36 sensors and 25 directions (1538 nodes is used in the BEM calculation)

4 NUMERICAL SIMULATION AND RESULTS

In this section, the results of numerical simulations are presented. The beamformer for a rigid spherical array is designed based on a BEM calculation and then the beam pattern of both analytical beamformer and BEM-based beamformer are plotted. The same approach is also applied to a rigid ellipsoidal array.

4.1 Beamformer for spherical microphone arrays

We assume that the number of microphones M is 16 and the number of arrival directions L is 36. The beamformer output, calculated from discretization of equation (3), is given by

$$z(\hat{\mathbf{y}}_{look}) = \sum_{m=1}^{M} p_T(\mathbf{x}_m) w(\mathbf{x}_m, \hat{\mathbf{y}}_{look}) * \Delta\Omega_{a,m} \tag{21}$$

The beam pattern can be generated by assuming that plane waves with unitary magnitude impinge on the array from all directions. In this simulation we assume that the measured pressure $p_T(\mathbf{x}_m)$ is given by equation (12). The weighting vector $w(\mathbf{x}_m, \hat{\mathbf{y}}_{look})$ is computed analytically from equation (12) and numerically by performing a BEM and then applied to equation (21) in order to obtain the beam pattern. Fig 5 and 6 show the beam pattern of the array at kr = 2 and 4 respectively.

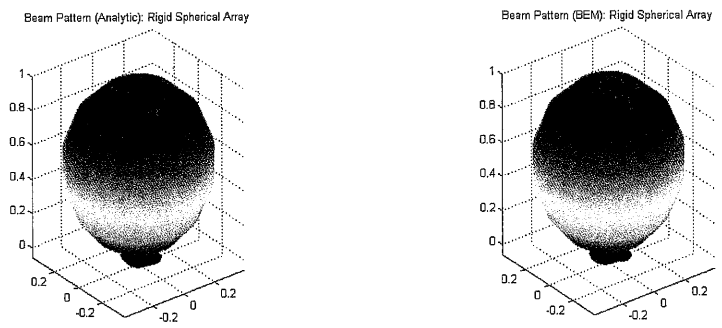

Fig 9. Beam pattern of a rigid spherical array at kr = 2: Analytical (left), BEM (right)

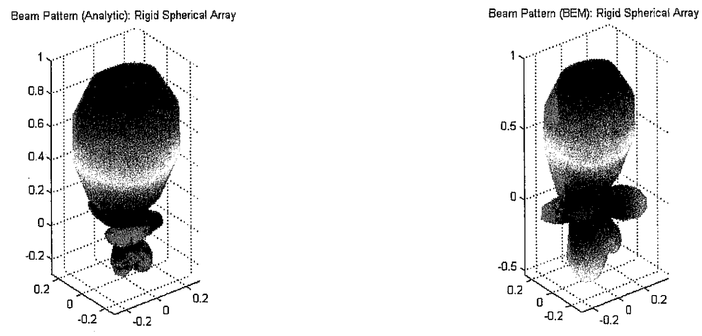

Fig 10. Beam pattern of a rigid spherical array at kr = 4: Analytical (left), BEM (right)

4.2 Ellipsoidal Array

The BEM is applied to study the beam pattern for an array that does not conform to a conventional geometry. In this study, a rigid ellipsoidal array is chosen as an example. 16 microphones are arranged on the rigid surface of an ellipsoid and 36 arrival directions are assumed for the computation of the weighting function $w(\mathbf{x}_m, \hat{\mathbf{y}}_{look})$. The ellipsoid equation in Cartesian coordinate is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad (22)$$

where the coefficients a , b ,and c are the length of the semi-principal axes. In this simulation we chose $b = 0.8a$ and $c = 0.6a$. The simulation is repeated as we did in the previous section. However we do not compute the pressure $p_T(\mathbf{x}_m)$ analytically but we compute both the total pressure and the weighting vector $w(\mathbf{x}_m, \hat{\mathbf{y}}_{look})$ using the BEM. The beam pattern was calculated using equation (26) and is displayed in fig 7. It has been found that the beam pattern of the rigid ellipsoidal array has an asymmetrical shape and the beam width along the x, y and z axes is affected by the length of the corresponding semi-principle axes.

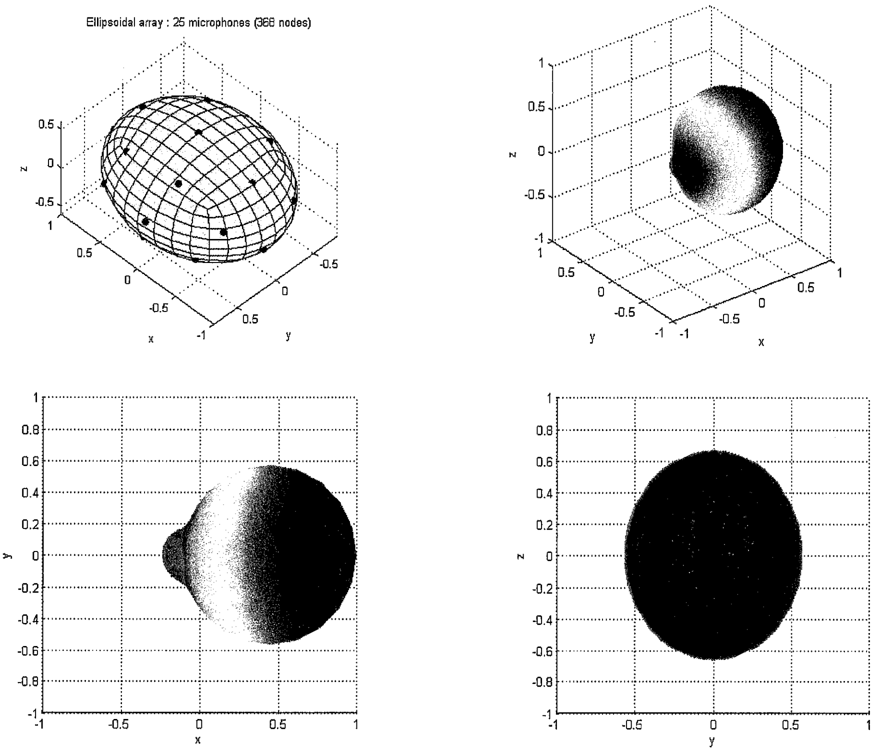

Fig 11. Beam pattern of a rigid ellipsoidal array at $kr = 4$

5 CONCLUSIONS

This paper has presented an application of the BEM to the design of a beamformer for microphone array applications. The use of the Herglotz wave function has been introduced and been expanded to a more general idea by changing the kernel of the integral operator from simple plane waves imping on a transparent array to acoustic transfer functions in which the scattering from a rigid object is included. A beamforming technique based on the inversion of the acoustic transfer function has been discussed, as well as the orthogonality property of the acoustic transfer function matrix and the errors arising from the numerical computation. The beamforming technique then has been applied to a spherical microphone array and the beam patterns of the analytical beamformer and of the BEM-based beamformer have been shown. The proposed technique has also been applied to an ellipsoidal microphone array and a representation of the beam pattern of the ellipsoidal array has been reported. It has been found that the length of the semi principle axes of the ellipsoid is related to the beam width of the ellipsoidal array in the corresponding directions. The framework developed here will be useful in the design of beamformer for microphone arrays with other unconventional shapes and more complex geometries.

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