UNDERWATER ACOUSTIC NORMAL MODE COUPLING DUE TO RESONANT SCATTERING

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1. ABSTRACT

The scattering of sound from underwater internal waves is investigated using a theory based on coupling between acoustic normal modes. The predictions of the coupled mode theory are tested by numerical simulation using a mathematically dissimilar parabolic equation propagation model. Trends seen in the outputs from the computer model are explained by the predictions of the coupled mode theory, illustrating the validity of the coupled mode approach

2. INTRODUCTION

Internal waves [1] are vertical displacements in the background density structure of the ocean. They may be caused by tides, currents, wind stresses or by the passage of ships. In regions far from mesoscale features such as fronts or eddies [2] internal waves can be regarded as a small perturbation on an otherwise horizontally stratified ocean. The disparate nature of the sources of internal waves means that away from any single, dominant source, they are best described as having random amplitudes. An empirically derived spectrum exists [3] that relates the amplitude and oscillation frequencies of waves and describes the statistical nature of the waves' amplitudes. There also exists a dispersion relation between the frequency and wavenumber of internal waves.

The density fluctuations caused by internal waves result in fluctuations in the sound speed of the ocean. Internal waves consequently interact with acoustic waves travelling through the ocean. This interaction takes the form of a scattering of sound from a distribution of random sound speed fluctuations. This ocean-acoustic interaction process has received considerable study [4] and a variety of mathematical approaches have been adopted. One such approach [5] used the method of normal modes [6] as the basis for a perturbative approach for the prediction of the acoustic pressure in the ocean, including the effects of scattering from internal waves. The approach described these effects in terms of the scattering of sound between normal modes using a matrix of coefficients, known as a coupling matrix. The production of an expression for the acoustic pressure involves the calculation of the eigenvalues of this matrix.

Previously [5], the matrix has been assumed to be Hermitian and its eigenvalues consequently purely real. In this paper, it is shown that the behaviour of normal mode amplitudes in some situations can only be described using a *symmetric* matrix with consequently complex eigenvalues. Predictions made using a symmetric matrix are observed in the results of a numerical model, including the effects of idealised internal waves.

3. MODE COUPLING DUE TO INTERNAL WAVES

The normal mode expression for the total acoustic pressure at ranges very much greater than an acoustic wavelength in a horizontally stratified ocean is of the form [6]

$$p(r,z)=\sum_{n=1}^{N}\Phi_n(z,\xi_n)\Phi_n(z_s,\xi_n)\frac{e^{-i(\xi_n r+\frac{\pi}{4})}}{\sqrt{\xi_n r}} \tag{1}$$

where p is the acoustic pressure, r is the receiver range, z is the receiver depth, N is the total number of propagating normal modes, Φ_n is the nth normal mode, ξ_n is the horizontal wavenumber of the nth mode, z_s is the depth of the acoustic source.

Thus, the acoustic field is expressed in terms of a series of cylindrically spreading waves with depth distributions described by the mode shapes, Φ_n. The contribution made to the total pressure by a mode of given number is determined not only by the amplitude of the nth mode at the receiver depth but also at the source depth.

The method of normal modes has been extended to situations including isotropic, homogenous internal wave fields [5]. In such situations, the amplitude of each mode is no longer determined only by the source and receiver depths. The refraction undergone by sound as it passes through the randomly varying internal waves leads to scattering of sound between normal modes. In this case, the expression (1) can be re-written as

$$p(r,z)=\sum_{n=1}^{N}\Phi_n(z,\xi_n)\frac{A_n(r)e^{-i\xi_n r}}{\sqrt{\xi_n r}} \tag{2}$$

where A_n is the amplitude of the nth propagating mode, including the effects of mode coupling. An expression is now required for the modal amplitudes as a function of range, including the influence of source depth and scattering due to internal waves. It has been shown [5] that the modal amplitudes satisfy the differential equation

$$\frac{dA_n}{dr}=\sum_{m=1}^{N}R_{nm}A_m \tag{3}$$

where R_{nm} is the coefficient describing the strength of the link between the nth and mth propagating modes. Equation (3) holds for all propagating modes and it is convenient to express the resulting system of simultaneous equations in matrix form

$$\frac{d\underline{A}}{dr}=\underline{\underline{R}}\,\underline{A} \tag{4}$$

where $\underline{A}$ is an N-element column matrix containing all the modal amplitudes, $\underline{\underline{R}}$ is an N by N square matrix containing the coupling coefficients R_{nm} for all values of n and m. This is referred to as the coupling matrix. The expression for the coupling coefficients is [5]

$$R_{nm}=-i\left(\frac{\omega}{c_{av}}\right)^2\frac{e^{i(\xi_m-\xi_n)r}}{\sqrt{\xi_n\xi_m}}\int_0^\infty\Phi_n(z)\Phi_m(z)\frac{\delta c}{c}dz \tag{5}$$

where ω is the angular frequency of the acoustic source, c_{av} is the depth averaged sound speed, c is the sound speed and δc is the sound speed fluctuation arising from the internal waves.

4. SYMMETRY OF COUPLING DUE TO RESONANT SCATTERING

Initial inspection of (5) suggests that the coefficient R_{nm} will be the complex conjugate of R_{mn} and that, consequently, the coupling matrix will be Hermitian. However, this is not the case if the scattering of sound is considered to be a resonant process. The importance of resonant scattering processes in the interaction between internal waves and underwater sound has been demonstrated previously [7]. In resonant scattering, the only portion of the sound speed fluctuation field which contributes to the scattering of sound between two propagating modes is that part of its Fourier spectrum which has wavenumber equal to the *difference* between the horizontal wavenumbers of the two modes. The importance of this assumption can be illustrated by expressing the fractional sound speed fluctuation in (5) in terms of its Fourier transform

$$\frac{\delta c}{c} = \int_{-\infty}^{\infty} dk \left\{ A_k \Theta_k(z) e^{ikr} \right\} \tag{6}$$

where k is the horizontal wavenumber variable for the sound speed fluctuation field, A_k is the amplitude of the wave with wavenumber k and Θ_k is the distribution with depth of the fractional sound speed fluctuation with horizontal wavenumber k. Substitution of this expression into (5) gives

$$R_{nm} = -i\left(\frac{\omega}{c_{av}}\right)^2 \frac{1}{\sqrt{\xi_n \xi_m}} \int_{-\infty}^{\infty} dk \left\{ A_k e^{i[k-(\xi_m-\xi_n)]r} \int_0^{\infty} \Phi_n(z)\Phi_m(z)\Theta_k dz \right\} \tag{7}$$

The importance of the resonance condition is now clear. For the internal wave with wavenumber equal to the difference in the horizontal wavenumbers of the two modes, the part of (7) which is oscillatory in range is removed. This allows the coupling coefficients to become merely constants in the coupled mode equation, (4), rather than functions of range themselves. The part of the internal wave field with resonant wavenumber is shown by (7) to be the only part of the spectrum that does not average over range to zero. Restriction to the resonant wavenumber removes the integral over k in (7) to give

$$R_{nm} = -i\left(\frac{\omega}{c_{av}}\right)^2 \frac{A_\Delta}{\sqrt{\xi_n \xi_m}} \int_0^{\infty} \Phi_n(z)\Phi_m(z)\Theta_\Delta dz \tag{8}$$

where Δ is the horizontal wavenumber equal to the difference between ξ_n and ξ_m. Equation (8) allows the coupling coefficients to be evaluated simply using numerical integration routines, given knowledge of the acoustic normal modes and internal waves. Inspection of (8) shows it to be unchanged by interchanging n and m. Thus the value of R_{nm} is now equal to R_{mn} and the coupling matrix $\underline{R}$ is symmetric, rather than Hermitian. Equation (8) also shows that the coupling coefficient is proportional to the amplitude of the internal wave with the resonant wavenumber. For coupling coefficients on the leading diagonal, R_{nn}, the resonant wavenumber is equal to zero. This is because the fluctuations in sound speed due to internal waves have zero mean and the amplitude $A_{\Delta=0}$ is therefore equal to zero. Thus, there is no coupling between the nth mode and itself. However, inspection of the coupled mode equation (4) shows that these on-diagonal elements relate the rate of change of modal amplitudes to themselves. In the presence of any attenuation (due to absorption for example) this rate of change will be non-zero. Indeed, it will be equal to the exponential decay coefficient describing the level of absorption for each mode. These on-diagonal elements of the matrix will be purely real while (8) shows that the off-diagonal elements of the matrix will be purely imaginary.

5. SOLUTION OF COUPLED MODE EQUATION

The consequences of the coupling matrix's symmetry are seen in the calculation of its eigenvalues. Hermitian matrices have purely real eigenvalues while the eigenvalues of symmetric matrices may be complex. Consideration an artificially simple 2-mode situation can show the physical consequences of this. In this case, the coupling matrix is given by

$$\underline{\underline{R}} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{bmatrix} \tag{9}$$

where R_{11} is the attenuation coefficient of the first mode with similar notation being used for the second mode. The symmetry condition has been used to reduce the number of coupling coefficients to one instead of two. For 2x2 matrices, the eigenvalues can be related simply to the elements of the matrix [9] to give

$$\begin{aligned} \alpha_1 &= \frac{1}{2}(R_{11}+R_{22}) + \frac{1}{2}\sqrt{(R_{11}-R_{22}) - 4|R_{12}|^2} \\ \alpha_2 &= \frac{1}{2}(R_{11}+R_{22}) - \frac{1}{2}\sqrt{(R_{11}-R_{22})^2 - 4|R_{12}|^2} \end{aligned} \tag{10}$$

where $\alpha_{1,2}$ are the first and second eigenvalues of the matrix. The nature of the eigenvalues of the matrix is thus dependent upon the relative sizes of the coupling coefficient and the difference in the attenuation coefficients of the two modes. For the limiting case of zero coupling, the eigenvalues of the matrix equal the two, purely real, attenuation coefficients. As coupling initially increases from zero, equation (10) shows that the eigenvalues remain purely real but the difference between them reduces as the two eigenvalues approach the average value of the two modal attenuation coefficients. For a critical level of coupling, given by

$$|R_{12}|_{crit} = \frac{1}{2}|R_{11} - R_{12}| \tag{11}$$

the eigenvalues are both equal to the average value of the modal attenuation coefficients. For levels of coupling above this critical value, (10) shows that the eigenvalues become complex as the argument of the square root function becomes negative. The real parts of the eigenvalues remain at the average value of the modal attenuation coefficients but increasing coupling leads to increasingly large imaginary parts in the eigenvalues. The two eigenvalues have equal magnitude but oppositely signed imaginary parts.

The physical consequences of the complex nature of the eigenvalues are illustrated by the solution of the coupled mode equation (3). Standard methods such as Laplace transforms allow the solution of such a differential equation to be given as

$$A_n(r) = \sum_{j=1}^{N} B_{nj} e^{\alpha_j r} \tag{12}$$

where B_{nj} are complex coefficients whose values can be found by satisfying boundary conditions at zero range. The exponential dependence of modal amplitudes on the eigenvalues leads to purely real eigenvalues resulting in decaying amplitudes with range. Complex eigenvalues result in modal amplitudes that not only decay with range but also oscillate.

6. NUMERICAL SOLUTION

This change in the behaviour of modal amplitudes was investigated using a numerical propagation model that solved a wide-angle version of the parabolic equation [9]. This method is not based on mode coupling and any agreement between its predictions and those of the mode coupling theory could be taken to support the theory strongly. To simplify the calculation of the mode coupling coefficients, an artificially simple environment was chosen. The water column was modelled as being, on average, isovelocity, having a mean sound speed of 1500m/s. A sound speed fluctuation was introduced that was limited to the top 80m of the water column. At depths greater than this, no fluctuation was allowed. The sound speed fluctuation was made to vary sinusoidally with range in the top part of the water column. The amplitude and wavenumber of the oscillation was varied to investigate the predictions made by the mode coupling theory. A rigid seabed was simulated in the numerical model by setting its sound speed to be 10,000m/s and the density to be 10,000g/cm^3.

It is recognised that the environment chosen was extremely unrealistic in almost every aspect. The environment was used so that the acoustic calculations would be as straightforward as possible. The solution of the coupled mode equation in this simple environment was, however, very similar to the solution which would be carried out in an environment with physically reasonable sound speed profile and internal wave field. The solution of the coupled mode equation requires only numerical values for coupling and modal attenuation coefficients. It was the consequences of variations of the relative sizes of these coefficients that were studied in the research reported here.

In an isovelocity environment, the acoustic normal modes are sinusoidal. The integral involved in the calculation of the coupling coefficients can be performed analytically, allowing the coupling coefficients to be related directly to the amplitude of the artificial sound speed fluctuations. In order to simplify the situation further, the acoustic frequencies studied were chosen so as to be low enough to allow only two normal modes to propagate [9]. For the cases illustrated here, a frequency of 15Hz was used.

To introduce attenuation to the system, an arbitrary value of seawater absorption coefficient was chosen. This was consistent with the artificiality of the sea water sound speed fluctuations and was set so as to give a reasonably large level of attenuation at the maximum range of 10km used in the numerical simulation. A value of 0.34 dB per acoustic wavelength was used.

The receiver depth for which results were calculated was selected to be 66.66m, corresponding to the depth of a node in the second propagating mode. This was done so that the variation of the amplitude of the first mode with depth could be directly displayed without interference from the second mode.

The baseline case studied was without simulated internal waves. In this case, the two modes propagate separately and their amplitudes simply decay. The rates at which they decay were determined by running the SUPERSNAP [9] normal mode model for the simple isovelocity environment under study. These values correspond to the on-diagonal elements of the mode-coupling matrix and were equal to

$$R_{11} = 4.04x10^{-4} m^{-1}$$
$$R_{22} = 5.83x10^{-4} m^{-1} \qquad (13)$$

The predictions of the normal mode model SUPERSNAP were compared with those of the parabolic equation model and good agreement was found.

The values of the modal decay rates were used to determine the critical level of coupling for the two-mode system. Critical coupling was predicted to occur for a value of R_{12}=8.95x10^{-6}m^{-1}. This value of coupling coefficient was predicted by mode coupling theory to occur when the sound speed fluctuations were given amplitude of 4.134ms^{-1} and a wavenumber equal to 1.9278x10^{-2}m^{-1}, the resonant condition for the two propagating modes. This was deduced using the SUPERSNAP model to determine the wavenumbers of the two modes. Initially, propagation was calculated by the parabolic equation model for sound speed fluctuations with amplitude two thirds of this predicted critical level. Results for the receiver at 66.66m are shown in figure 1.

The figure shows the results for the sub-critical coupling case in the solid line with the results for the zero coupling case in a dotted line. The rate of decay of the first mode is shown to increase due to the coupling as the solid line shows progressively greater loss than the dotted curve with increasing range. Note that the modal amplitude still decreases monotonically with range. The results shown in figure 1 illustrate that the introduction of sound speed fluctuations with amplitude 2.756ms^{-1} and resonant wavenumber led to noticeable differences in propagation loss. The validity of the resonant scattering assumption was investigated by running the parabolic equation model in an environment with sound speed fluctuations of the same amplitude but with a non-resonant wavenumber of 1.0x10^{-2}m^{-1}. It was found in this case that the rate of decay of the first mode was unchanged by the presence of the sound speed fluctuations. This observation supported the use of the resonant scattering condition for the calculation of normal mode coupling.

Figure 2 shows the predictions of the parabolic equation model for the critical coupling case. In this situation, the increase in loss with range is still monotonic but the rate of increase of loss is greater than for the previous, lower coupling case. Simple curve fitting revealed that the decay rate for the mode was equal to the average value of the two initial modal decay rates.

Figure 3 shows the results for a super-critical coupling condition, achieved by setting the amplitude of the sound speed fluctuations to be 8.268ms^{-1}. Note that the amplitude of the first mode as illustrated by these results no longer decreases monotonically. Instead, there is an oscillation in modal amplitude with losses increasing in a region corresponding to a minimum in the modal amplitude. For ranges beyond these minima, the propagation loss decreases, indicating an increase in the amplitude of the first mode. These results show how the amplitude of the first mode oscillates with range for super-critical coupling.

As a final investigation of the predictions of mode coupling theory, a further super-critical coupling case was produced by setting the amplitudes of the sound speed fluctuations to be 11.024ms^{-1}. The results of the parabolic equation model for this case are given in figure 4. It is shown that increasing the level of coupling decreases the range at which the minimum in the amplitude of the first mode occurred, as predicted by the mode coupling theory.

7. SUMMARY

The scattering of sound from internal waves in the ocean has been investigated using a mathematical approach based on normal mode coupling, using a resonant scattering condition. The predictions of this theory have been numerically investigated using a parabolic equation solving computer program. Trends seen in the output of this model for increasing levels of artificial internal wave activity have been explained using the predictions of the coupled mode theory. The agreement observed between the theory's predictions and the outputs from the mathematically dissimilar computer propagation model are taken to support the use of the coupled mode theory in investigations of the scattering of sound by underwater internal waves.

8. REFERENCES

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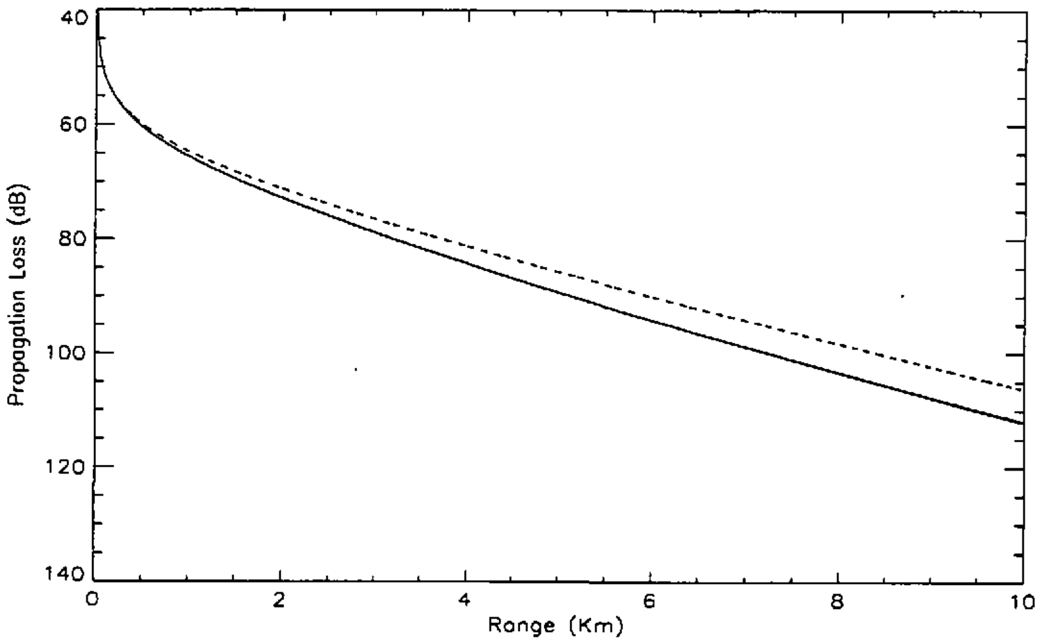

Figure 1 *Propagation loss versus range curves for 66m receiver with 2.756m/s amplitude fluctuations (solid line) a nd without (dotted line).*

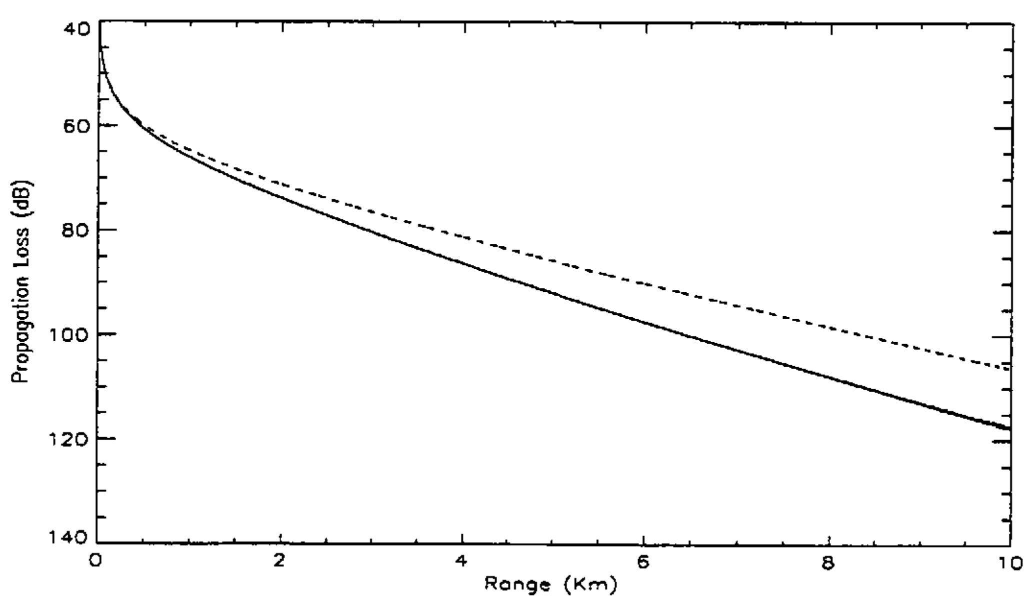

Figure 2 *Propagation loss versus range curves for 66m receiver with 4.134m/s amplitude fluctuations (solid line) and without (dotted line)*

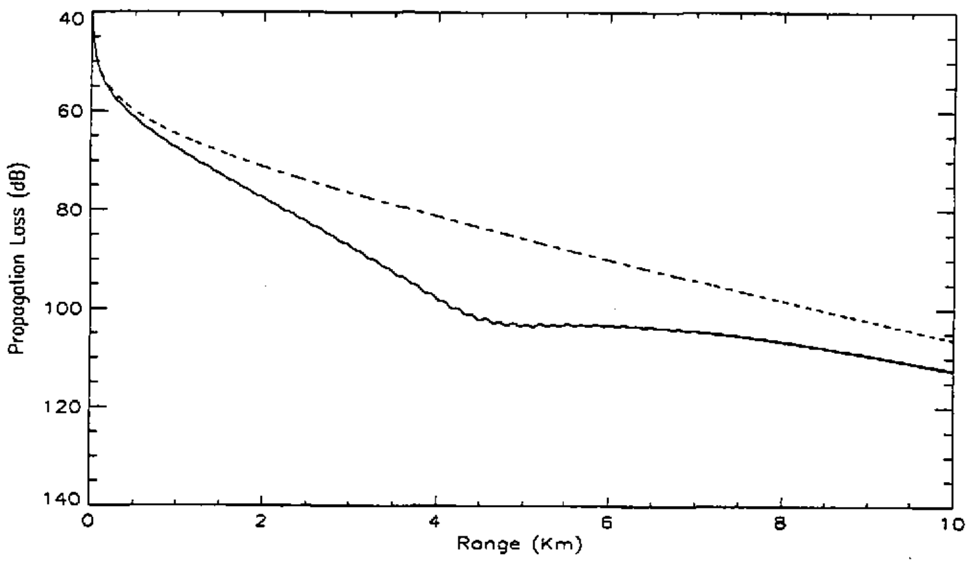

Figure 3 *Propagation loss versus range curves for 66m receiver with 8.268m/s amplitude fluctuations (solid line) and without (dotted line).*

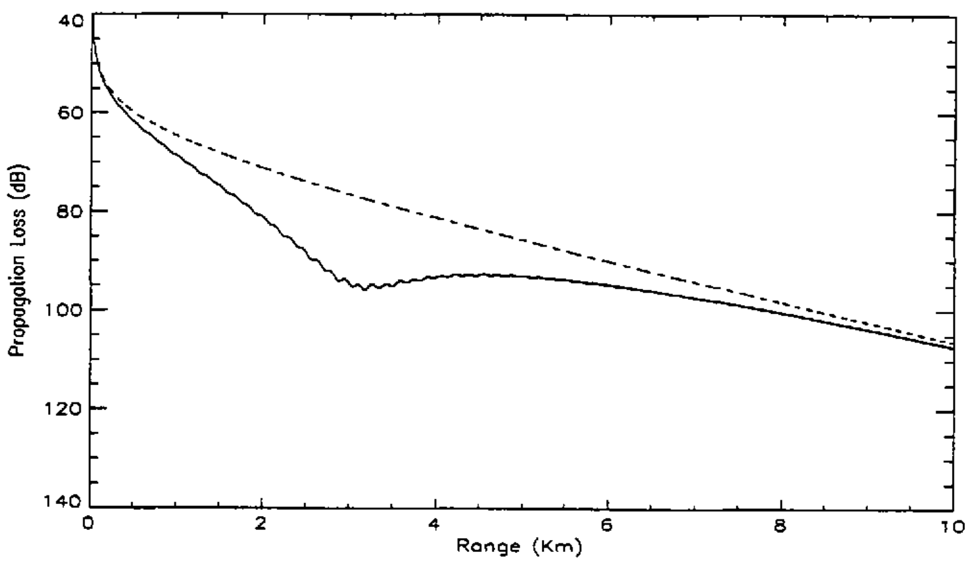

Figure 4 *Propagation loss versus range curves for 66m receiver with 11.024m/s amplitude fluctuations (solid line) and without (dotted line)*