APPLYING INVERSE METHODS TO DISTRIBUTED SOURCE REGIONS

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1 ABSTRACT

Cross-spectrum-based inverse methods can be readily applied to acoustic sources that can be adequately modelled as a number of discrete monopoles. However, there are many situations, particularly in the fields of vibroacoustics and aeroacoustics, where the sources are distributed continuously in space over a finite area (or volume). This paper is concerned with the practical problem of applying inverse methods to such distributed source regions and includes simulations of spatial sampling criteria, matrix conditioning and some suggestions for optimising source discetisation.

2 INTRODUCTION

Continuous, distributed acoustic sources are commonplace in the fields of vibroacoustics and aeroacoustics. Vibroacoustics is concerned with the radiation or reception of sound by structures which often consist of shells or plates which support complex and spatially-dependent vibration fields. Many aeroacoustic sources are similarly distributed in space. For simple source distributions, which can be described in idealised coordinate systems, analytical models of sound radiation can be used to predict the radiated sound field or used as a source model for inverse problems. However, in general, the source region has to be divided into discrete elemental areas over which the source field is assumed to be uniform, each element is then replaced by a simple source, such as a monopole or dipole, of equivalent strength. For this simplification to be valid, the size and position of the elements must be carefully chosen to minimise errors.

3 SPATIAL SAMPLING IN THE FORWARD PROBLEM

There is a fundamental difference between the required spatial sampling of source regions for the forward problem, where a known source distribution is used to estimate the radiated sound field, and the inverse problem, of interest in this paper, where a known sound field is used to estimate the source distribution that generates it. In the forward problem, not all components of the source distribution may contribute to the radiated sound field. For example, a plate may vibrate with a surface wavelength that is small compared with an acoustic wavelength at the same frequency. In this case, only parts of the vibration field that are near to the edges of the plate contribute significantly to the radiated (far) field - the radiation from the rest of the plate being surpressed by nearby out-of-phase source regions. If the vibration field is measured at an insufficient number of discrete points, the estimate of the radiated sound field will be in error due to spatial aliasing of the vibration field. Thus spatial sampling of the source region in the forward problem must satisfy both an acoustic criterion and a source distribution one.

Work on the applications of vibroacoustic reciprocity [1, 2] has shown that errors due to spatial aliasing of the vibration field can be very significant when dealing with the sound radiated by complex structures, and that they can be minimised if spatial averaging of the vibration field is carried out. Figure 1 shows the results of a computer simulation of the sound field radiated by a thin vibrating plate excited by a point force. In this simulation, the sound field is estimated from a knowledge of the vibration of the plate and compared with a mode-based analytic solution of the same (see [1]). The plate is divided into a number of elements with centres spaced 80mm apart — about half an acoustic wavelength at 2kHz — and the sound field estimated using a discrete approximation to the Rayleigh Integral,

$$p_m = j\,\frac{\omega\rho_0}{2\pi}\sum_n \frac{u_n\,S_n}{r_{m,n}}\,e^{-j\omega r_{m,n}/c_0}\ , \qquad (1)$$

where $r_{m,n}$ is the distance from the centre of the elemental area n to the far-field sensor point m, and S_n is the area of element n having vibration velocity u_n, ρ_0 and c_0 are the static density and speed of sound and ω is the angular frequency of interest.

In Figure 1, the mean error in the estimate of the radiated sound field when the vibration is assumed to be known only at points at the centre of each element is compared to that when the vibration field is integrated over each element prior to application of Equation (1). The point sampled estimate shows very high errors even though the sample spacing is within the normally-accepted Nyquist criterion of 2 samples per acoustic wavelength. The reduction in error when the point-sampled vibrations are replaced by area-integrated ones demonstrates that most of the error is due to spatial aliasing of the vibration (source) field. If these errors in the forward problem are to be minimised, the spatial details of the source field, as well as the acoustic field need to be taken into account in any source sampling decisions.

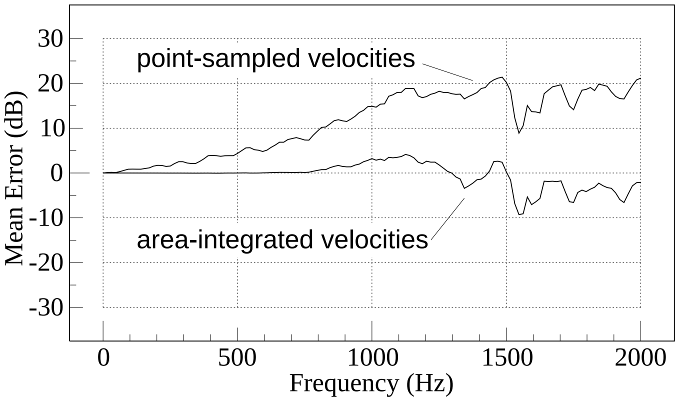


Figure 1 Mean Error in the Estimate of the Sound Field Radiated by a Thin Vibrating Plate Excited by a Point Force using Point−Sampled and Area-Integrated Vibration Velocities Spaced 80mm Apart

4 SPATIAL SAMPLING IN THE INVERSE PROBLEM

In contrast to the forward problem, when applying inverse methods to distributed source problems, both the spatial sampling of the sensors and the assumed source positions have to be taken into account. However, in many cases it may be accepted at the outset that non-radiating source components are of no interest, as they may not contribute to the sound field at the sensors. In these cases only acoustic spatial sampling criteria need be applied and only those source components that contribute to sound radiation are then quantified. This section details some computer simulations carried out in order to identify the source sampling criteria required for application of the inverse method to different types of source regions.

4.1 Continuous Line Sources

Consider a line source of length L positioned at the centre of a polar array consisting of 37 sensors covering 180 degrees at 5 degree increments along an arc of radius $100L$. The line source is aligned with the $-90^{\bullet}$ and $+90^{\bullet}$ receivers as shown in Figure 2.

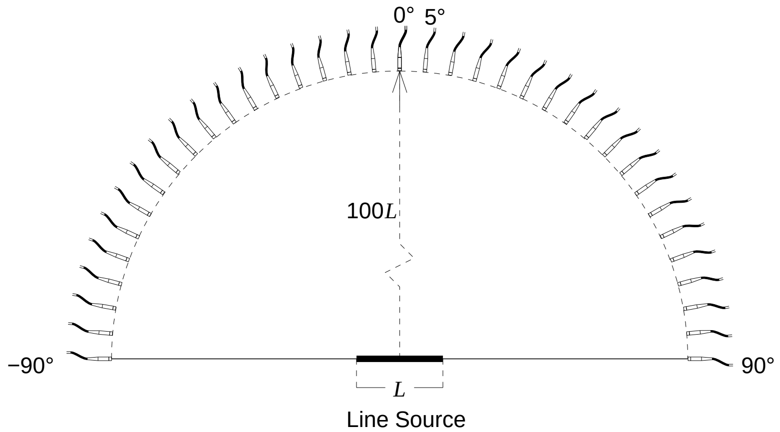


Figure 2 Geometry for Line Source Simulations

The radiation from the continuous line source is calculated by approximating the line by 1000 point monopoles, the amplitude, phase and mutual coherence of which are set depending on the source field of interest and used to generate a source spectral matrix $\boldsymbol{S}_{\hat{q}\hat{q}}$. The matrix of pressure cross spectra at the sensors is then

$$\boldsymbol{S}_{\hat{p}\hat{p}} = \boldsymbol{G}\,\boldsymbol{S}_{\hat{q}\hat{q}}\,\boldsymbol{G}^{\mathrm{H}} , \tag{2}$$

where the superscript $^{\mathrm{H}}$ refers to the Hermetian (conjugate) transpose and $\boldsymbol{G}$ is the matrix of (free-space) Green functions linking each monopole to each receiver according to their relative positions, thus:

$$g_{m,n} = j\,\frac{\omega\rho_0}{4\pi}\,\frac{e^{-j\omega r_{m,n}/c_0}}{r_{m,n}} , \tag{3}$$

The purpose of this simulation is to investigate the errors associated with discretisation of the continuous source. To this end, the line source is divided into a small number of elemental sources and the inverse method is applied to the pressure cross spectra defined in Equation (2) to yield an estimate of a simplified source cross-spectral matrix

$$\tilde{\boldsymbol{S}}_{\hat{q}\hat{q}} = \tilde{\boldsymbol{G}}^{+} \boldsymbol{S}_{\hat{p}\hat{p}} \tilde{\boldsymbol{G}}^{+\mathrm{H}} , \qquad (4)$$

where $\tilde{\boldsymbol{G}}^{+}$ is the pseudo inverse of the simplified Green function matrix $\tilde{\boldsymbol{G}}$ which links the centre of each source element to each receiver. An estimate of the pressures at the sensors using the simplified source distribution is then

$$\tilde{\boldsymbol{S}}_{\hat{p}\hat{p}} = \tilde{\boldsymbol{G}} \tilde{\boldsymbol{S}}_{\hat{q}\hat{q}} \tilde{\boldsymbol{G}}^{\mathrm{H}} , \qquad (5)$$

which can be compared to the exact radiated field from Equation (2) to give a means square dB error between the exact and reconstructed pressure fields, thus:

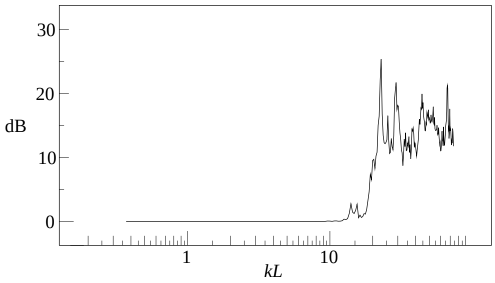
$$error = \frac{1}{M} \sum_{m=1}^{M} 10 \, log_{10} \left\{ \left| \frac{\tilde{p}_{m,m}}{p_{m,m}} \right| \right\} \text{ dB.} \qquad (6)$$

where $\tilde{p}_{m,m}$ is the mth diagonal component of $\tilde{\boldsymbol{S}}_{\hat{p}\hat{p}}$ etc., and M is the number of sensors. This error is then investigated for a number of different source element sizes and source field descriptions.

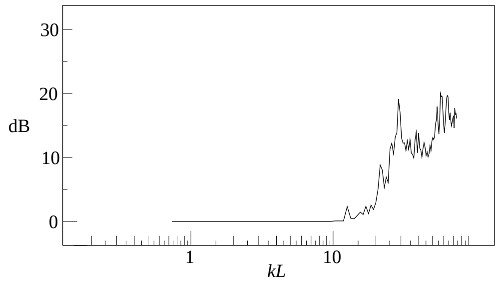
4.1.1 Deterministic Line Sources

Many real line sources have a source strength amplitude that is uniform along the length of the source but a phase distribution that corresponds to that of a travelling wave(s). Travelling waves that propagate with phase speeds much greater than the speed of sound radiate sound as if the whole line were in phase, those with speeds comparable with the speed of sound radiate sound at an angle to the line, and those having speeds much lower than the speed of sound are inefficient radiators of sound [3]. It is therefore reasonable to expect that the phase speed of the travelling wave will have a bearing on the errors introduced when the source distribution is sampled. Figures 3a—d show the dB error in the reconstructed pressure field as a function of non-dimensional frequency $kL\,(= \omega L / c_0)$ using 8 source elements and source phase speeds of $100c_0$, $2c_0$, c_0 and $c_0/2$ respectively. Figures 4a—d show the dB error in the reconstructed pressure field as a function of non-dimensional frequency for a source phase speed of $2c_0$ when the number of source elements is 2, 4, 8 and 16 respectively.

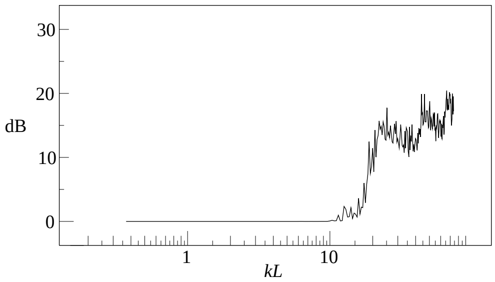
The example results shown in Figures 3 and 4, along with a number of similar results (not shown here), demonstrate that the error in the reconstructed sound field is strongly dependent upon frequency, but only weakly dependent upon the speed of the source travelling wave. There are two frequency regions on each plot separated by a fairly steep cut-off. The error in the reconstructed sound field changes from low values below the cut-off to high values above it. If the cut-off frequency is defined as that frequency where the error is 3dB, the mean value of cut-off frequency corresponds to the non-dimensional value of $\omega\delta x / c_0 \approx 2$, where δx is the length of a single element, with a range of 1·25 to 2·5. These numbers correspond approximately to wavelength-to-element length ratios of $5 > \lambda / \delta x > 2{\cdot}5$, with a mean of about 3 elements per wavelength. It is interesting to note that the lower figure accords with the Nyquist criterion for sampling waveforms and the higher figure the minimum number of elements-per-wavelength recommended for numerical (BEM, FEM) acoustics modelling.



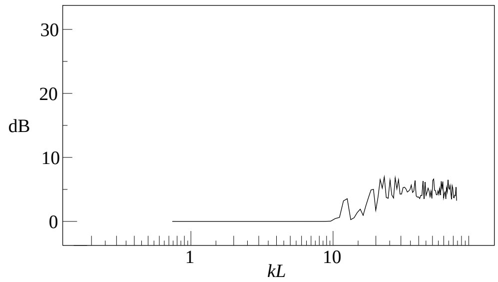
a) Phase Speed $c_x = 100c_0$



b) Phase Speed $c_x = 2c_0$

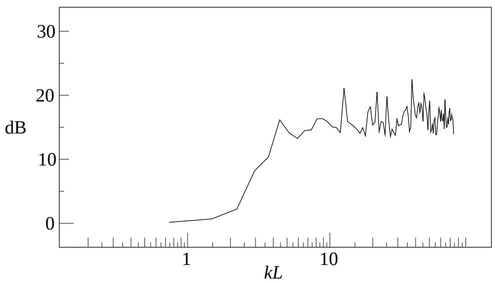


c) Phase Speed $c_x = c_0$

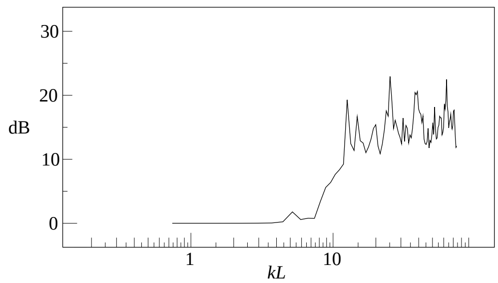


d) Phase Speed $c_x = c_0/2$

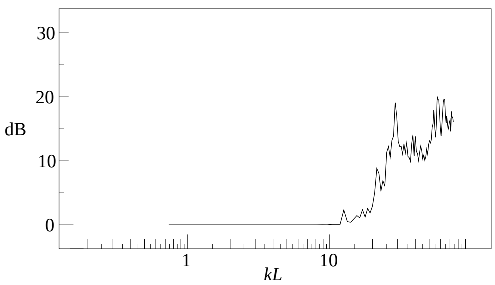
Figure 3 dB error in Reconstructed Pressure for 8 Source Elements for Various Source Travelling Wave Phase Speeds



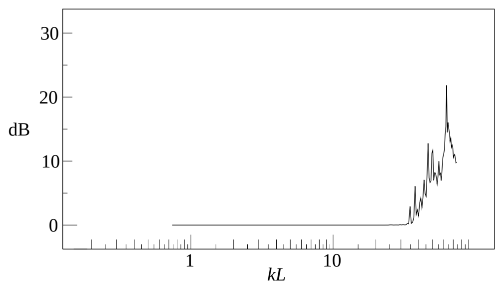
a) 2 Source Elements



b) 4 Source Elements



c) 8 Source Elements



d) 16 Source Elements

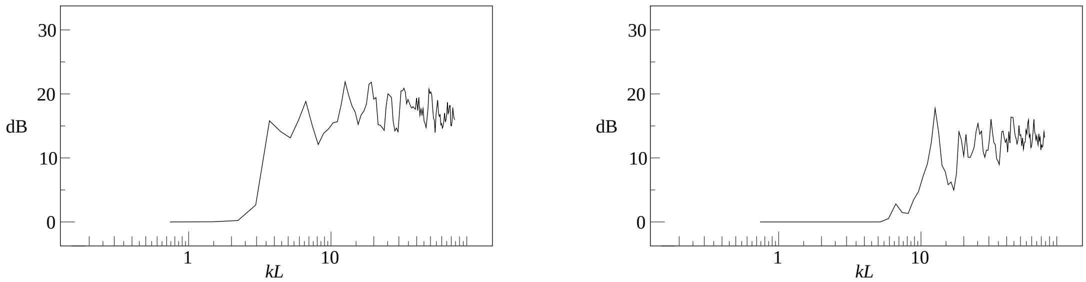
Figure 4 dB error in Reconstructed Pressure for a Source Travelling Wave Speed of $2c_0$ *for Various Numbers of Source Elements*

4.1.2 Stochastic Line Sources

Many aeroacoustic sources can be modelled as a stochastic line source. In this case, the correlation length of the source may be short compared to the length of the source, or even zero. For this type of source distribution, the source cross-spectral matrix takes the form of a diagonal band, the width of which is determined by the correlation length (note width of one for zero correlation length). Simulations similar to those in section 4.1.1 show that the cut-off frequency observed above for deterministic sources is independent of correlation length and that the error above cut-off increases with increasing correlation length. The spatial sampling criterion identified above for deterministic line sources can therefore also be applied to stochastic line sources.

4.2 Continuous Plane Surface Sources

In this section, the simulations detailed above for line sources are extended to plane (2 dimensional) source distributions. Figures 5a and 5b shows the error in the reconstructed sound field as a function of non-dimensional frequency when a square source surface of side length L is divided into 4 and 16 source elements respectively. The source distribution is deterministic with wave speeds across the plane of $100c_0$.



a) 4 Source Elements (2 × 2) *b) 16 Source Elements (4 × 4)*

Figure 5 dB Error in Far-Field Sound Pressure for a Continuous Deterministic Plane Surface Source with $c_x = c_y = 100c_0$ for different numbers of source elements.

A comparison between Figure 5 and Figures 3 and 4 shows that the spatial sampling criterion identified for line sources can also be applied to plane surface sources.

5 MATRIX CONDITIONING AND SPATIAL SAMPLING

All of the simulations presented so far have been the results of processing perfect, noise- and error-free data limited only by machine precision. In reality, data taken from measurements will contain uncertainties, and the importance or otherwise of these uncertainties can be estimated by considering the condition number of the Green function matrix which is inverted [4, 5, 6]. Figure 6 shows the condition number of the Green function matrix for the continuous plane source simulation in Section 4.2 above for the 16 source element case.

A comparison between Figures 5b and 6 show that the reduction in error in the reconstructed sound field at low frequencies is accompanied by a rise in the condition number of the Green function matrix. For a given source element size, there is a narrow range of frequencies over which the errors due to spatial aliasing are small and the condition numbers are low. If the source elements are too small, the individual Green functions in the matrix will be too similar to each other, the condition number will be high, and the success of the inversion process will be sensitive to uncertainties in the data. If the source elements are too large, errors in the reconstruction of the radiated pressure field will occur due to spatial aliasing. Applying the 'safe' spatial sampling criterion of $k\delta x = 1{\cdot}25$ from Section 4 leads to condition numbers of greater than 1000 in this case, which may be too high for use with real data; however, applying $k\delta x = 2{\cdot}5$ (equivalent to 2·5 elements per wavelength) leads to more acceptable condition numbers.

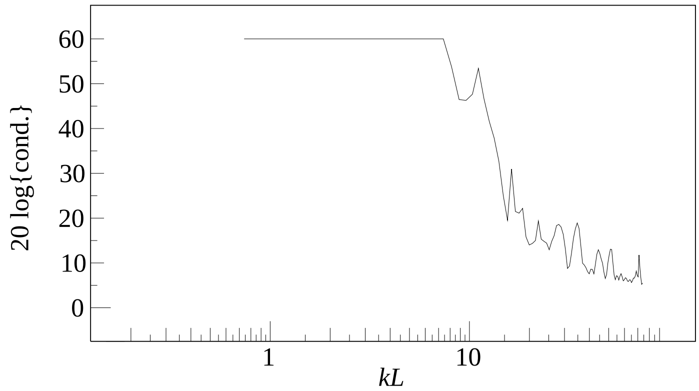


Figure 6 Condition Number of the Green Function Matrix for the Simulated Plane Surface Source with 16 Source Elements

6 OPTIMISING SOURCE SPACING

In order to minimise the errors associated with data uncertainty and spatial aliasing, the size of the source elements must be optimised for the particular frequency of interest. If a large number of small elements are chosen to minimise spatial aliasing at high frequencies, the requirements for a smaller number of larger elements at low frequencies can be realised either by grouping elements together, or by improving the conditioning of the Green function matrix using regularisation or by discarding small singular values [4, 5, 6].

7 CONCLUSIONS

- Computer simulations of the errors associated with the application of inverse methods to continuous distributed sources have revealed criteria for choosing the optimum element size.
- Investigations into deterministic and stochastic line sources and deterministic plane sources have shown that for the inverse problem, the optimum spatial sampling is largely independent of the source structure or geometry.
- Consideration of the sensitivity of the matrix inversion process to uncertainties in data via condition numbers indicates that the source element size must be optimised for the frequency of interest.
- The optimum spatial sampling criterion for the examples investigated here is 2·5 elements per wavelength.

8 REFERENCES

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9 ACKNOWLEDGEMENT

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