ACOUSTIC DECOY USING GRADED METASURFACE: ANOMALOUS SCATTERING OF LOW-FREQUENCY SOUND

L. Schwan Acoustics Research Centre, University of Salford, U.K.
O. Umnova Acoustics Research Centre, University of Salford, U.K.
C. Boutin ENTPE, Université de Lyon, LGCB, UMR CNRS 5513, France

1 INTRODUCTION

While the theory of homogenization for continuous periodic media is well established since the late 1970s[1,2] and applications of media with inner resonance can be found as early as 1985 in elastodynamics[3], the use of micro-structured composites spread widely in most fields of physics only since the early 2000s under the attractive name "meta-materials". Indeed, a (periodic) array of micro-structures can be described as an homogenous medium with effective material properties, provided that the wavelength strongly exceeds the size of the Representative Elementary Volume (REV) of the array (condition of scale separation). In case of resonant micro-structures, the effective properties can reach unconventional values leading, for instance, to bandgaps in the low-frequency range[3-5]. The fundamental equations of physics can be adapted for materials with such exotic properties: Newton's Law[6], Snell-Descartes' Law[7], and others. Note however that they are apparent properties at the macroscopic scale of heterogeneous media and do not call the classic form of fundamental principles into question, whether at the local scale of heterogeneous media or for homogeneous media.

By spatially-grading effective properties, the so-called invisibility cloaks[8] and devices for wavefront manipulation[7,9,10] could be realized. However, a dichotomy usually exists in the literature between: (1) the continuous but graded effective properties; and (2) the "micro"-structures designed considering (numerically) the reflection/transmission of a plane wave in normal incidence on a surface array assuming its strict in-plane periodicity[9,10]. This raises the issue of the effects that the grading and the angle of incidence may have on the effective properties. The former means that, strictly-speaking, the periodicity is lost, while the latter might be of particular interest when considering anomalous reflection. Another issue in the so-called Generalized Snell-Descartes's Law[7] concerns the amplitude of the waves reflected from a graded surface and the influence of the resonators' damping since it accounts only for phase discontinuities.

In this paper, we report the homogenization of a graded surface array (graded metasurface) by means of the method of two-scale asymptotic homogenization. In this approach, the restrictions of the strict periodicity and plane-wave in normal incidence disappear: the method requires only a local periodicity by which local fields are periodic at a certain degree of approximation while varying at the macroscopic scale. However, in the usual homogenization approach[2,11], a strict material periodicity is assumed while the macroscopic fluctuations are issued from the quasi-periodic fields. In the present case, a material quasi-periodicity is assumed in addition to the quasi-periodicity of the fields, providing nonetheless an overall quasi-periodicity in the system, which allows its homogenization. In particular, this material quasi-periodicity results from a geometrical and/or rheological modulation of its generic cell over long distances, i.e. of the order of the wavelength. The homogenization of the Long-Scale Graded Metasurface (LSGM) leads to the definition of an effective surface admittance, graded over long-scale distances, and which explicitly accounts for the resonators behavior (including the damping). The design of the admittance is shown to enable anomalous scattering.

In Section 2, the homogenization of the LSGM is performed using the two-scale asymptotic scheme. In Section 3, the example of anomalous scattering from a cylinder is studied making an acoustic decoy out of it.

2 TWO-SCALE ASYMPTOTIC HOMOGENIZATION

In this section, the propagation of small airborne acoustic perturbations is studied in the presence of a scattering object, the boundary of which is covered with a LSGM. The analysis is performed under the ambient conditions (atmospheric pressure $P_e = 1.013 \times 10^5$Pa, density $\rho_e = 1.2$ kg/m^3, adiabatic constant$\gamma = 1.4$, sound speed $c = \sqrt{(\gamma P_e/\rho_e)} \approx 343.8$ m/s) for harmonic waves with the angular frequency ω (time dependence $e^{-i\omega t}$).

2.1 Long-Scale Graded Metasurface under scale separation

The LSGM consists of the two-dimensional arrangement (in an otherwise three-dimensional space) of volume cells Ω_n, the centroids C_n of which are located at the points x_n included in the surface $\mathcal{S}$ (unit normal vector $\mathcal{N}$), see Figure 1. Each cell Ω_n includes a rigid backing (boundary Γ_n^0) and $s = 1..N_s$ linear acoustic resonators (boundary Γ_n^s), the nature of which is not specified at this stage. They can be Helmholtz resonators, quarter-wavelength tubes folded in a bulky shape or others.

The scale separation between the cells arrangement and the acoustic field is assumed. If ℓ_n is the characteristic size of the cell Ω_n and $\ell = \max \ell_n$ is the largest cell size, then the sound wavelength $\lambda = 2\pi c/\omega$ is supposed to be much larger than ℓ. The scale separation is quantified by the small scale parameter $\epsilon = 2\pi\ell/\lambda \ll 1$. Local scale parameters $\epsilon_n = 2\pi\ell_n/\lambda < \epsilon \ll 1$ can also be defined for each cell. The scale separation implies the existence of two characteristic lengths in the system: the cells characteristic size ℓ and the reduced wavelength $L = \lambda/(2\pi)$. As a result, the usual space variable x can be normalized by either of them, providing the two variables: $\overline{x} = x/L$ (the macroscopic/long-scale variable) and $\overline{y} = x/\ell$ (the microscopic/local variable).

The metasurface is graded over long-scale distances: it means that both the geometry and the properties of the structures (rigid backing and resonators) in the cells Ω_n are modulated over distances of the order of the reduced wavelength $\mathcal{O}(L)$. Likewise, the surface $\mathcal{S}$, to which the centroids C_n belong, can display a curvature with a long radius $\mathcal{O}(L)$. Those long-scale modulation implies that, locally, the cell arrangement can be considered periodic with a good approximation with the cell Ω_n playing the role of a *local REV*, while, at the long-scale, the periodicity is disturbed. This overall quasi-periodicity in the system in the presence of a macroscopic field allows the use of two-scale asymptotic homogenization methods.

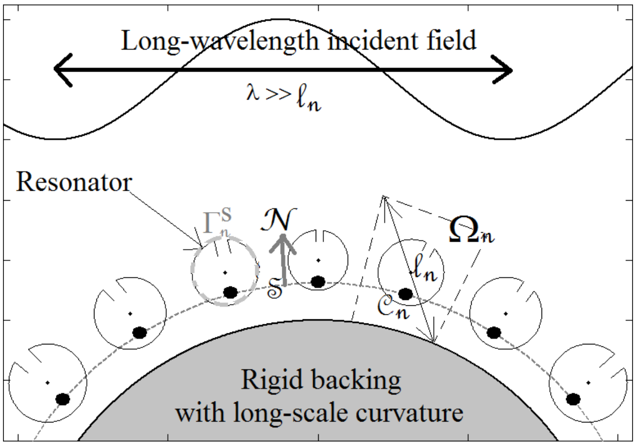

Figure 1: A long-wavelength field is incident on a long-scale graded metasurface arranged at the vicinity of the surface of a scatterer with a long-scale curvature. The cells Ω_n are shown here with a single Helmholtz resonator in it. The long-scale modulation of the metasurface concerns in this figure: the spacing between the resonators, the radius of the resonators, and the inner-dynamics of the resonator (width and length of the resonator neck).

2.2 Long-wavelength field and boundary layer field

Another particularity of the present study, compared to the classic homogenization of bulk heterogeneous media, is the loss of (quasi-)periodicity in the normal direction of the metasurface. The homogenization is then performed by means of a boundary layer[12,13]. Indeed, when the LSGM is submitted to an incident wave with the pressure p_i and particle velocity v_i which vary significantly over distances $\mathcal{O}(L)$, the field scattered by the LSGM can be decomposed in two parts: the field (p_s, v_s), which varies significantly over long distances $\mathcal{O}(L)$; and the field $(p^\star, v^\star)$ which varies over short distances $\mathcal{O}(\ell)$ while being modulated over long distances $\mathcal{O}(L)$. The field (p_s, v_s) is the coherent field that prevails at distances $\mathcal{O}(L)$ from the metasurface, while the field $(p^\star, v^\star)$ is made of evanescent surface waves which fade away at distances $\mathcal{O}(\ell)$ from the metasurface and hence form a Boundary Layer (BL). In the following, the superposition $p = p_i + p_s$ and $v = v_i + v_s$ of the coherent field (p_s, v_s) and the incident field (p_i, v_i) is called the Long Wavelength (LW) field. Both LW and BL fields satisfy the equations of mass and momentum conservation:

$$\operatorname{div} v = i\omega p/(\gamma P_e) \;\; ; \;\; i\omega\rho_e v = \operatorname{grad} p \tag{1a}$$

$$\operatorname{div} v^\star = i\omega p^\star/(\gamma P_e) \;\; ; \;\; i\omega\rho_e v^\star = \operatorname{grad} p^\star \tag{1b}$$

while their superposition must satisfy the boundary conditions at the boundary Γ_n^s (outward normal vector $\mathbf{n}_n^s$) of each structure $s = 0..N_s$ (rigid backing and resonators) in each local REV Ω_n. Those boundary conditions are described as follows. In response to the total pressure $p + p^\star$ that acts at the surface, the structure s in the local REV Ω_n produces the particle velocity $V_n^s = \mathcal{R}_n^s(p + p^\star)$ at its boundary Γ_n^s, where $\mathcal{R}_n^s$ is a linear operator that describes the *inner dynamics* of the structure. The superposition of the LW and BL particle velocity fields $v + v^\star$ at the surface Γ_n^s must balance the particle velocity V_n^s prescribed by the resonator. This is described by the following equations

$$V_n^s = \mathcal{R}_n^s(p + p^\star) \quad \text{on } \Gamma_n^s \tag{2a}$$

$$(v + v^\star) \cdot \mathbf{n}_n^s = V_n^s \cdot \mathbf{n}_n^s \quad \text{on } \Gamma_n^s \tag{2b}$$

It is assumed that the boundary layer plays a significant role in the mass conservation, i.e. the BL velocity field $v^\star$ is supposed to be of the same order as the LW velocity field v. This means that the effect of the metasurface on the scattered field is significant.

2.3 Two-scale asymptotic formulation

The method of two scale asymptotic homogenization proceeds as follows[2,11]. Firstly, the dependence of the fields on the normalized space variables $\overline{x}$ and/or $\overline{y}$ is introduced. The LW fields (p, v) and the vector $\mathcal{N}$ are described by the long-scale variable $\overline{x}$ only. The BL fields are described using both variables $\overline{x}$ (for the long-modulation) and $\overline{y}$ (for the local periodicity). The particle velocity V_n^s and the vector $\mathbf{n}_n^s$ normal to the micro-structures' boundary Γ_n^s depend on the local variable $\overline{y}$:

$$p(\overline{x})\,;\;\; v(\overline{x})\,;\;\; \mathcal{N}(\overline{x})\,;\;\; p^\star(\overline{x}, \overline{y})\,;\;\; v^\star(\overline{x}, \overline{y})\,;\;\; V_n^s(\overline{y})\,;\;\; \mathbf{n}_n^s(\overline{y})\,. \tag{3}$$

Secondly, the long-scale dependence $(\overline{x})$ of the LW and BL fields and of the normal vector $\mathcal{N}$ are downscaled to describe their variations within each cell Ω_n at subwavelength distances from the centroids $x_n = L\overline{x_n} = \ell\overline{y_n}$. This is done using Taylor expansions[6,14] around the centroid points $\overline{x_n}$:

$$p(\overline{x}) = p(\overline{x_n}) + \epsilon\,[\nabla_{\overline{x}} p(\overline{x_n})] \cdot (\overline{y} - \overline{y_n}) + \mathcal{O}(\epsilon^2\, p(\overline{x_n})) \tag{4a}$$

$$v(\overline{x}) = v(\overline{x_n}) + \epsilon[\nabla_{\overline{x}} v(\overline{x_n})] \cdot (\overline{y} - \overline{y_n}) + \mathcal{O}(\epsilon^2 v(\overline{x_n})) \tag{4b}$$

$$\mathcal{N}(\overline{x}) = \mathcal{N}(\overline{x_n}) + \epsilon[\nabla_{\overline{x}} \mathcal{N}(\overline{x_n})] \cdot (\overline{y} - \overline{y_n}) + \mathcal{O}(\epsilon^2\, \mathcal{N}(\overline{x_n})) \tag{4c}$$

$$p^\star(\overline{x}, \overline{y}) = p^\star(\overline{x_n}, \overline{y}) + \epsilon\,[\nabla_{\overline{x}} p^\star(\overline{x_n}, \overline{y})] \cdot (\overline{y} - \overline{y_n}) + \mathcal{O}(\epsilon^2 p^\star(\overline{x_n}, \overline{y})) \tag{4d}$$

$$v^\star(\overline{x}, \overline{y}) = v^\star(\overline{x_n}, \overline{y}) + \epsilon\,[\nabla_{\overline{x}} v^\star(\overline{x_n}, \overline{y})] \cdot (\overline{y} - \overline{y_n}) + \mathcal{O}(\epsilon^2 v^\star(\overline{x_n}, \overline{y})) \tag{4e}$$

where $\overline{x} = \overline{x_n} + \epsilon(\overline{y} - \overline{y_n})$ and $\nabla_{\overline{x}}$ is the *del* operator with respect to the variable $\overline{x}$. Thirdly, the local equilibrium of the system is established in each local REV Ω_n assuming that the BL fields are locally Σ_n-periodic regarding their second argument, where Σ_n is the intersection of Ω_n with the surface $\mathcal{S}$. Consequently, the equations (1b) are rescaled using the variable $\overline{y}$ of local description:

$$\epsilon^{-1}\,\nabla_{\overline{y}} \cdot v^\star = \; iL\omega p^\star/(\gamma P_e) \quad ; \quad iL\omega\rho_e v^\star = \epsilon^{-1}\,\nabla_{\overline{y}} p^\star \tag{5}$$

where $\nabla_{\overline{y}}$ is the *del* operator with respect to $\overline{y}$. Finally, the fields are expanded asymptotically in powers of $\epsilon \ll 1$ (superscripts in brackets indicate the order of the terms in the expansions):

$$p(\overline{x}) = p^{(0)}(\overline{x}) + \epsilon\, p^{(1)}(\overline{x}) + \epsilon^2 p^{(2)}(\overline{x}) + \cdots \tag{6a}$$

$$v(\overline{x}) = v^{(0)}(\overline{x}) + \epsilon\, v^{(1)}(\overline{x}) + \epsilon^2 v^{(2)}(\overline{x}) + \cdots \tag{6b}$$

$$\mathcal{N}(\overline{x}) = \mathcal{N}^{(0)}(\overline{x}) + \epsilon\, \mathcal{N}^{(1)}(\overline{x}) + \epsilon^2 \mathcal{N}^{(2)}(\overline{x}) + \cdots \tag{6c}$$

$$p^\star(\overline{x}, \overline{y}) = p^{\star(0)}(\overline{x}, \overline{y}) + \epsilon\, p^{\star(1)}(\overline{x}, \overline{y}) + \epsilon^2 p^{\star(2)}(\overline{x}, \overline{y}) + \cdots \tag{6d}$$

$$v^\star(\overline{x}, \overline{y}) = v^{\star(0)}(\overline{x}, \overline{y}) + \epsilon\, v^{\star(1)}(\overline{x}, \overline{y}) + \epsilon^2 v^{\star(2)}(\overline{x}, \overline{y}) + \cdots \tag{6e}$$

$$V_n^s(\overline{y}) = V_n^{s(0)}(\overline{y}) + \epsilon\, V_n^{s(1)}(y) + \epsilon^2 V_n^{s(2)}(y) + \cdots \tag{6f}$$

The asymptotic and Taylor expansions (6) and (4) are substituted back into equations (5) of mass and momentum conservation and in boundary conditions (2). Terms of the same orders are collected to result in problems that can be solved successively in increasing order of powers of ϵ.

2.4 Homogenization: effective metasurface admittance

At the leading order ϵ^0, the problem to solve reads:

$$\nabla_{\overline{y}} \cdot v^{\star(0)} = 0 \quad (7a)$$

$$\nabla_{\overline{y}} p^{\star} = 0 \quad (7b)$$

$$V_n^{s(0)} = \mathcal{R}_n^s\left(p^{(0)} + p^{\star(0)}\right) \quad \text{on } \Gamma_n^s \quad (7c)$$

$$(v^{(0)} + v^{\star(0)}) \cdot \mathbf{n}_n^s = V_n^{s(0)} \cdot \mathbf{n}_n^s \quad \text{on } \Gamma_n^s \quad (7d)$$

where $p^{\star(0)}(\overline{x},\ \overline{y})$ and $v^{\star(0)}(\overline{x},\ \overline{y})$ are Σ_n-periodic in $\overline{y}$ and fade away as $\overline{y} \cdot \mathcal{N}^{(0)}(\overline{x_n}) \to \infty$ and

$$p^{(0)}(\overline{x}) = p^{(0)}(\overline{x_n}) \quad (8a)$$

$$v^{(0)}(\overline{x}) = v^{(0)}(\overline{x_n}) \quad (8b)$$

$$\mathcal{N}^{(0)}(\overline{x}) = \mathcal{N}^{(0)}(\overline{x_n}) \quad (8c)$$

$$p^{\star(0)}(\overline{x}, \overline{y}) = p^{\star(0)}(\overline{x_n}, \overline{y}) \quad (8d)$$

$$v^{\star(0)}(\overline{x}, \overline{y}) = v^{\star(0)}(\overline{x_n}, \overline{y}) \quad (8e)$$

Combining (7b) with the evanescence condition, the BL pressure is found to be negligible at the leading order, i.e. $p^{\star(0)} = 0$. Besides, equation (7a) shows that the boundary layer can be considered as incompressible. Both sides of equation (7a) are $\overline{y}$-integrated over the column $\mathcal{V}_n = \Sigma_n \times \mathcal{N}^{(0)}(\overline{x_n})$ of air located in and above the local REV Ω_n. Note that, at the leading order, the surface Σ_n is plane since its normal $\mathcal{N}^{(0)}(\overline{x_n})$ is constant. Applying the divergence theorem and accounting for periodicity and evanescence of $v^{\star(0)}(\overline{x_n},\ \overline{y})$, the following result is derived:

$$\int_{\mathcal{V}_n} \nabla_{\overline{y}} \cdot v^{\star(0)} \ d\mathcal{V}_{\overline{y}} = -\sum_s \int_{\Gamma_n^s} v^{\star(0)} \cdot \mathbf{n}_n^s \, d\Gamma_{\overline{y}} = 0. \quad (9)$$

On the other hand, since the LW velocity field $v^{(0)}(\overline{x_n})$ is locally uniform on $\mathcal{V}_n$:

$$\int_{\mathcal{V}_n} \nabla_{\overline{y}} \cdot v^{(0)} \ d\mathcal{V}_{\overline{y}} = -\sum_s \int_{\Gamma_n^s} v^{(0)} \cdot \mathbf{n}_n^s \, d\Gamma_{\overline{y}} + |\Sigma_n|_{\overline{y}} \, v^{(0)} \cdot \mathcal{N}^{(0)}(\overline{x_n}) = 0. \quad (10)$$

Here $|\Sigma_n|_{\overline{y}} v^{(0)} \cdot \mathcal{N}^{(0)}(\overline{x_n})$ is the contribution as $\overline{y} \cdot \mathcal{N}^{(0)}(\overline{x_n}) \to \infty$ where $|\Sigma_n|_{\overline{y}} = \int_{\Sigma_n} d\Sigma_{\overline{y}}$. Besides, the integration of equation (7d) over the boundary of the micro-structures provides:

$$\sum_s \int_{\Gamma_n^s} v^{(0)} \cdot \mathbf{n}_n^s \, d\Gamma_{\overline{y}} + \sum_s \int_{\Gamma_n^s} v^{\star(0)} \cdot \mathbf{n}_n^s \, d\Gamma_{\overline{y}} = \sum_s \int_{\Gamma_n^s} V_n^{s(0)} \cdot \mathbf{n}_n^s \ d\Gamma_{\overline{y}}. \quad (11)$$

Combining equations (9-10) gives the following mass conservation law for the LW field:

$$\left[v^{(0)} \cdot \mathcal{N}^{(0)}\right](\overline{x_n}) = \sum_s \frac{1}{|\Sigma_n|_{\overline{y}}} \int_{\Gamma_n^s} V_n^{s(0)} \cdot \mathbf{n}_n^s \ d\Gamma_{\overline{y}} \quad (12)$$

This means that, at the point $\overline{x_n}$, the LW velocity $v^{(0)}$ must balance the flux produced by all the micro-structures in the local REV Ω_n, per unit area of the two-dimensional local period Σ_n. The velocity $V_n^{s(0)}$ is produced by the microstructure s in response to the pressure field acting on it. Since $p^{\star(0)} = 0$, only the uniform LW pressure $p^{(0)}(\overline{x}) = p^{(0)}(\overline{x_n})$ acts on the microstructures. Using equation (7c) and the linearity of $\mathcal{R}_n^s$, the following relation holds:

$$\frac{1}{|\Sigma_n|_{\overline{y}}} \int_{\Gamma_n^s} V_n^{s(0)} \cdot \mathbf{n}_n^s \ d\Gamma_{\overline{y}} = -\Upsilon_n^s \ p^{(0)}(\overline{x_n}), \quad (13)$$

where Υ_n^s is the effective surface admittance of the structure s at the point $\overline{x_n}$ defined as:

$$\Upsilon_n^s = -\frac{1}{|\Sigma_n|_{\overline{y}}} \int_{\Gamma_n^s} [\mathcal{R}_n^s(1)] \cdot \mathbf{n}_n^s \ d\Gamma_{\overline{y}} \quad (14)$$

Substituting equations (13-14) in (12), the following boundary condition is found:

$$\left[v^{(0)} \cdot \mathcal{N}^{(0)}\right](\overline{x_n}) = -\Upsilon(\overline{x_n}) \, p^{(0)}(\overline{x_n}) \quad \text{where} \quad \Upsilon(\overline{x_n}) = \textstyle\sum_s \Upsilon_n^s. \quad (15)$$

The property (15) is satisfied at all the discrete points $\overline{x_n}$ of the surface $\mathcal{S}$ which form a fine mesh of $\mathcal{S}$ under the scale separation. Hence the relation (15) is extrapolated to the whole surface $\mathcal{S}$:

$$v^{(0)} \cdot \mathcal{N}^{(0)} = -\Upsilon \, p^{(0)} \quad \text{on } \mathcal{S} \quad (16)$$

where the function $\Upsilon(\overline{x} \in \mathcal{S})$ is the continuous surface admittance taking the values $\Upsilon(\overline{x_n}) = \sum_s \Upsilon_n^s$ at the discrete points $\overline{x_n}$. In the following, the superscripts (0) will be omitted.

To conclude this section, the homogenization model leads to the fact that the long-scale graded metasurface can be described by an equivalent admittance $\Upsilon(\overline{x} \in \mathcal{S})$ that varies only at the long-scale and the acoustic characteristics of which are directly inherited from the local properties of the modulated micro-structures, see equation (14). If this admittance is used at the boundary of an object and is modulated over long-scale distances to sculpt the scattered field in a desired way, such a metasurface can offer the opportunity to create acoustic decoys. This is what will be shown in the next section, where the field scattered by a cylinder will be tuned by means of the admittance $\Upsilon(\overline{x})$.

3 ANOMALOUS SCATTERING FROM A CYLINDER

3.1 Design of the metasurface admittance: general principles

To simplify the design of the metasurface admittance Υ, each cell Ω_n is assumed to include a single Helmholtz resonator arranged upon a rigid surface. The Helmholtz resonator has a cavity with the volume $V = \mathcal{O}(|\Sigma_n \ell_n|)$, and a straight neck with the aperture A and the length b. The Helmholtz resonator is a Single Degree of Freedom oscillator with the mass $M = \rho_e A\, b$ and stiffness $K = \gamma P_e A^2/V$: Newton's Second Law provides the normalized admittance $\rho_e c \Upsilon(x_n)$ in the form:

$$\rho_e c\, \Upsilon(x_n,\omega) \;=\; \frac{-i2\eta_n\omega_n\omega}{\omega_n^2 - i\,2\xi_n\omega_n\omega -\, \omega^2} \tag{17}$$

where the eigenfrequency ω_n of the resonator at the point x_n and the parameter η_n are given by:

$$\omega_n = \sqrt{\frac{K}{M}} = c\sqrt{\frac{A}{bV}} \quad ; \qquad \eta_n = \frac{\rho_e c\, A^2}{2\,|\Sigma_n| M\,\omega_n} \;=\; \frac{1}{2}\frac{V}{|\Sigma_n|}\frac{\omega_n}{c} = \mathcal{O}(\epsilon_n) \ll 1 \tag{18}$$

and $\xi_n \ll 1$ is the damping coefficient. Recalling that the LW fields consist of the incident and scattered fields (p_i, v_i) and (p_s, v_s), equation (16) can be re-written as follows:

$$\rho_e c\, \Upsilon(x_n,\omega) = \mathcal{F}(x_n,\omega) \quad \text{ where } \quad \mathcal{F}(x_n,\omega) = \frac{1}{-i\,\omega/c}\,\frac{\nabla p_i(x_n)\cdot\mathcal{N}_n \;+\; \nabla p_s(x_n)\cdot\mathcal{N}_n}{p_i(x_n) + p_s(x_n)}. \tag{19}$$

where $\mathcal{N}_n = \mathcal{N}(x_n)$. The function $\mathcal{F}$ depends only on the nature of the LW fields. Equating (17) and (19) leads to the explicit relation between the micro-structures' properties and the LW fields:

$$\rho_e c\, \Upsilon(x_n,\omega) \;=\; \frac{-i2\eta_n\omega_n\omega}{\omega_n^2 - i\,2\xi_n\omega_n\omega -\, \omega^2} = \mathcal{F}(x_n,\omega) \tag{20}$$

Suppose that all the eigen-frequencies ω_n are close to the frequency ω_0, i.e. $\omega_n = \omega_0(1+\delta_n)$ with $|\delta_n| \ll 1$. Since $\xi_n \ll 1$ and $\eta_n \ll 1$, equation (20) becomes, at the first order:

$$\frac{-i2\eta_n\omega_0\omega}{\omega_0^2(1+2\delta_n) - i\,2\xi_n\omega_0\omega -\, \omega^2} = \mathcal{F}(x_n,\omega) \tag{21}$$

In the low and high frequency range, $\omega \ll \omega_0$ and $\omega \gg \omega_0$, the left-hand-side tends to zero: the admittance of the metasurface is negligible compared to that of air, resulting in the same scattered field as the one that the scatterer would produce if its surface were rigid. For frequencies ω close to the frequency ω_0, i.e. $\omega = \omega_0(1+\nu)$ with $|\nu| \ll 1$, equation (21) tends to:

$$\frac{-i\,\eta_n}{\delta_n - \nu - i\,\xi_n} = \mathcal{F}(x_n,\omega) \;\; , \quad \omega = \omega_0(1+\nu) \tag{22}$$

Equating both modulus and phase leads to:

$$\frac{\eta_n}{\xi_n} = \frac{|\mathcal{F}(x_n,\omega)|^2}{\mathrm{Re}(\mathcal{F}(x_n,\omega))} \quad ; \qquad -\arctan\left(\frac{\delta_n - \nu}{\xi_n}\right) \;= \arg\mathcal{F}(x_n,\omega) \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]. \tag{23}$$

Equation (23) shows that detuning the resonators' eigenfrequency by δ_n is responsible for shaping the argument of the normalized admittance $\rho_e c\Upsilon = \mathcal{F}$ while the parameter η_n tunes its amplitude. However, the additional constraints $|\mathcal{F}(x_n,\omega)| = \mathcal{O}(\eta_n/\xi_n)$ and $\arg\mathcal{F}(x_n,\omega) \in \left[-{}^{\pi}/_{2},\, {}^{\pi}/_{2}\right]$ (on the top of the scale separation, long-scale modulation and long-scale curvature requirements) make that, with passive structures (here, Helmholtz resonators), not any admittance contrast and phase-shift can be reached, and thus not any scattered field can be sculpted from an object submitted to an incident field.

3.2 Scattering from a cylinder with a long-scale graded surface admittance

The LSGM is applied at the boundary of an infinitely long cylinder (radius R) and its scattering in response to an incident field p_i propagating with the wavenumber $k = \omega/c$ is studied. The analysis is performed using the polar coordinate system (O, r, θ) where the angle θ is counted from the X-axis and O is the center of the cylinder (Figure 2). The boundary $r = R$ of the cylinder has the normalized admittance $\rho_e c\,\Upsilon(\theta)$. The incident field p_i, the admittance $\rho_e c\,\Upsilon(\theta)$ and the scattered fields p_s being 2π-periodic with θ, they can be decomposed into the following Fourier series:

$$p_i = \sum_{m=-\infty}^{m=+\infty} S_m J_m(kr)\, e^{im\theta} \;\; ; \quad \rho_e c\Upsilon = \sum_{m=-\infty}^{m=+\infty} A_m e^{im\theta} \;\; ; \quad p_s = \sum_{m=-\infty}^{m=+\infty} B_m H_m(kr)\, e^{im\theta} \tag{29}$$

where J_m and H_m are the Bessel and Hankel functions of the first kind and order m. The coefficients A_m are the Fourier coefficients of $\rho_e c\Upsilon$; while for a plane wave $p_i = \exp(ikrcos(\theta - \theta_i))$ propagating at an angle θ_i with the X-axis, the coefficient S_m are given by the Jacobi-Anger expansion[15]:

$$A_m = \frac{1}{2\pi}\int_0^{2\pi} \rho_e c\Upsilon(\theta) e^{-im\theta}\, d\theta \quad ; \quad S_m = i^m e^{-im\theta_i}. \tag{30}$$

As for the scattering coefficients B_m, they are given by the admittance condition $(\nabla p_i + \nabla p_s) \cdot \mathcal{N} = -ik\rho_e c\Upsilon \cdot (p_i + p_s)$ at the boundary $r = R$. Using the Cauchy product of two series yields:

$$B_m H_m'(kR) + i\sum_{q=-\infty}^{q=+\infty} A_{m-q} B_q H_q(kR) \;=\; -\left(S_m J_m'(kR) + i\sum_{q=-\infty}^{q=+\infty} A_{m-q} S_q J_q(kR) \right) \tag{31}$$

where J_m' and H_m' are the derivatives of J_m and H_m with respect to their argument. Equation (31) provides a system of linear equations for the scattering coefficients B_m if the admittance $\Upsilon(\theta)$ is known. This latter is shaped by the properties of the resonators at the periphery of the cylinder.

3.3 Resonators at the periphery of the cylinder

The $n = 1..N$ Helmholtz resonators of the metasurface are realized using slotted cylinders with a long duct rolled around their cylindrical cavity (see Figure 2). All the resonators are supposed to have the same outer radius a, the same duct width e and the same damping coefficient $\xi \ll 1$. The resonators are positioned at the distance $r = R + a$ from the center of the cylinder, and at the discrete set of angles θ_n. The angular opening of the period ℓ_n at θ_n is $\Theta_n = \ell_n/R = (\theta_{n+1} - \theta_{n-1})/2 \ll 1$, with $\theta_{1-1} = \theta_N$ due to the 2π-periodicity around the cylinder. The period widths ℓ_n are normalized as $\ell_n = \ell_0\alpha_n$ where $\ell_0 = R\Theta_0$ is a characteristic period size and $\alpha_n = \mathcal{O}(1)$ is a modulation function. Besides, the resonance frequency $\omega_n = \omega_0(1 + \delta_n)$ of the resonator at θ_n is tuned using equation (18) by changing the length $b_n = b_0/(1 + \delta_n)^2$ of the duct, where the duct length b_0 provides the resonance frequency ω_0 at which anomalous scattering is sought. Since the duct is wrapped around the cavity, the angle $\beta_0 = b_0/(a - e/2)$ is defined. The scale parameter ϵ_n and the parameter η_n take the form $\epsilon_n = \epsilon_0\alpha_n$ and $\eta_n = \eta_0(1 + \delta_n)/\alpha_n$ where $\epsilon_0 = k_0\ell_0 \ll 1$ and $\eta_0 = ec/(2\ell_0 b_0\omega_0)$ are related to a uniform ℓ_0-periodic metasurface with resonators having the duct length b_0. In that uniform configuration, the dimensions of the system depend on $\epsilon_0 = k_0\ell_0$, $\eta_0 = \mathcal{O}(\epsilon_0)$ and β_0 when normalized by $k_0 = \omega_0/c$:

$$k_0 a = \epsilon_0\,\frac{1 + \epsilon_0\eta_0\beta_0}{1 - \epsilon_0\eta_0\beta_0}\sqrt{\frac{2\eta_0}{\pi\epsilon_0}} \;; \quad k_0 b = \frac{\epsilon_0\beta_0}{1 - \epsilon_0\eta_0\beta_0}\sqrt{\frac{2\eta_0}{\pi\epsilon_0}} \;\; ; \quad k_0 e = \frac{2\epsilon_0^2\eta_0\beta_0}{1 - \epsilon_0\eta_0\beta_0}\sqrt{\frac{2\eta_0}{\pi\epsilon_0}} \tag{32}$$

while $k_0 R = \mathcal{O}(1)$ for the curvature of the cylinder to be negligible at the scale of the periods and $N_0 = 2\pi k_0 R/\epsilon_0$ is the number of resonators at the periphery of the cylinder. The design method proposed hereafter relies on the two following steps: (1) the design of the uniform metasurface under the scale separation; and (2) the modulation of that uniform metasurface into the graded one.

3.4 Example of anomalous scattering

The uniform configuration of the metasurface (normalized admittance $\rho_e c\Upsilon_0$) is designed as follows: the scale separation is satisfied at the frequency ω_0 with the scale parameter $\epsilon_0 = 0.3 \ll 1$;

resonators are weakly-damped with the damping ratio $\xi = 4\%$; an impedance matching is achieved between air and the metasurface at the resonance ω_0 with $\eta_0 = \xi$, see equation (17); the cylinder has the radius $k_0 R = 1$; and the resonators' duct is wrapped around the cavity over the angle $\beta_0 = 3\pi/2$. That leads to $N_0 = 2\pi k_0 R/\epsilon_0 = 21$ resonators arranged at the angles $\theta_n^0 = n\,2\pi/N_0$ at the periphery of the cylinder, and to the resonators' dimensions: $k_0 a = 0.098$; $k_0 b = 0.436$; $k_0 e = 0.010$. For instance, at the frequency $\omega_0/(2\pi) = 165$ Hz, the wavenumber is $k_0 \approx 3$ rad/m and thus $R \approx 33.2$ cm; $\ell_0 \approx 10$ cm; $a \approx 3.2$ cm; $b \approx 14.5$ cm; $e \approx 3.3$ mm.

An extreme way to modify the scattered field is to suppress it. That is achieved is the admittance condition in equation (16) can be satisfied by the incident field itself. Unfortunately, that would require $\rho_e c\Upsilon(\theta) = -\cos(\theta - \theta_i)$ which is real negative (and not realizable with passive structures) for $\pi/2 < \theta - \theta_i < 3\pi/2$, i.e. in the shadow zone of the cylinder. However, such an admittance profile in the 'enlightened' zone of the cylinder might avoid back-scattering from the cylinder. That is the effect we would like to achieve here at the frequency ω_0 for plane waves propagating in the directions $\theta_i = 0$ or $\theta_i = \pi$. Thus, the resonators are all the same ($\delta_n = 0$) for the normalized admittance to be real positive at ω_0 while their spacing $\ell_n = \ell_0 \alpha_n$ is modulated as follows to shape the admittance $\rho_e c\,\Upsilon(\theta_n) = 1/\alpha_n$ with a scale parameter $\epsilon_n = \ell_0 \alpha_n \leq 0.6$:

$$\alpha_n = 1/|\cos(\theta_n)| \quad \text{if} \quad \theta_n \in \left[0 \pm \frac{\pi}{3}\right] \cup \left[\pi \pm \frac{\pi}{3}\right] \quad \text{and} \quad \alpha_n = 1/2 \text{ otherwise} \tag{33}$$

The position $\theta_{n+1} \approx \theta_n + \epsilon_0 \alpha_n$ of the resonators and the normalized admittance $\rho_e c\Upsilon(\theta)$ that realizes approximately the above profile at the resonance frequency ω_0 are given in Figure 2. The field scattered at the resonance by the cylinder in response to an incident wave propagating in the direction $\theta_i = 0$ is given in Figure 2 with: a rigid boundary; the uniform admittance $\rho_e c\Upsilon_0 = 1$; and the modulated admittance $\rho_e c\Upsilon(\theta)$. The drastic reduction of the scattering towards the $x < 0$ is evidenced with the long-scale graded admittance, compared to the cylinder with a rigid boundary, while the graded admittance provides a better impedance matching than the uniform admittance.

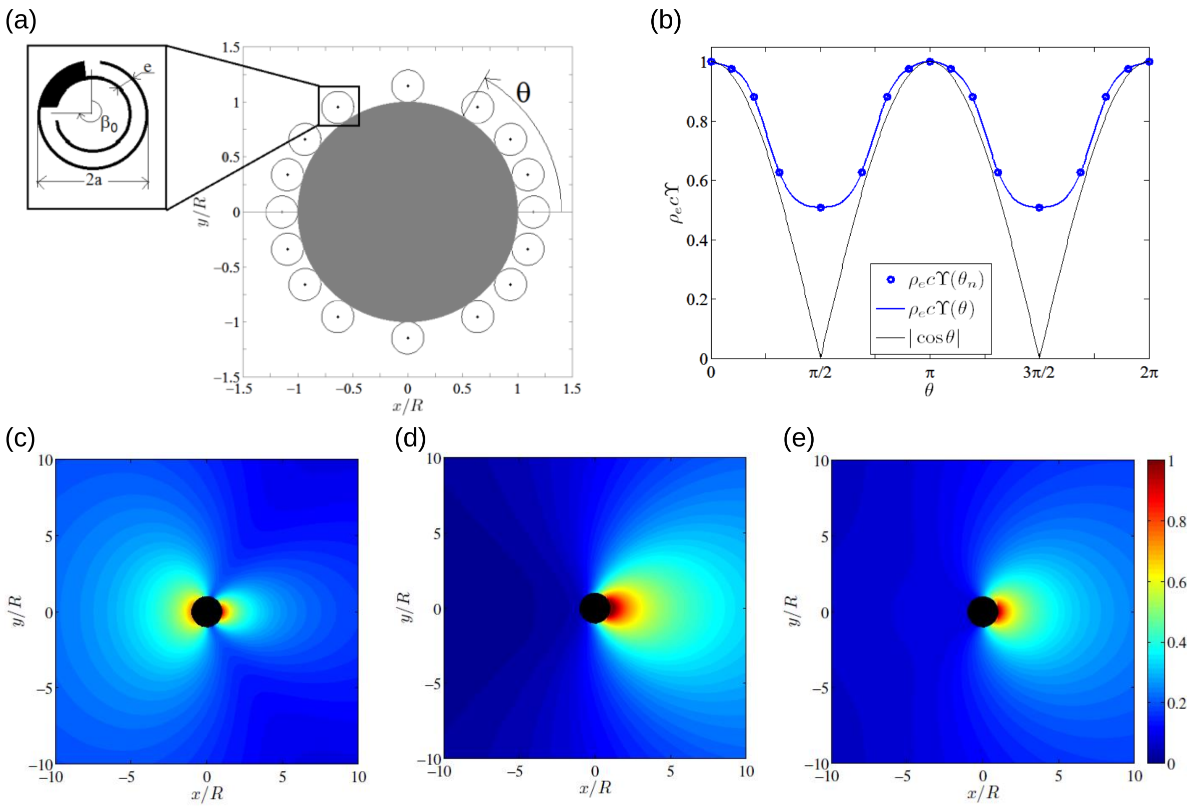

Figure 2: (a) Helmholtz resonator arranged at the periphery of the cylinder (not at scale); (b) Normalized admittance at resonance ω_0 on the cylinder boundary (spline extrapolation between the angles θ_n); (c), (d) and (e): Amplitude of the field scattered by the cylinder at the frequency ω_0 with: a rigid surface, a uniform admittance $\rho_e c\Upsilon_0 = 1$, or the graded admittance depicted in (b).

4 CONCLUSION

In this paper, the homogenization of a metasurface with long-scale graded properties has been reported. The specificity of such a metasurface lies in the fact that the arrangement of resonant structures is only quasi-periodic, i.e. the geometrical and rheological properties of the micro-structures are modulated over distances of the order of the long-wavelength. The homogenization leads to the macroscopic description of such a metasurface in the form of an equivalent admittance, the long-scale graded characteristics of which are directly inherited from the local properties of the modulated micro-structures. The use of such a graded admittance at the surface of a scatterer (with long-scale surface curvature) has been shown to enable the sculpture of the field scattered by the object in response to an incident field at the resonance frequency. This offer the opportunity to create acoustic decoys, whereby a scatterer would scatter a field that it is not meant to scatter.

The homogenization model presented here provides a rigorous theoretical and analytical framework whereby the local resonant structures are considered in their actual environment. In particular, it remains valid for any incident wavefield that satisfies the scale separation, it is 3D by nature, and it accounts for the damping within the resonators. It also provides a unified framework for long-scale graded metasurfaces whether applied at a plane for Generalized Snell-Descartes's Law, or at the surface of a scatterer for scattering cancellation. The model remains valid for uniform, i.e. not graded, metasurface.

The analytical formulation allows parametric studies (effect of the frequency, angle of incidence, etc) at very low computational cost and straightforward analytical designs. It also allows considering more complex arrangements than the one presented in this paper, such as combining several oscillators of different nature in a period. The leading order description can also be improved by considering higher order terms in the asymptotic expansion (when the scale separation is not sharp).

5 REFERENCES

1. A. Bensoussan,J.-L. Lions and G. Papanicolaou. 'Asymptotic analysis for period structures'. AMS Chelsea Publishing.(1978).
2. E. Sanchez-Palencia. 'Non-homogeneous media and vibration theory'. Lectures Notes in Physics vol. 127. Springer-Verlag Berlin Heidelberg. (1980).
3. J.-L. Auriault and G. Bonnet. 'Dynamique des composites élastiques périodiques'. Arch. Mech. 37(4–5), 269–284. (1985).
4. J.-L. Auriault and C. Boutin. 'Long wavelength inner-resonance cut-off frequencies in elastic composites materials'. Int. J. Solid and Structures 49:3269-3281. (2012).
5. A. Krynkin, *et al*.. 'Analytical approximations for low-frequency band gaps in periodic arrays of elastic shells'. J. Acoust. Soc. Am. 133:781–791. (2013).
6. C. Chesnais,et al. 'Effects of the local resonance on the wave propagation in periodicframe structures: Generalized Newtonian mechanics'.J. Acoust.Soc.Am.132:2873–2886. (2012).
7. N. Yu *et al.*. 'Light propagation with phase discontinuities: Generalized Laws of reflection and refraction'. Science 334:333-337. (2011).
8. U. Leonhardt. 'Optical conformal mapping'. Science 312:1777-1780. (2006)
9. B.-I. Popa and S. A. Cummer. 'Design and characterization of broadband acoustic composite metamaterials'. Phys. Rev. B 80, 174303. (2009).
10. Y. Li et *al*.. 'Reflected wavefront manipulation based on ultrathin planar acoustic metasurfaces'. Sci. Rep. 3:2546. (2013).
11. J.-L. Auriault, C. Boutin, and C. Geindreau. 'Homogenization of Coupled Phenomena in Heterogenous Media'. ISTE and Wiley. (2009).
12. C.L. Holloway and E.F. Kuester, 'Equivalent boundary conditions for a perfectly conducting periodic surface with a cover layer', Radio Science 35(3) 661-681 (2000)
13. C. Boutin and P. Roussillon, 'Wave propagation in presence of oscillators on the free surface', Int. J. of Eng. Sci. 44 180-204 (2006)
14. G. Moreau and D. Caillerie. 'Continuum modeling of lattice structures in large displacement applications to buckling analysis'. Computers and Structures 68:181-189. (1998)
15. H. Bateman and A. Erdelyi. 'Higher transcendental functions'. Vol. 2. McGraw-Hill. (1953)