A NEW CELL MODEL FOR PREDICTING THE ACOUSTICAL PROPERTIES OF GRANULAR MATERIALS.

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1. INTRODUCTION

Models for the acoustical properties of porous materials can be classified as 'internal' and 'external' flow models. External flow models consider the fluid flow around constituent particles rather than in hypothetical pores. They require knowledge of porosity, particle shape and the properties of packing. Coupled phase theory [1], which applies for wavelengths that are large compared with individual particles, is an example of an external flow model. When calculating the coupling drag term $D(\omega)$ and the heat transfer term $S_H(\omega)$ in concentrated suspensions and granular materials, interactions between neighbouring particles can be taken into account using cell models [2]-[4]. Cell models are based on hypothetical fluid – filled regions or 'cells' surrounding each particle and assume that the particle affects fluid flow only inside this cell. The influence of neighbouring particles is taken into account through boundary conditions on the outer surface of the cell. In a highly-concentrated system the applicability of the cell model approach becomes questionable. Neighbouring solid particles will be in contact and hence will penetrate the hypothetical cell around each particle. Nevertheless, in this paper, we will show that adjustment of the cell radius enables reasonable values for dynamic drag parameters as well as for impedance of a material consisting of rigidly – fixed stacked spheres.
In the material with solid phase density and heat capacity much greater than those of fluid, the complex compressibility and complex density depend on the heat transfer term $S_H(\omega)$ and the viscous drag term $D(\omega)$ respectively:

$$C(\omega)=(\frac{\partial\rho}{\partial p})_T\frac{1}{\rho_0}\left(1-\rho_0\frac{\alpha^2}{(\frac{\partial\rho}{\partial p})_T}\frac{T_0}{\rho_o c_p}\left(1+\frac{S_H(\omega)}{-i\omega\phi\rho_0 c_p}\right)^{-1}\right)$$

$$\rho(\omega)=\rho_0+\frac{D(\omega)}{-i\omega\phi}, \qquad (1)$$

where ρ_o, T_0, α and c_p are fluid density, temperature, the volume thermal expansion coefficient and heat capacity at constant pressure, ϕ is the porosity.

2. COMPLEX DENSITY IN CELL MODEL.

In an oscillatory flow of incompressible viscous fluid with macroscopic fluid velocity $\vec{v}e^{-i\omega t}$ around a fixed spherical particle the fluid velocity field around a sphere can be represented by [5]:

$$\vec{u}=[\vec{v}+curlcurl(f(r)\vec{v})]e^{-i\omega t} \quad ,$$

here *f(r)* is the potential function which obeys the following equation:

$$\frac{d}{dr}\left(\Delta\Delta f-\frac{\omega\rho_0}{i\eta}\Delta f\right)=0, \tag{2}$$

where $\Delta=\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}$ is the radial part of the spherical laplacian operator and η is the fluid viscosity.

In terms of the potential function the non-slip boundary conditions on the particle surface can be rewritten as:

$$\frac{df}{rdr}\Big|_R=\frac{1}{2},\quad \Delta f\big|_R=\frac{3}{2}. \tag{3}$$

In the Kuwabara/Strout model [2], [4] the vorticity of the fluid velocity is assumed to be zero at the outer cell boundary. In terms of the potential function that means that

$$\frac{d\Delta f}{dr}\Big|_b=0, \tag{4}$$

where b is the cell radius.

We derive an additional boundary condition on the outer cell surface from matching the macroscopic fluid velocity v with the z component of the fluid particle velocity averaged over the cell volume. Hence

$$\langle u_z\rangle_V=\frac{2\pi}{\frac{4}{3}\pi\left(b^3-R^3\right)}\int_0^\pi\int_R^b\left(u_r(r,\theta)\cos\theta-u_\theta(r,\theta)\sin\theta\right)r^2\sin\theta d\theta dr=v,$$

it gives

$$\frac{df}{rdr}\Big|_b=\frac{\Theta}{2}, \tag{5}$$

where

$$\Theta=\left(\frac{R}{b}\right)^3.$$

The drag force, *F*, parallel to the direction of the velocity *v* is calculated from:

$$F=\int\left(-P\cos\theta+\sigma_{rr}\cos\theta-\sigma_{r\theta}\sin\theta\right)df=\frac{4}{3}\pi R^3 v\eta\left(-3\frac{d\Delta f}{rdr}\Big|_R+\Delta\Delta f\big|_R-\frac{\omega\rho_0}{i\eta}\right)$$

where P is pressure, σ_{rr} and $\sigma_{r\theta}$ are the components of stress tensor, integration is performed over particle surface.

The drag term is calculated as the sum of the drag forces on individual particles.

As a result of solving equation (2) with boundary conditions (3)-(5) we get:

$$D(\omega)=+i\omega\rho_0(1-\phi)\left(1+\frac{3}{2}\left(\Theta^{-1}-1\right)\frac{\exp\left(2s(\Theta^{-\frac{1}{3}}-1)\right)A_1+A_2}{\exp\left(2s(\Theta^{-\frac{1}{3}}-1)\right)B_1-B_2}\right), \tag{6}$$

where

$$A_1=\left(s\Theta^{-\frac{1}{3}}-1\right)\left(s^2+3s+3\right),\ A_2=\left(s\Theta^{-\frac{1}{3}}+1\right)\left(s^2-3s+3\right),\ s=\sqrt{\frac{\eta}{i\omega\rho_0}}R$$

$$B_1 = \left(s\Theta^{-\frac{1}{3}} - 1\right)\left(-s^2\left(\Theta^{-1} - 1\right) + 3s + 3\right), \qquad B_2 = \left(s\Theta^{-\frac{1}{3}} + 1\right)\left(s^2\left(\Theta^{-1} - 1\right) + 3s - 3\right)$$

3. DYNAMIC DRAG PARAMETERS IN CELL MODEL

Dynamic drag parameters – DC permeability k_0, tortuosity α_∞ and characterisrtic viscous dimension Λ are calculated from the low – and high – frequency limits of complex density [6]:

$$\lim_{\omega\to 0} \frac{\rho(\omega)}{\rho_0} = \frac{\phi}{k_0}\frac{i\eta}{\omega\rho_0}, \quad \lim_{\omega\to\infty} \frac{\rho(\omega)}{\rho_0} = \alpha_\infty + \left(\frac{i\eta}{\omega\rho_0}\right)^{\frac{1}{2}} \frac{2\alpha_\infty}{\Lambda}.$$

Comparison of these expressions with low – frequency limit of complex density in cell model gives:

$$k_0 = \frac{2}{9}\frac{\phi^2}{(1-\phi)(1-\Theta)\Omega_k}R^2, \quad \text{where } \Omega_k = \frac{5}{5 - 9\Theta^{\frac{1}{3}} + 5\Theta - \Theta^2} \qquad (7)$$

Expressions for tortuosity and characteristic viscous dimension are:

$$\Lambda = \frac{4(1-\Theta)\phi\alpha_\infty}{9(1-\phi)}R, \qquad \alpha_\infty = 1 + \frac{1-\phi}{2\phi} \qquad (8)$$

The latter expression coincides with well – known result for tortuosity obtained by Berryman [7]. So far we have not defined the parameter Θ – the ratio of the particle volume to the cell volume. In the original version of Kuwabara/Strout model it was taken to be equal to the volume fraction of the solid phase in the material:

$$\Theta = 1 - \phi. \qquad (9)$$

Instead, in [8] we propose use of larger spherical cells based on spheres that circumscribe the unit cells of the packing. In this case for simple cubic (SC) packing:

$$\Theta = \frac{3}{\sqrt{2}\pi}(1-\phi) \cong 0.675(1-\phi), \qquad (10)$$

Chapman and Higdon have calculated the drag parameters and the dynamic permeability of different packings of identical spherical particles numerically [9]. In Figure 1A, their numerical predictions of DC permeability for SC packing for the range of porosities up to the close packing limit $\phi_c = 0.4764$ are compared with cell model results. If the model with modified cell radius (10) is used, there is tolerable agreement (within 18%) even at the porosity corresponding to close packing. With the unmodified cell radius (9), the agreement with numerical results is good only for very dilute systems. In Figure 1B, the corresponding comparisons for characteristic viscous dimension are presented. The results of cell model with modified radius are closer to numerical values than those with the unmodified radius, except for close packing. Overall, the error is not more than 20% for the modified cell model, whereas with the unmodified theory and some values of porosity, it exceeds 40%. In Figure 1C the cell model predictions of formation factor $F = \frac{\alpha_\infty}{\phi}$ are compared with numerical results. Since the value of formation factor does not depend on parameter Θ in the cell model, both versions give the same results. The

deviation between the numerical results and the cell model estimates does not exceed 12% even for close packing.
In further comparisons the relationship (10) between cell radius parameter and porosity will be used.

4. RELATIONSHIP BETWEEN VISCOUS AND THERMAL PARAMETERS OF SPHERICALLY - GRAINED MEDIA

Now consider the effect of the surrounding particles on the heat transfer from the sphere subject to an oscillating temperature $T_1 e^{-i\omega t}$ in the surrounding fluid.
The amplitude of temperature distribution *T(r)* in the fluid can be found from the equation of heat transfer :

$$\Delta T - (\frac{\omega \rho_0 c_p}{ik}) T = 0, \qquad (11).$$

On the particle surface

$$T|_R = T_1, \qquad (12)$$

and, according to the 'mirror approximation' [10], zero temperature flux may be assumed on the external boundary of a hypothetical spherical cell of radius *b* :

$$\frac{dT}{dr}|_b = 0. \qquad (13)$$

The heat transfer term S_H can be found as a sum of heat transfer rates from single particles [11]:

$$S_H = \frac{-3(1-\phi)k}{R} \frac{\frac{d\left(\frac{T}{T_1}\right)}{rdr}|_R}{1-\frac{\overline{T}}{T_1}}, \qquad (14)$$

The volume averaged temperature of the fluid in the cell is:

$$\overline{T} = \frac{3}{b^3 - R^3} \int_R^b T(r) r^2 dr = \frac{3\Theta}{1-\Theta} \frac{k}{i\omega\rho_0 c_p} \frac{dT}{rdr}|_R .$$

From the symmetry of equations and boundary conditions (2) – (4) and (11) – (13) it is possible to get a relationship between potential function laplacian Δf and dimensionless temperature. Taking (6) and (14) into account, this gives the relationship between the drag coefficient $D(\omega)$ and the heat transfer coefficient $S_H(\omega)$, i.e.

$$S_H(\omega) = \frac{2c_p}{3N_{pr}} (1-\Theta) \left[D(\omega N_{pr}) + i\omega\rho_0 N_{pr} \frac{(1-\phi)}{2} \right].$$

Consequently, due to (1), the relationship between complex density and compressibility is:

$$C(\omega) = \frac{1}{\rho_0} (\frac{\partial \rho}{\partial p})_T (1 - \frac{\alpha^2 T_0}{c_p (\frac{\partial \rho}{\partial p})_T F_2(\rho(\omega N_{PR})}), \qquad (15)$$

where $F_2(\rho(\omega)) = \frac{2}{3}(1-\Theta)\left[\frac{\rho(\omega)-\rho_0}{\rho_0} - \frac{1-\phi}{2\phi}\right]+1$.

In view of the well-known high-frequency limit of compressibility, the function $F_2(\rho(\omega))$ can also be written as

$$F_2(\rho(\omega)) = \frac{2}{3}(1-\Theta)\left[\frac{\rho(\omega)-\rho(\omega\to\infty)}{\rho_0}\right]+1. \qquad (15')$$

The relationship (15) is analogous to those obtained by Stinson [12] for cylindrical tubes.
Johnson [6] has proposed expression for the dynamic density of porous material:

$$\frac{\rho(\omega)}{\rho_0} = \alpha_\infty\left(1+\frac{i\eta\phi}{\omega\rho_0 k_0\alpha_\infty}\left(1+\frac{\omega\rho_0 4\alpha_\infty^2 k_0^2}{i\eta\Lambda^2\phi^2}\right)^{\frac{1}{2}}\right), \qquad (16)$$

It has been shown by Champoux and Allard [13] and Lafarge et al [14], that, for porous materials of arbitrary internal structure, the dynamic compressibility can be calculated from

$$C(\omega) = \frac{1}{P_0}\left(\gamma-(\gamma-1)\left(1+\frac{i\eta\phi}{\omega\rho_0 k_0' N_{pr}}\left(1+\frac{\omega\rho_0 4k_0'^2 N_{pr}}{i\eta\Lambda'^2\eta\phi^2}\right)^{\frac{1}{2}}\right)^{-1}\right), \qquad (17)$$

where the parameters Λ' and k_0' are the characteristic thermal dimension and steady state thermal permeability, $N_{pr} = \frac{\eta c_p}{k}$ is the Prandtl number, γ is the ratio of the specific heats.

Application of the relationship (15') to (16) and (17) gives specific relationships between the viscous and thermal characteristics for granular materials:

$$k_0' = \frac{3}{2(1-\Theta)}k_0, \quad \Lambda' = \frac{3}{2\alpha_\infty(1-\Theta)}\Lambda. \qquad (18)$$

5. COMPARISON WITH IMPEDANCE DATA

If the expressions for complex density and compressibility are known, the propagation constant K, the characteristic impedance Z and the normalised impedance Z_1 of a layer of thickness *d* can be calculated from:

$$Z(\omega) = \left(\frac{\rho(\omega)}{C(\omega)}\right)^{0.5}, K(\omega) = \omega\ (\rho(\omega)C(\omega))^{0.5}, \ Z_1(\omega) = \frac{Z(\omega)}{\rho_0 c_0\phi}\coth(iK(\omega)d),$$

In Figure 2, A, B the predictions of the complete cell model (given by (6) and relationship (15)) and predictions of Johnson/Allard model combined with relationships (18) will be compared with impedance data for the layer of glass beads. Where experimental values of tortuosity and flow resistivity $\sigma = \frac{\eta}{\kappa_o}$ are not available they were calculated using (7) and (8). In general, two-parameterscell model gives similar agreement with data to that given the Johnson/Allard model while requiring fewer parameters.

6. CONCLUDING REMARKS

The condition on the radial fluid velocity component at the cell boundary assumed in the Kuwabara/Strout cell model has been modified so as to match the cell averaged particle velocity and the macroscopic fluid velocity. Together with 'the mirror approximation' on heat transfer, it has been used to allow for hydrodynamic and thermal interactions between particles in these media. A new similarity relationship between dynamic viscous and thermal characteristics for granular materials has been derived on the basis of these developments. This has lead to the development of a complete cell model for the acoustical properties of granular materials. In general, it requires only three parameters: the mean radius of the constituent particles, porosity and an adjustable cell radius. The results of the complete cell model depend strongly on the assumptions made about the inner structure of the material and consequently on the cell radius parameter. Elsewhere [8], it has been shown that a useful value for the cell radius is that of an imaginary spherical cell circumscribing a unit cell in a SC lattice arrangement. With this assumption, the new cell model requires knowledge of porosity and grain radius only. The new similarity relationship has been applied also to empirical Johnson/ Allard model for the acoustical properties of porous media. The cell model approach enables derivation of explicit relationships between the viscous and thermal permeabilities and viscous and thermal characteristic lengths of granular materials. The cell model version of the semi-empirical model requires 5 parameters: porosity, radius of the particles, tortuosity, dc permeability and the cell radius. The complete cell model achieves at least as good agreement with data as that obtained with the modified empirical model but with fewer parameters.

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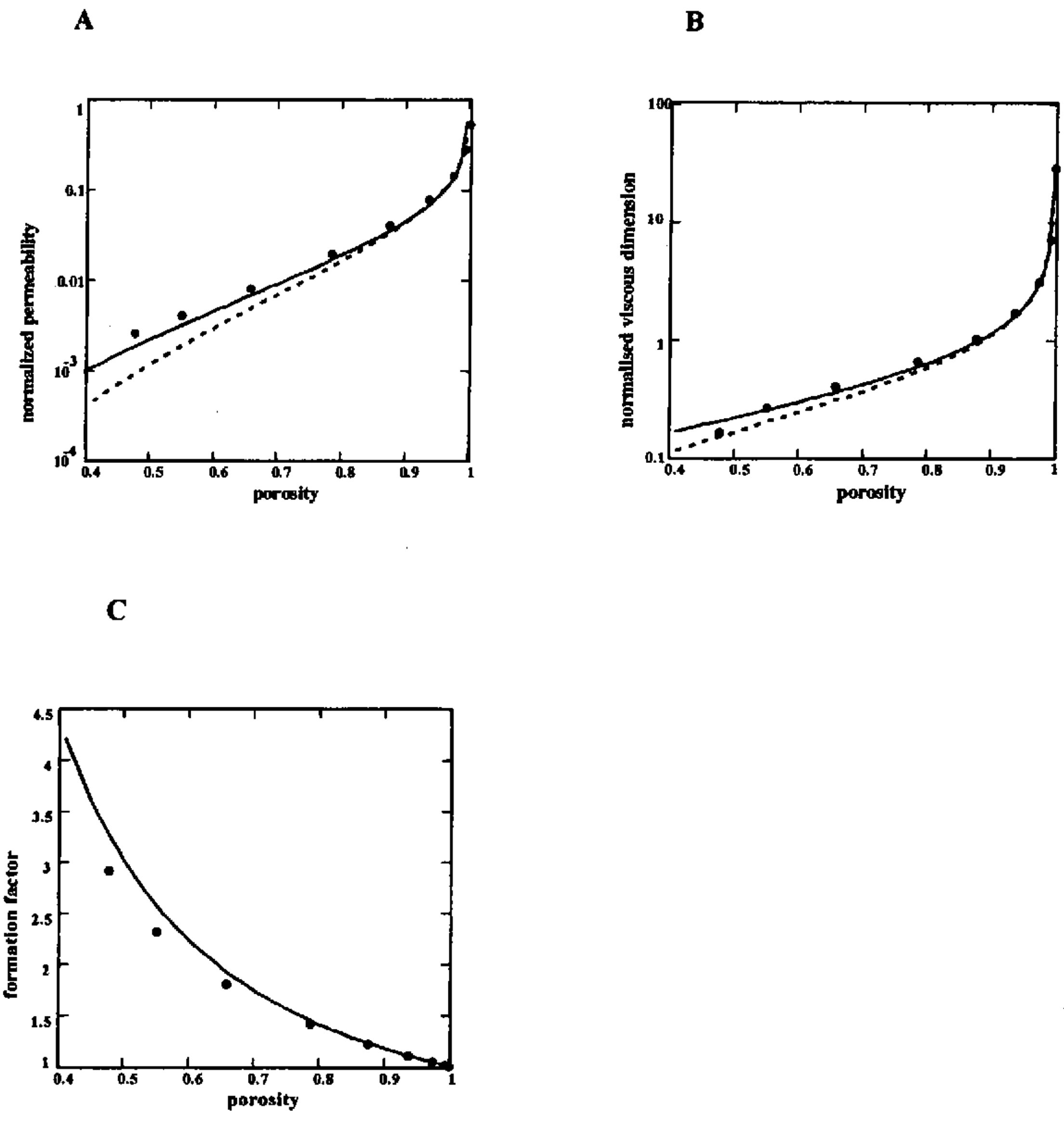

Figure 1. Dimensionless DC permeability $\frac{k_0}{d^2}$ ($d = \sqrt[3]{\frac{4\pi}{3(1-\phi)}}R$ is the base vector of the lattice) (A), characteristic viscous dimension $\frac{\Lambda}{d}$ (B) and formation factor $F = \frac{\alpha_\infty}{\phi}$ (C) for SC packing as a function of porosity, circles – numerical results [9], solid line – cell model (10), dashed lines – cell model (9).

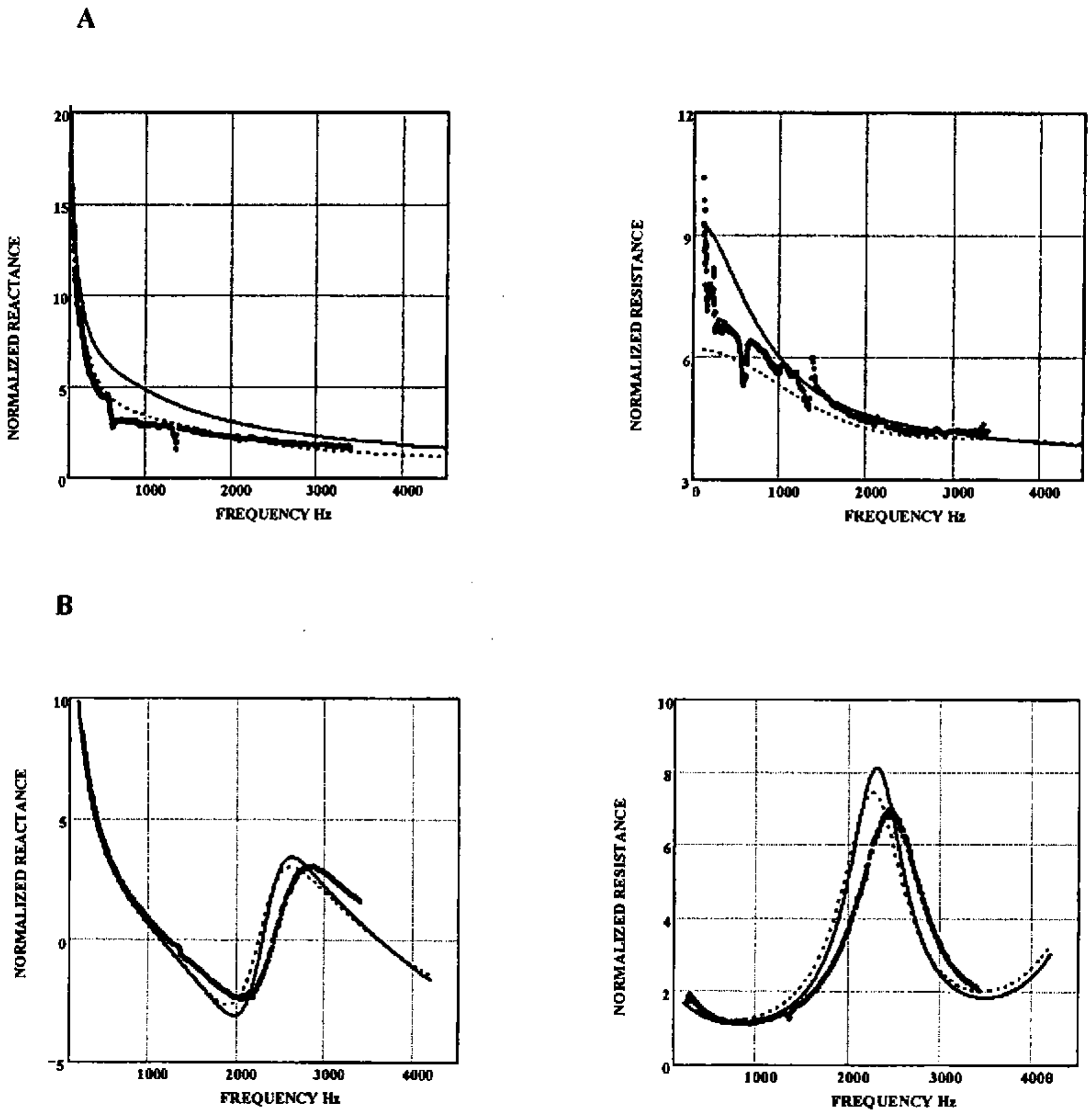

Figure 2. Comparison between predictions of normal surface impedance for a layer of glass beads and impedance tube data. Porosity is 0.39, calculated value of tortuosity is 1.78, layer thickness is 0.0475*m*, mean bead radius is either $5.625\times10^{-4}m$ (A) or $1.7\times10^{-4}m$ (B), measured value of flow resistivity is either 13400mks rayls/m (A) or 240000mks rayls/m (B). Solid line – modified characteristic dimension model, dashed lines – cell model.