

SOURCES LOCALISATION IN ELLIPSOIDS BY BEST RATIONAL APPROXIMATION IN PLANAR SECTIONS

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1 INTRODUCTION

In this paper, we consider a classical inverse problem for the Laplace operator consisting in recovering finitely many dipolar sources located in a 3D domain from measurements of their potential under prescribed current flux on the boundary.

For instance, in medical engineering, the inverse EEG (ElectroEncephaloGraphy) problem consists in detecting pointwise dipolar current sources, modeling epileptical foci located in the brain, from measures of the electrical potential on the scalp⁸.

The problem is also of interest in geophysics, more precisely for the determination of the mass-density distribution of the Earth, which can be approximated by an ellipsoid, or for that of equipotential surfaces (geoids), from measurements of the gravitational potential at the Earth's surface^{9,10}.

Concerning the EEG issue, the so-called spherical model, where the 3D domain is a ball has been handled³. It is shown that this recovery issue reduces to a sequence of 2D inverse problems, each of which consisting in recovering the branch-points and singularities of some holomorphic function in a disk. These 2D problems may then be solved using best rational or meromorphic approximation algorithms¹. Finally, those 2D singularities are strongly linked with the original dipolar sources, that they allow to approximately locate².

Though there are a number of possible approaches to this inverse problems, that mainly deal with iterative resolution of the associated direct problem^{7,11}, we describe here an efficient algorithm based on rational approximation in families of planar cross-sections of Ω . Those are costless since they run with boundary data only.

We will investigate in this paper the case of an ellipsoidal domain. Concerning the EEG inverse problems, this is a first step towards a more realistic geometry for the brain.

We will show that if the 3D domain is an ellipsoid, then the issue can still be approached by a sequence of 2D inverse problems in a family of ellipses (cross-sections), hence in a family of disks as in the spherical case, using the fact that the boundary of an ellipse is the image by a rational function of the unit circle. We will then face once again the issue of recovering the singularities that a holomorphic function has in a disk.

Let $v(X)$ be the difference of potential with respect to X in the closure of Ω , where the boundary $\partial\Omega$ is an ellipsoid. The following equation with mixed boundary conditions arises when studying the EEG inverse problem, and assuming the conductivity of the head to be constant:

$$\begin{aligned}\Delta v &= F && \text{in } \Omega \\ v &= g && \text{on } \partial\Omega \\ \partial v / \partial n &= \varphi && \text{on } \partial\Omega\end{aligned}$$

where $g \in L^2(\partial\Omega)$ and $\varphi \in L^2(\partial\Omega)$ are the given measurements of the difference of potential and of the current flux on (the scalp) $\partial\Omega$, with $\int_{\partial\Omega} \varphi \, ds = 0$ and $F = \sum_{1 \leq k \leq m} \langle p_k, \nabla \delta_{c_k} \rangle$ is the distribution corresponding to the pointwise dipolar sources located at $c_k \in \Omega$ with moments $p_k \in \mathbb{R}^3$.

Since, $E(X) = 1/(4\pi \|X\|)$ is a fundamental solution of the Laplacian in \mathbb{R}^3 , we get that

$$(1) \quad v(X) = h(X) + u(X) \quad \text{with } X \in \Omega,$$

where h is a harmonic function in \mathbb{R}^3 and

$$(2) \quad u(X) = \sum_{1 \leq k \leq m} \langle p_k, X - c_k \rangle / (4\pi \|X - c_k\|^3).$$

The decomposition of a harmonic function in a neighborhood of the ellipsoid in terms of ellipsoidal harmonics using the knowledge of the function and of the normal derivative gives an explicit decomposition of this function. This explicit decomposition gives exactly the singular term.

The following step now consists in recovering the singularities of $u^2(x, y, z_p)$ for a family of cross sections $\{z = z_p\}$ of Ω from its values on the 2D boundaries (ellipses).

This is done by using a rational mapping ϕ_p from these ellipses $\{z = z_p\} \cap \partial\Omega$ to the unit circle T , and then by means of best meromorphic approximation of the function $f_p = [u|_{\{z=z_p\}} \circ \phi_p]^2$ on T .

The poles of such an approximant allow to approximately locate the $2m$ singularities of f_p in the unit disk and, finally, the sources c_k .

Let us precise that the singularities of $u^2(x, y, z_p)$ induced by the dipolar sources will appear also as polar singularities of order 3, whence in the expression of $u(x, y, z_p)$, they appear only as ramified singularities of order $3/2$. This is a framework in which the above rational approximation algorithms are more efficient.

The main steps of the recovery algorithm that we consider in this work are thus summarized as follows:

Step 1: get the singular part $u|_{\partial\Omega}$ from the data $v|_{\partial\Omega} = g$, $(\partial v / \partial n)|_{\partial\Omega} = \varphi$, using ellipsoidal harmonics¹².

Step 2: find (approximately) the singularities (branch-points) $\zeta_{k,p}$ of $f_p = [u|_{\{z=z_p\}} \circ \phi_p]^2$ using:

(2.i) a rational correspondance ϕ_p between the ellipse $\{z = z_p\} \cap \partial\Omega$ and the circle T ,

(2.ii) best approximation on T by rational functions in the disk.

Step 3: recover c_k from the estimated $\zeta_{k,p}$.

In section 2, we briefly recall some results about best rational approximation and potential theory in the complex plane³, that are to the effect that the poles of the approximants converge (in a sense to be made precise) to the singularities as the degree increases; this is the basis of Step (2.ii). We will then be back to the inverse problem in section 3, and explain how to localize the sources inside the ellipsoid from related 2D singularities in disks, which is Step 3, using the fact that the boundaries of ellipses are the images under rational mappings of the unit circle, for Step (2.i). We end up in section 4 with a numerical illustration of the above scheme.

2 ABOUT MEROMORPHIC APPROXIMATION IN THE UNIT DISK

2.1 Best meromorphic approximation

Let us introduce some more notations. Let H^2 be the familiar Hardy spaces of the unit disk D :

$$H^2 = \{ f \in L^2(\mathcal{T}), f(\exp(i\theta)) = \sum_{0 \leq n} a_n \exp(in\theta), (a_n) \in l^2 \}$$

Let $R_n = R_n(D)$ be the set of rational fractions having at most n poles in D .

Introduce the set $H_n = H^2 + R_n$ of meromorphic function with at most n poles in D .

Let now $f \in L^2(\mathcal{T})$. A best meromorphic $L^2(\mathcal{T})$ -approximant to f with at most n poles is a function $\{R_n \in H_n\}$ such that:

$$(3) \quad \|f - R_n\|_{L^2(\mathcal{T})} = \inf_{\{R \in H_n\}} \|f - R\|_{L^2(\mathcal{T})}$$

Existence and uniqueness properties of solutions to (3) are discussed in⁴. Constructive algorithms to generate local minima can be obtained using Schur parametrization¹³.

2.2 Behaviour of poles

Let $A_\varepsilon \subset D$ be the annulus: $A_\varepsilon = \{z \in D, 1 - \varepsilon < |z|\}$, and BP be the class of functions that are continuous in the closure of A_ε and holomorphic in A_ε , for some $\varepsilon > 0$, and that can be analytically extended to D excepted for finitely many poles and branch-points.

Theorem 1^{4,14} If $f \in BP$, there exists a unique connected open V_f in the closure of D with $A_\varepsilon \subset \subset V_f$ and such that f extends holomorphically to V_f which has the properties that $D \setminus V_f$ is of minimal Green capacity among such sets and that V_f contains every such sets. Furthermore, $D \setminus V_f$ consists of the poles and branchpoints of f and of finitely many analytic cuts (between branchpoints, with no loops).

Denote by $(s_{j,n})_{j=1,\dots,n}$ the poles in D , counted with their multiplicities, of the solution $R_n = R_n(f)$ to (3). Define the sequence of their counting probability measure by: $\mu_n = \mu_n(f) = (\sum_{1 \leq j \leq n} \delta_{s_{j,n}})/n$.

Theorem 2⁷ If $f \in BP$ is not single-valued, then the measure μ_n converges weak-* to the Green equilibrium distribution of $D \setminus V_f$ as $n \rightarrow \infty$.

We use theorem 2 to approach the inverse sources problem, which consists in recovering singularities that appear both as branchpoints and poles of f . Indeed, the counting measure μ_n will asymptotically charge the endpoints of $D \setminus V_f$, because the equilibrium distribution is infinite there. By theorem 1, $D \setminus V_f$ includes the poles and branchpoints of f , whence computing μ_n for increasing values of n will allow to approximately locate them. Of course, one has to keep in mind that computations are in practice restricted to limited values of n , thus the quality of the recovery scheme will strongly depend on how fast the poles converge.

3 LOCALIZATION OF THE SOURCES

Recall that our aim is to recover the sources $c_k \in \Omega$ being given the values of the function v of the form (1-2) and that of $\partial v / \partial n$ on the ellipsoid $\partial\Omega$. We assume that we have already computed the singular part u of the potential on $\partial\Omega$ from the boundary data using ellipsoidal harmonics (for the details of the step 1 of the algorithm, we refer to¹²). We explain how to handle the steps 2 and 3.

3.1 Expressions on the boundary of the slices $z = z_p$: from ellipses to circles: Step (2.i)

This step consists in recovering the singularities of $u^2(x, y, z_p)$ for a family of cross sections $\{z = z_p\}$ of Ω from its values on the 2D boundaries (ellipses). This is done by using a rational mapping ϕ_p from the ellipses $\{z = z_p\} \cap \partial\Omega$ (of radii $a_{1,p}$ and $a_{2,p}$) to the unit circle T , and then, by means of best rational approximation of the function $f_p = [u|_{\{z=z_p\}} \circ \phi_p]^2$ on T . Again, the poles of the approximant allow to approximately locate the $2m$ singularities of f_p in D and, finally, the sources $c_k = (x_k, y_k, z_k)$. In every slice $\{z = z_p\}$, let $\phi_p(\zeta) = \alpha_p \zeta + \beta_p / \zeta$ for $\zeta \in T$, where $\alpha_p = (a_{1,p} + a_{2,p})/2$ and $\beta_p = (a_{1,p} - a_{2,p})/2$. We show, in the single source situation, that $u^2(\phi_p(\zeta), z_p)$ coincides on T with a rational function whose numerator is a polynomial of degree 8, and whose denominator is $Q_{1,p}^3(\zeta)$, a polynomial of degree 12 (up to multiplication by a constant).

Whenever $m > 1$, $u^2(\phi_p(\zeta), z_p)$ coincides on T with a function whose numerator has both (triple) poles and branched singularities whereas its denominator is $4(4\pi)^2 \prod_{1 \leq k \leq m} Q_{k,p}^3(\zeta)$, where

$$Q_{k,p}(\zeta) = \alpha_p \beta_p \zeta^4 + \varpi_{3,k,p} \zeta^3 + \varpi_{2,k,p} \zeta^2 + \text{conj}(\varpi_{3,k,p}) \zeta + \alpha_p \beta_p$$

with
and

$$\varpi_{2,k,p} = \alpha_p^2 + \beta_p^2 + |w_k|^2 + h_{k,p}^2, \quad \varpi_{3,k,p} = -(\alpha_p \text{conj}(w_k) + \beta_p w_k),$$

$$h_{k,p} = z_p - z_k, \quad w_k = x_k + iy_k.$$

3.2 About the zeroes of $Q_{k,p}$ and w_k, z_k : Step (2.i) (ctn.) and Step 3.

Proposition 1 At fixed p , the polynomial $Q_{k,p}$ has two roots ζ_1 and ζ_2 inside D such that $\arg \zeta_{1,k,p} = -\arg \zeta_{2,k,p}$, and two roots outside D given by $1/\text{conj}(\zeta_{1,k,p})$ and $1/\text{conj}(\zeta_{2,k,p})$.

Remark 1 Since $\zeta_{1,k,p} = r_{1,k,p} \exp(i\theta_{k,p})$ and $\zeta_{2,k,p} = r_{2,k,p} \exp(-i\theta_{k,p})$, the quantity

$$(\alpha_p + \beta_p) / (\alpha_p - \beta_p) \text{Arctg}((r_{1,k,p} r_{2,k,p} - 1) / (r_{1,k,p} - r_{2,k,p}) / (r_{1,k,p} r_{2,k,p} + 1) / (r_{1,k,p} + r_{2,k,p}) \text{tg } \theta_{k,p})$$

does not depend on p . This allows us, at least in principle, to compute w_k and then z_k , from the knowledge of $\zeta_{1,k,p}, \zeta_{2,k,p}$ at some p . But we may also choose a variational approach, using $\zeta_{1,k,p}, \zeta_{2,k,p}$ for all p , which will be more robust in numerical computations, see section 3.4.

3.3 How to get $(\zeta_{k,p})_{k,p}$ using ARL2: Step (2.ii).

Let $f_p(\zeta) = [u(\phi_p(\zeta), z_p)]^2$, for each p . We now look for a best rational approximant $R_{p,N}$ of degree N to f_p , solution to (3) on the circle T which is sent by ϕ_p onto the ellipse $\{z = z_p\} \cap \partial\Omega$.

For all p , $f_p \in BP \subset C(T)$ and possesses $2m$ triple poles in D that coincide with branched singularities $(\zeta_{1,k,p}, \zeta_{2,k,p})_{k=1,\dots,m}$.

It thus follows from the results of section 2 that, for each p , the N poles, say s_j , of the best rational approximant $R_{p,N}$ will asymptotically locate $\zeta_{i,k,p}$, $i=1, 2$, $k=1,\dots,m$, near the singularities $\zeta_{i,k,p}$, $i=1, 2$, $k=1,\dots,m$: this allows us to approximately locate them.

3.4 How to get c_k from $(\zeta_{k,p})_p$: Step 3 (ctn)

Let us now assume that the singularities $\zeta_{i,k,p}$, $i=1,2$, $k=1,\dots,m$, inside the unit disk of the above function are (approximately) known from the above step, and that we are given $\zeta_{i,k,p}$, $i=1, 2$, $k=1,\dots,m$.

For each p , we then have $2m$ triple poles inside the unit disk. We first class these $2m$ poles by pairs for each p , using the remark 1: we take a pair $(\zeta_{1,k,p}, \zeta_{2,k,p})$ such that their arguments are opposed. They correspond to a first source whose projection on the plane $z = z_p$ has affix w_1 .

Doing the same thing for the other pairs, $k \in \{2, \dots, m\}$, and then for every p , and using the above invariants, we can sort the different pairs along the horizontal sections $z = z_p$, since the pairs $(\zeta_{1,k,p}, \zeta_{2,k,p})$ correspond to the same (still unknown) source w_k , for all p . We thus get the family $(\zeta_{1,k,p}, \zeta_{2,k,p})_p$ for each k .

Hence, for each p and k , we can form the polynomials

$$Q_{k,p}(\zeta) \approx q_{k,p}(\zeta) = \alpha_p \beta_p \zeta^4 + \varpi_{3,k,p} \zeta^3 + \varpi_{2,k,p} \zeta^2 + \text{conj}(\varpi_{3,k,p}) \zeta + \alpha_p \beta_p$$

with

$$\varpi_{2,k,p} \approx \varpi_{2,k,p} = \alpha_p^2 + \beta_p^2 + |w_k|^2 + h_{k,p}^2$$

hence, for fixed k , the expression

$$(4) \quad \varpi_{2,k,p} - (\alpha_p^2 + \beta_p^2) \approx \varpi_{2,k,p} - (\alpha_p^2 + \beta_p^2)$$

is minimal among the indices p if and only if the slice $\{z = z_p\}$ contains the source w_k .

Because also $\varpi_{3,k,p} \approx \varpi_{3,k,p}$, the discussion of remark 4.1 allows us to reconstruct w_k .

4 NUMERICAL ILLUSTRATION

In order to provide a preliminary numerical illustration of the above scheme (implemented with Matlab7), let us work with an ellipsoid with average semi-axes equal to $a_1 = 3$, $a_2 = 2$, and $a_3 = 1$. We will consider $z_p = -1, \dots, 1$ with an increment of .1.

Assume further that we are in a single source situation ($m = 1$) where $c_1 = (.2, .5, -.3)$, $p_1 = (.7, 0, -.7)$. In this case, we computed the singular part u given of the associated solution to the EEG inverse problem. Figures 1 to 3 are related to this case.

Figures 1 is related to the two values of p corresponding to the slices $z_p = -.3$, $z_p = .8$. The singularities of f_p are the $*$, while the 6 poles of its best rational approximant $R_{p,6}$ are the black points (there are 3 poles around each $*$, and this accounts for the fact that those are triple poles of f_p). in this situation, the poles are so close to each other that it is pretty hard to distinguish between them all. Figure 2 shows (in abscissa) the behavior of the criterion (4), which takes its minimal value at the slice $z_p = -.3$ containing the source c_1 . Figure 3 shows both the exact and the computed position of c_1 in Ω .

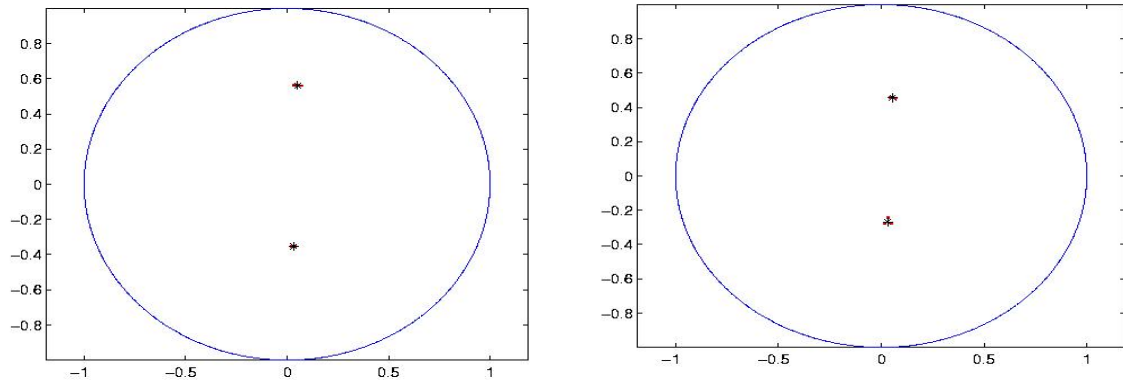


Figure 1: Slices $z_p = -.3$ and $z_p = .8$

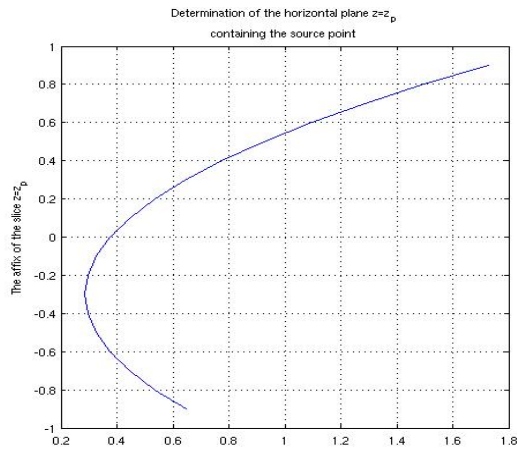


Figure 2: The function $\varpi_{2,k,p} - (\alpha_p^2 + \beta_p^2)$ (x axis) w.r.t. the height z_p of the slice (y axis)

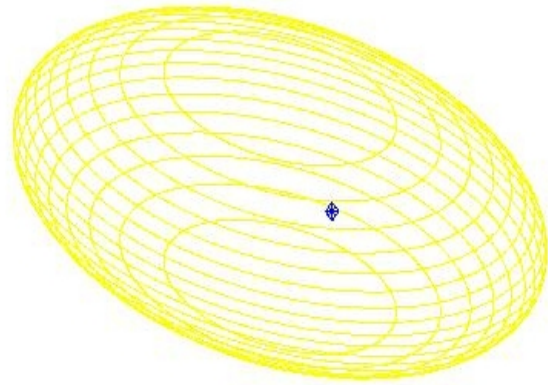


Figure 3: Exact and reconstructed source

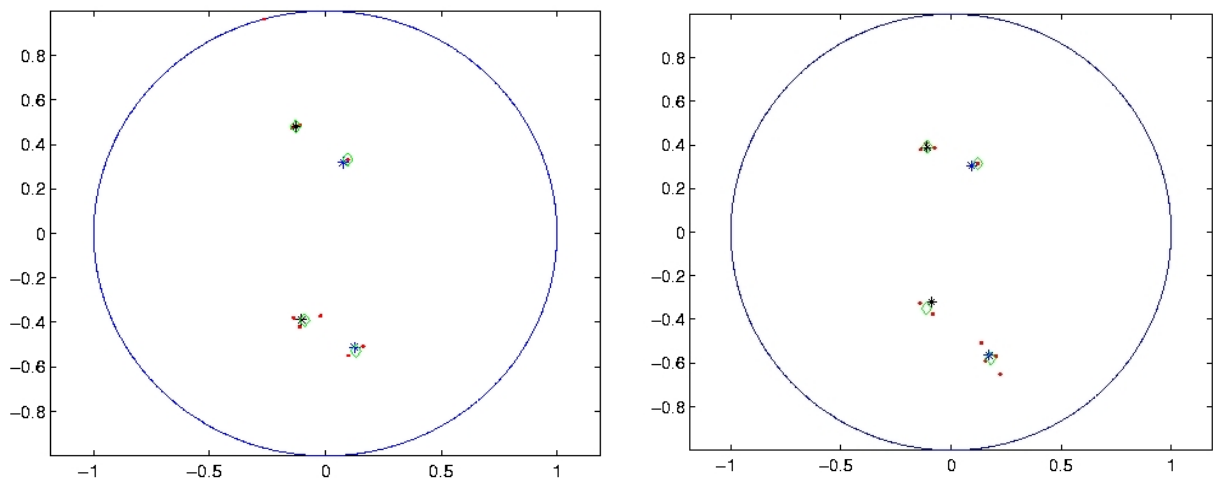


Figure 4 : Singularities $\zeta_{1,k,p}$, $\zeta_{2,k,p}$ for $k=1,2$
at $z_p = -.5$ and $R_{p,10}$ at $z_p = .7$ and $R_{p,10}$

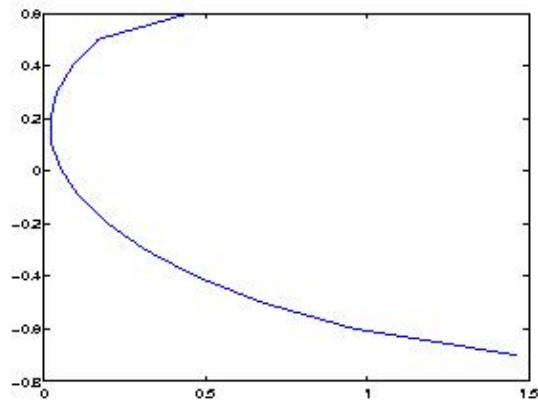


Figure 5: The function $\varpi_{2,k,p} - (\alpha p^2 + \beta p^2)$ (x axis) w.r.t. the height z_p of the slice (y axis)

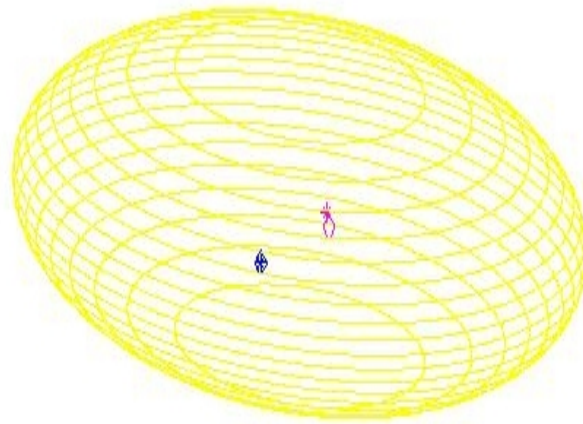


Figure 6: Exact and reconstructed source

Considering a situation with two sources ($m = 2$) at $c_1 = (-.5, .2, -.5)$ and $c_2 = (.5, -.5, .3)$, and associated moments $p_1 = (10, 0, 0)$, $p_2 = (0, 1, 0)$. We computed the singularities of the function f_p for two values of p corresponding to the slices $z_p = -.5$ and $z_p = .7$. They are plotted as $*$ in Figure 4, where the poles of $R_{p,10}$ are the black points. We can thus estimate the positions by computing the barycenters of those 4 groups of poles. We also show the behavior of the criterion (4), in Figure 5 for $k = 2$, derived from numerical simulations. One can see that the computed minimum of (4) related to the source c_2 is achieved for $z_p = .2$ (close but not equal to its true value $.3$). Finally, Figure 6 shows both the exact and the reconstructed sources.

5 CONCLUSION

A number of algorithmical and computational aspects of the present study remain to be approached. Among them, we can list the computation of the moments, but also questions related to the selection of the estimated singularities. From this numerical point of view, in order to get more precision, we have to refine the increment between two consecutive slices, to run the algorithm along slices parallel to others directions, and then to cluster the estimated singularities.

A numerical limitation of the present computation is that it requires to compute rational approximants of huge degree. This comes from the fact that the algorithm is mostly dedicated to the recovery of simple poles. This point is reinforced by the fact that we have $2m$ singularities to be recovered on each slice, which is already twice the complexity of the spherical situation where there was only m singularities in each slice³. Further, the convergence properties are only to the effect that we asymptotically approximate the singularities. More efficient would be a scheme constrained to find triple poles.

Despite these numerical limitations, the extension of the present work to more general geometries should in principle be feasible, at least for revolution domains whose plane sections by the family $\{z = z_p\}$ are so-called quadrature domains, as described in⁶.

Also, for the EEG application, the preliminary cortical mapping step should be considered in this ellipsoidal setup, using decompositions on ellipsoidal harmonics. This step consists in solving a Cauchy type issue for the potential v , which is practically given by pointwise measures (electrodes) on the scalp, together with a vanishing current flux there, and has to be transmitted as a harmonic function up to the boundary of the brain, where the sources are sought. A classical model is to consider the head as made of a number of layers of constant conductivity, brain, skull, scalp. A feasible way to handle this recovery problem directly in 3D, and no longer using this slicing process and 2D approximation schemes, would perhaps be to use quaternionic analysis. A number of definitions of analyticity are available in this framework and an appropriate one has to be chosen. These issues will be the topics of further work.

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