INVERSE ACOUSTIC OBSTACLE SCATTERING FOR FAR FIELD DATA WITHOUT PHASE

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1 INTRODUCTION

In this paper we present a solution method for the inverse problem of time-harmonic acoustic wave scattering where the shape of an obstacle is reconstructed from a given incident field and the modulus of the far field pattern of the scattered field. The approach originates from the method suggested by Kress and Rundell[5] and it is based on a pair of nonlinear and ill-posed integral equations to be solved for the shape of the unknown boundary. The considered problem arises in many practical applications such as nondestructive testing, ultrasound tomography and seismic imaging.

Mathematically the problem can be formulated as follows. Let $D \subset \mathbb{R}^2$ be a simply connected bounded domain with C^2 boundary Γ. We consider the scattering of plane wave incidence $u^i = e^{ikx \cdot d}$ with wave number $k > 0$ and direction d of propagation. Now, for a sound-soft obstacle the direct scattering problem is to find a total field $u = u^i + u^s$ such that u^s solves the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D} \tag{1}$$

fulfills to the Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \Gamma \tag{2}$$

and in addition, the scattered field u^s satisfies the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \qquad r = |x|. \tag{3}$$

This condition ensures uniqueness of the solution to the scattering problem and implies the asymptotic behavior of an outgoing cylindrical wave

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|} \right) \right\}, \qquad |x| \to \infty,$$

uniformly in all directions $\hat{x} = x / |x|$.

Inverse Problem:

Given the modulus of far field pattern $|u_\infty|$ for one incident wave u^i determine the boundary Γ of the sound-soft scatterer D.

For the numerical solution of this problem Kress and Rundell[3] suggested a Newton method. They also have shown that only the shape of the obstacle can be recovered from the modulus of the far field pattern for one incident plane wave, but not the location. The reason for this consists in the translation invariance of the modulus of the far field data for the plane wave incidence, i.e. for the shifted domain $D_\varepsilon := \{x + \varepsilon h : x \in D\}$ with $h \in \mathbb{R}^2$, the far field pattern $u_{\infty,\varepsilon}$ fulfills the equality

$$u_{\infty,\varepsilon}(\hat{x}) = e^{ik\varepsilon h\cdot(d-x)}u_{\infty}(\hat{x}), \qquad \hat{x}\in\Omega, \tag{4}$$

where Ω is a unit circle.
For the shape reconstruction we employ the iterative method, which was first suggested by Kress and Rundell[5] for the inverse boundary value problem for harmonic functions and then extended to inverse scattering problems[3]. In the next section we derive the system of nonlinear integral equations and formulate the iterative algorithm based on it. We conclude by presenting numerical examples that illustrate feasibility of the method. More detailed description of the method can be found in[2].

2 NONLINEAR INTEGRAL EQUATIONS METHOD

Due to Huygen's principle, Theorem 3.12 in[1], the total field has the integral representation

$$u(x) = u^i(x) - \int_\Gamma \frac{\partial u}{\partial \nu}(u)\Phi(x,y)ds(y), \quad x\in \mathrm{R}^2\setminus\overline{D}, \tag{5}$$

where ν is outward unit normal vector to Γ and $\Phi(x,y) = i/4H_0^{(1)}(k\,|\,x-y\,|)$ is a fundamental solution to the Helmholtz equation with the Hankel function $H_0^{(1)}$ of order zero and of the first kind. The corresponding far field pattern of the scattered field u^s is given by

$$u_\infty(\hat{x}) = -\gamma\int_\Gamma \frac{\partial u}{\partial \nu}(u)e^{-ik\hat{x}\cdot y}ds(y), \qquad \hat{x}\in\Omega,\ \ \gamma = e^{i\pi/4}/\sqrt{8\pi k}. \tag{6}$$

For further analysis and in order to introduce integral operators we assume that the boundary Γ has a C^2-smooth and 2π-periodic star-like parameterization

$$z(t) = r(t)(\cos t, \sin t),\ \ t\in[0,2\pi] \tag{7}$$

with $|\,z'(t)\,|>0$ for $t\in[0,2\pi]$. The unit circle Ω is parameterized by $z_\infty(t) = (\cos t,\sin t)$.
We introduce the parameterized single layer operator $A_z : L^2[0,2\pi]\to L^2[0,2\pi]$ by

$$(A_z\varphi)(t) = \frac{i}{4}\int_0^{2\pi} H_0^{(1)}(k\,|\,z(t)-z(\tau)\,|)\varphi(\tau)d\tau,\ \ t\in[0,2\pi],$$

and define the parameterized far field operator $A_{z,\infty} : L^2[0,2\pi]\to L^2[0,2\pi]$ by

$$(A_{z,\infty}\varphi)(t) = \gamma\int_0^{2\pi} e^{-ikz_\infty(t)\cdot z(\tau)}\varphi(\tau)d\tau, \qquad t\in[0,2\pi].$$

The nonlinear dependence of the operators $A_z, A_{z,\infty}$ on the boundary curve Γ is indicated by the subindex z. Now, from the equalities (5), (6) and the boundary condition (2) we derive a system of two-by-two nonlinear equation for the unknowns z and $\varphi = |\,z'\,|\,\frac{\partial u}{\partial \nu}\circ z$

$$A_z\varphi = w_z \tag{8}$$

and

$$\overline{A_{z,\infty}\varphi}\,A_{z,\infty}\varphi = |\,w_\infty\,|^2\ . \tag{9}$$

Here $w_z = u^i\circ z$ and $|\,w_\infty\,| = |\,u_\infty\,|\circ z$ are parameterized incident field and the modulus of the far field pattern, respectively.
It is easy to check that the solution to the inverse scattering problem satisfies the system (8)-(9) and vice versa; in this sense the inverse problem and the system of nonlinear integral equation are equivalent[2].

To solve the system of nonlinear equations we suggest an iterative method involving the full linearization of (8)-(9) with respect to both variables: the boundary parameterization z and the density φ. The operators $A_z, A_{z,\infty}$ are Fréchet differentiable[6] and their derivatives can be found by formally differentiating their kernels with respect to z

$$(A_z^{'}(\varphi)\varsigma)(t) = \frac{ik}{4}\int_0^{2\pi} H_0^{(1)'}(k\,|\,z(t)-z(\tau)\,|)\frac{[z(t)-z(\tau)]\cdot[\varsigma(t)-\varsigma(\tau)]}{|\,z(t)-z(\tau)\,|}\varphi(\tau)d\tau,\;\; t\in[0,2\pi],$$

and

$$(A_{z,\infty}^{'}(\varphi)\varsigma)(t) = -ik\gamma\int_0^{2\pi} e^{-ikz_\infty(t)\cdot z(\tau)} z_\infty(t)\cdot\varsigma(\tau)\varphi(\tau)d\tau,\; t\in[0,2\pi],$$

where $\varsigma(t) = q(t)(\cos t, \sin t),\;\; t\in[0,2\pi]$. Similarly one obtains

$$(w_z{}'\varsigma)(t) = ike^{ikd\cdot\varsigma(t)}, \quad t\in[0,2\pi].$$

Finally, by the product rule we calculate the derivative of $\overline{A_{z,\infty}}\,A_{z,\infty}$ in the direction ς

$$(\overline{A_{z,\infty}\varphi}\,A_{z,\infty}\varphi)'\varsigma = 2\Re(\overline{A_{z,\infty}\varphi}\,A_{z,\infty}^{'}(\varphi)\varsigma).$$

Since the integral operators are linear with respect to φ the linearization of (8)-(9) leads to

$$A_z\psi + A_z^{'}(\varphi)\varsigma - w_z^{'}\varsigma = w_z - A_z\varphi \qquad (10)$$

and

$$2\Re\left(\overline{A_{z,\infty}\varphi}\,(A_{z,\infty}\psi + A_{z,\infty}^{'}(\varphi)\varsigma)\right) = |\,w_\infty\,|^2 - |\,A_{z,\infty}\varphi\,|^2\,. \qquad (11)$$

Iterative scheme:

Step1. We choose an initial guess for the boundary Γ parameterized by z and find the density φ from (8).

Step2. Given a current approximation for z and φ, we solve the linear system (10)-(11) and update $z = z + \varsigma$ and $\varphi = \varphi + \psi$

Step3. The Step2 is repeated until a suitable stopping criteria is satisfied, e.g. Morozov's discrepancy principle.

We note that the linearized system inherits the ill-posedness of the inverse problem; therefore one has to incorporate a regularization method, e.g. Tikhonov regularization.

3 NUMERICAL EXAMPLES

To approximate the radial function r in (7) and its update q we use the space of trigonometric polynomials of degree less than or equal to K, that is,

$$q(t) = a_0 + \sum_{m=1}^{K} a_m \cos mt + \sum_{m=1}^{K} b_m \sin mt, \qquad t\in[0,2\pi]. \qquad (12)$$

As it was shown[2, 4] we cannot recover the coefficients a_1 and b_1 in (12), therefore we set them to be equal to the correct values. This helps us to compare the reconstructed curve with the correct one, although the procedure works for other values of a_1, b_1 and this leads to a shifted curve. Thus, the only $2K-1$ coefficients are left to determine.

The synthetic data were obtained by solving (1)-(3) via the combined double- and single-layer approach, that avoids committing an inverse crime. Perturbed data $|\,u_\infty^\delta\,|^2$ were constructed in the following way

$$|\,u_\infty^\delta\,|^2 = |\,u_\infty\,|^2 + \delta\frac{\left\|\,|\,u_\infty\,|^2\right\|_{L^2}}{\|\eta\|_{L^2}}\eta, \qquad (13)$$

where η is normally distributed random variable and δ is the relative noise level.

To approximate the integral operators $A_z, A_{z,\infty}, A_z^{'}, A_{z,\infty}^{'}$ which have at most logarithmic singularity in their kernels we use $2M$ equidistant quadrature points for the trapezoidal rule and the logarithmic singularity quadratute[1]. Since the right-hand side of the equation (11) and the unknown ς are real valued, but the right-hand side of (10) and the unknown ψ are complex valued, therefore we split the density ψ into its real and imaginary components and consider the three-by-three system of linear real valued equations. We apply to this system a fully discrete collocation at the same $2M$ equidistant points and obtain a $6M \times 6M$ linear system for the $2K-1$ coefficients (a_m, b_m) from (12) and the $4M$ approximate values for $\psi(t_0),\ldots,\psi(t_{2M-1})$ with $t_j = \pi j / M$, $j = 0,\ldots,2M-1$. In the examples below we use an $L^2 \times H^2 \times H^2$ penalty term in the regularization. The wave number k is set to be equal to 1, $K = 20$ and $M = 32$. The initial guess is taken to be the circle indicated by dotted · · · lines in the figures. The exact boundary contour is displayed as a solid line and the reconstructed contour from exact data is displayed as a dashed - - - line. The reconstructed contours from noisy data with minimal and maximal relative error are displayed by dashed - - - and dash-dot - · -· lines, respectively. The arrow in the figures shows the direction of the incoming wave.

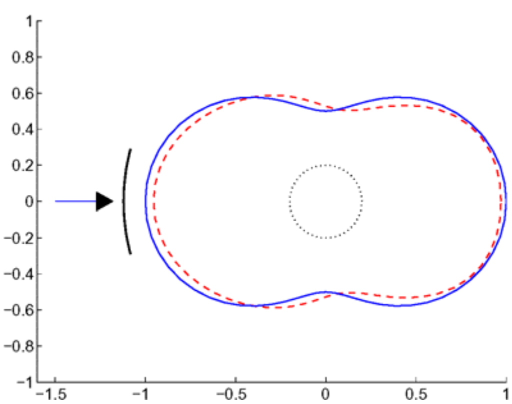
We present the reconstruction of a peanut shaped contour with radial function

$$r(t) = \frac{1}{2}\sqrt{3\cos^2 t + 1}, \quad t \in [0, 2\pi],$$

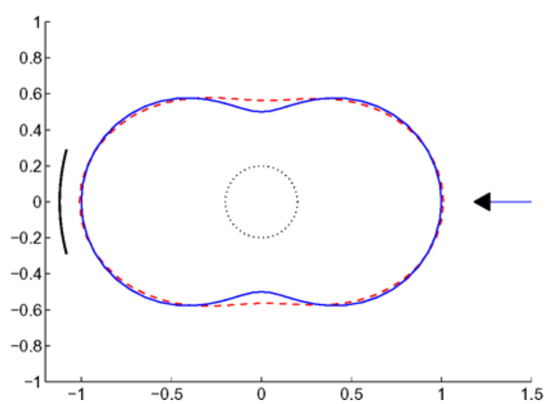
from the modulus of the far field data given in a range of angles centered around the direction $(-1, 0)$ for incidence with directions $d = (1,0)$ and $d = (-1,0)$. The measured aperture is displayed in the figures by a circular arc for the corresponding angle. As a stopping criteria for the iterative procedure we choose

$$\frac{\left\| |w_\infty|^2 - |A_{z,\infty}\varphi|^2 \right\|_{L^2}}{\left\| |w_\infty|^2 \right\|_{L^2}} \le \tau, \tag{14}$$

for some sufficiently small parameter $\tau > 0$ depending on the noise level.
In Figure 1 the reconstructed curves from the exact modulus of the far field data in the range of angle $\pi/6$ are shown. The regularization parameters were chosen to be $\alpha_n = 0.0001(2/3)^n$ for the density ψ and $\beta_n = 0.01(2/3)^n$ for the boundary update ς. In the stopping criteria (14) the parameter $\tau = 10^{-7}$ was used. The iteration started with an initial guess as a circle of radius 0.2.



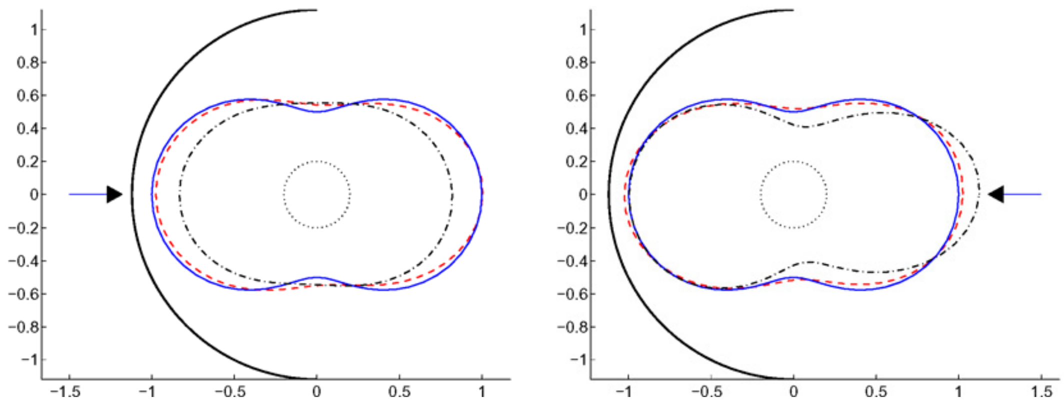
a) $d = (1,0)$, 76 its.



b) $d = (-1,0)$, 44 its.

FIGURE 1: Reconstruction of the sound-soft peanut shaped obstacle, exact data

During the numerical experiments we observed that the accuracy of reconstruction depends only marginally on the radius of the initial guess for the incoming wave with the direction $d = (-1,0)$. We also noticed that generally for a small aperture in the scattering direction one needs twice as many iterations as for an aperture in the backscattering direction.
For further investigations 10 different sets of perturbed data were generated in the form (13) with $\delta = 0.03$. The regularization parameters were changed to $\alpha_n = 0.0001(10/11)^n$, $\beta_n = 0.01(10/11)^n$ and $\tau = 0.0299$. In Figure 2 the reconstructions with the minimal and maximal relative errors for the aperture π and two different incident directions are presented.



a) best - 11 its., worst – 4 its   b) best - 5 its., worst – 3 its

FIGURE 2: Reconstruction of the sound-soft peanut shaped obstacle, 3% noise level

For the reconstruction from the perturbed data we observed the following behavior. For the large radius of initial guess one can reconstruct obstacles with reasonable accuracy up to the aperture $\pi/4$ for both directions of the incoming way. The error of the reconstruction is smaller for the aperture on the backscattering side and for the aperture on the scattering side the error level decreases with decreasing of the radius of initial guess.

Summarizing, the numerical experiments show accurate reconstructions from the modulus of the far field data given in different ranges of angles with reasonable stability against noisy data. To obtain a reconstruction with a certain accuracy from the modulus of the exact far field data we need more iterations in comparison with a reconstruction from the full data, specially for a small radius of the initial guess. The proposed method was extended to the case of sound-hard obstacles with modulus of the far field pattern as a data and accurate reconstructions were obtained[2].

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