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THE FORMAL BASIS OF TRANSFER FUNCTION MEASUREMENT IN ACOUSTICS AND AUDIO

P Darlington

Department of Acoustics and Audio Engineering
University of Salford
Salford M5 4WT UK

ABSTRACT

This paper is presented in support of a tutorial on practical measurement techniques for acoustics and audio engineering. The paper provides a description of the measurement of the frequency response function of a system. Motives for describing and measuring the performance of reproduced sound systems and components in the frequency domain are reviewed and the frequency response of a linear, time-invariant system is introduced by consideration of sinewave testing. Broadband methods for estimating the frequency response function (based upon power spectral estimates) are presented and their practical application and limitations are discussed.

1 INTRODUCTION

The electronic and electroacoustic components of a sound reproduction system are artifacts of technology. The violin is another technological product which serves to generate sound. However, the violin evolved before the scientific age, whereas a sound reproduction system is inconceivable without modern science; the violin is low technology and the sound reproduction system is high technology. As well as needing mature science to enable them, high technology products are distinguished by the way in which objective measurement of their behaviour is used during their design, installation, operation and maintenance. This paper considers contemporary measurement techniques applied in the area of reproduced sound. The theoretical background for the measurement of the transfer function of linear, time-invariant systems is considered, avoiding mathematical detail wherever this will not detract from the explanation. The discussion particularly aims to identify assumptions and limitations associated with the measurement techniques.

2 DYNAMICS - the physics of interest

Monotony is "the wearisome sameness of effect; (the) lack of interesting variety". Those things which are monotonous are static and lifeless. Interesting phenomena are changeable - indeed our perceptual systems are designed to detect change. The physics of non-monotonous systems is called dynamics and it is the dynamics of reproduced sound systems and components that we are concerned with measuring. Interestingly, the word monotony has musical roots (as it literally means sameness of pitch) and it was with the objective study of music that the science that spawned our reproduced sound systems began.

Dynamics as we know it today could be said to originate with the contributions of Isaac Newton. Newton not only expressed the laws of motion of (low speed) physical objects - laws which are used, for example, to describe the motion of a loudspeaker cone. Newton also contributed to the development of calculus (although Leibniz and Newton were engaged in a famous wrangle over the intellectual property rights to the idea). Calculus is the mathematical language of non-monotonous systems, because all dynamic systems are described by differential equations.

Whenever we seek to make a measurement in audio and acoustic systems, we are attempting to measure the input and output of a differential equation. The input and output of an amplifier, or any other piece of electronics, are related by a differential equation. The physical dynamics of a loudspeaker cone or a microphone diaphragm are dictated by differential equations. Finally, although acoustic systems are characterised by partial differential equations (expressing the fact that acoustic variables are dynamic with respect to time and space), our practical measurements in acoustics represent attempts to measure inputs and outputs of differential equations.

The solution of differential equations is not an easy task, as anyone who has taken a mathematics A-level will testify. Unfortunately, mathematics teachers are reluctant to admit that the basic technique is guesswork! Studying the solution of differential equations amounts to learning guess strategies which are appropriate to different types of differential equation, in much the same way as chess students memorise "gambits" - those strategies which have been found to work by people who have tried (and succeeded) before.

Those differential equations which control the idealised performance of a (component of a) reproduced sound system are ordinary differential equations with constant coefficients. The systems they describe are Linear, Time Invariant (LTI) systems - systems which are linear (i.e. not generating distortion) and whose performance does not change with time. Although this only relates to idealised performance of a reproduced sound system, the mathematics of LTI systems is the only mathematical dynamics that is currently fully understood. It also motivates the only class of measurements and analysis that are founded on sturdy logical foundations. Those measurements are based upon a system of "guessing" the solutions of the equations controlling LTI systems named in memory of two French mathematicians.

3 FOURIER, LAPLACE and the FREQUENCY DOMAIN

In 1822 Jean-Baptiste-Joseph Fourier published a document that was to have a huge impact on the whole of mathematics and natural science, not least acoustics and audio engineering. The impact of the publication was, perhaps, surprising given its title: "*Theorie analytique de la chaleur*". In the work described in this significant document, Fourier was studying the conduction of heat along a metal rod. What was important about the work was not the thermodynamics - Fourier described how arbitrary functions can be represented as an infinite sum of goniometric functions. In the context of an audio signal, a function (of time) can be represented as a sum of constituent sinusoidal and cosinusoidal functions (of time) each with different frequency.

There are two important reasons for the impact of Fourier's work in acoustics. Firstly, the frequencies of the constituent sinusoidal and cosinusoidal components of a signal analyzed by Fourier's methods elicit different pitch responses in human listeners. This means that description of a signal in terms of Fourier analysis fits well with the operation of human audition. Secondly, each of the component sine-waves of a Fourier representation of a complex audio signal is an excellent gambit for the solution of the differential equations which control LTI systems, as demonstrated below.

The general form of differential equation behind an LTI system, input x and output y, is:

$$y(t) = \sum_{i=0}^{N} \alpha_i \frac{d^i(x(t))}{dt^i} - \sum_{k=1}^{N} \beta_k \frac{d^k(y(t))}{dt^k} \qquad (1)$$

If we consider one of Fourier's constituent sinèwave components of the input:

$$x(t) = A\sin(\omega t) = A\sin(2\pi f t) \qquad (2)$$

(which, because of the linearity constraint, will cause an output at the same frequency, with potentially different magnitude and phase):

$$y(t) = B\sin(\omega t + \Phi) \qquad (3)$$

and substitute these into the differential equation (1), we find that it is reduced to an algebraic equation:

$$B\sin(\omega t + \Phi) = \sum_{i=0}^{N} \alpha_i \omega^i \sin(\omega t + \frac{i\pi}{2}) - \sum_{k=1}^{N} \beta_k \omega^k B\sin(\omega t + \Phi + \frac{k\pi}{2}) \qquad (4)$$

Equation (4), however daunting it may seem, can be solved by conventional algebra - all the guesswork is gone !

Fourier's method of analysis is seen not only to chime with the operation of our ears (and hence with the long standing interest of physical acousticians from Pythagoras to Helmholtz with the relationship between the pitch percept and the harmonic relationships observed in vibrating strings and air columns). It also decomposes a signal into constituents which simplify the solution of the very equations which characterise reproduced sound systems, at least in the idealised linear, time-invariant sense.

Fourier analysis takes a signal which is dynamic with respect to time and transforms it, such that the manner in which the signal's constituent sinusoidal components are distributed in frequency can be seen - the information carried in the signal is viewed in the Frequency Domain. Frequency domain techniques lie at the heart of contemporary measurement in audio and acoustics, as a result of the benefits of Fourier analysis (and its generalisation named for Pierre Simon, marquis de Laplace). The benefits of working in the frequency domain become apparent when we consider the most familiar frequency domain descriptor of a reproduced sound system (or component) - the Frequency Response function, or "transfer function".

4 FREQUENCY RESPONSE

In our measurements of acoustic, electroacoustic and electronic systems, we are considering relationships between input and output signals of a system. The system, being dynamic, is described by a differential equation and the benefits of Fourier analysis accrue when the systems are linear and time invariant. We shall now examine a typical measurement, in an attempt to understand what is meant by "linear" in this context.

Consider the situation illustrated in Figure 1, in which an input signal x(t) is applied to a system, H. The resulting output is y(t).

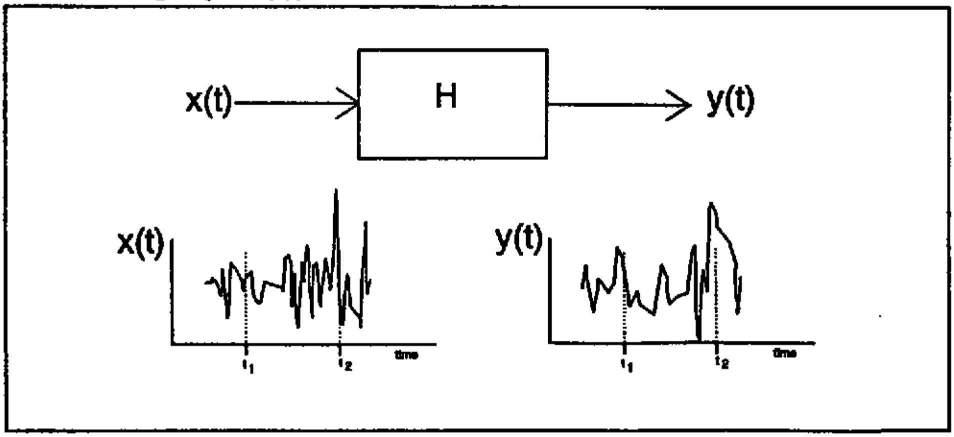

Figure 1 Input / output relationships of system H

Both input and output of H are dynamic - they change with respect to time. If we seek to characterise the system, H, we would measure x and y and look for the relationship between them. Such a relationship would describe the system, as it is the system H that causes the output to be different from the input. The time-dependence of x and y complicates our attempts to describe H - when should we attempt to relate x and y ?

If we look at the two instants of time shown in figure 1, t_1 and t_2, we would find that, for a general LTI system:

$$\frac{y(t_1)}{x(t_1)} \neq \frac{y(t_2)}{x(t_2)} \qquad (5)$$

This worrying result prevents us from finding a simple linear relationship between x and y, that is:

$$y(t) \neq H.x(t) \qquad (6)$$

in which H is a constant number. Given equation (6), in what sense can LTI systems (as described by equation (1)) be called linear ? It is at this point that the frequency domain methods of Fourier and Laplace come to our rescue.

4.1 Sine wave response

If we repeat the measurement suggested by Figure 1, but ensure that the input x(t) is a sinusoid, the situation reported as Figure 2 results.

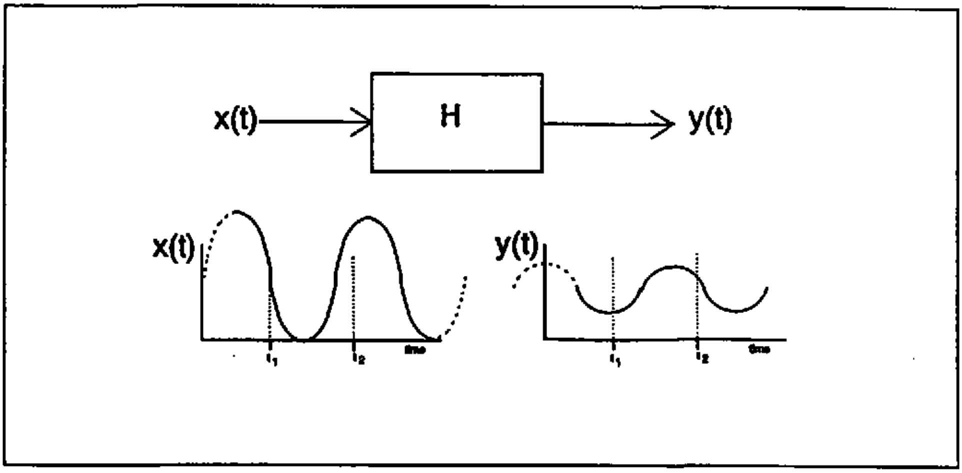

Figure 2 Sinewave testing of an LTI system.

In the situation depicted as figure 2 equation (5) still does not hold for general cases. However, we know from equations (2)-(4) that the output is a sinewave, at the same frequency as the input, with a potentially different magnitude and phase. It is the magnitude of a sinusoid applied at the input in which the response of the general LTI system is linear. That is to say, for a sinusoidal input of amplitude A, the magnitude of the output is a linear function of A:

$$y(t) = B\sin(\omega t + \Phi_\omega) = H_\omega A\sin(\omega t + \Phi_\omega) \quad (7)$$

in which H_ω is the ratio between output and input magnitudes when the test frequency is ω. This is a fixed property of the system H at frequency ω and is determined by the values of the α's and β's in equation (1). H_ω is the gain of the system at frequency ω. The phase shift between input and output in Figure 2, denoted as Φ_ω in equation (7) is a fixed property of the system H at frequency ω and is independent of input amplitude. Φ_ω is also determined by the values of the α's and β's in equation (1).

> *The behaviour of an LTI system when excited by a sinusoidal input at frequency ω is completely determined by the pair of numbers H_ω and Φ_ω*

This characterisation of H can be made experimentally by applying a sinusoidal input x(t) and observing the sinusoidal response, y(t) - conventional sinewave testing. The ratio of input and output magnitudes can be made by measuring the RMS value of each signal and finding the ratio of the RMS levels. The phase measurement can be made with a phase meter or an oscilloscope. The input signal should be applied at such a level that the system could be expected to remain linear and care must be taken to avoid including background noise (and any distortion products !) in the evaluation of the RMS level of the output.

Although we rarely play pure sinewaves through reproduced sound systems (except for testing), it has been shown that Fourier's methods allow any signal to be reduced to a sum of sinusoidal components. If we measure H_ω and Φ_ω at the frequency of all of the Fourier components of a signal, we can calculate the response to the individual components. Further, a defining property of linear systems is that the response to a sum of inputs is equal to the sum of the responses to the inputs applied individually. Thus, knowledge of the response of an LTI system to sinusoidal inputs at all frequencies gives understanding of the response to any input signal. Knowing values of gain and phase response at all frequencies is equivalent to knowing two <u>functions</u> of frequency, $H(\omega)$ and $\Phi(\omega)$, which, as a pair, form the frequency response function, or transfer function, of the system H.

Although it is formally necessary to measure a sinewave response at every frequency to exactly characterise a frequency response function, modern digital measurement techniques only estimate this function at a discrete number of frequencies.

5 Stepping Back into Time - THE IMPULSE RESPONSE

We saw in the example presented as Figure 1 that it is difficult to find a relationship in time between the input and output of a general LTI system. However, knowledge of the frequency response function of a system allows us to specify one very important time domain response of an LTI system which has special significance to the perceived quality of reproduced sound systems and the analysis of the acoustics of auditoria and listening spaces.

The frequency response function tells us how a system responds to signals having a specified combination of sinusoidal components. For example, if the input signal has sinusoidal components of amplitude a(ω) and phase 0 (for simplicity), then the output will have sinusoidal components of amplitude b(ω)=a(ω).H(ω) and phase Φ(ω); the system gain multiplies the amplitudes of the input Fourier components and the system phase is added to the phases of the input Fourier components.

If the amplitudes of all the components of a special input signal are equal, $a_\delta(\omega)=1$, (and the phases all 90°, $\Phi_\delta(\omega)=90$, for reasons that are not important here) then the amplitudes of the components of the response to this special input are specified by the frequency response function alone; $b_\delta(\omega)=a_\delta(\omega).H(\omega)=H(\omega)$. The special input signal with equal amplitude Fourier components at all frequencies is called a "delta function", or impulse, and the frequency response function contains all the information required to determine the response of an LTI system to an idealized impulsive input - the impulse response.

6 PRACTICAL MEASUREMENT WITH BROADBAND SIGNALS

The frequency response of a system was introduced in section 4.1 in the context of sinewave measurements. Measuring the response to a number of different sinusoidal frequencies one at a time is clearly time consuming. Most contemporary measurement techniques address all frequencies at once - a broadband approach. In order to describe this approach we shall first introduce some new notation for the Fourier transform of a signal.

In section 5 the amplitudes and phases of the individual Fourier components of y(t) were written as b(ω) and Φ(ω) respectively -they are the magnitude and phase components of the Fourier Transform of y(t). We shall write the Fourier Transform of y(t) as Y(ω), a complex valued function of frequency:

$$b(\omega) = |Y(\omega)| \qquad \Phi(\omega) = \angle(Y(\omega)) \qquad (8)$$

For any general signal f(t) there exists a Fourier transform, F(ω), expressed mathematically by:

$$F(\omega) = \int_{-\infty}^{\infty} f(t).e^{-j\omega t}\,dt \tag{9}$$

Using the mathematical operation described in equation (9) it is possible to derive the Fourier transforms of any pair of input and output signals from a system. As the Fourier transform describes the amplitude and phase of the sinusoidal component of the signal at ω, a new formal definition of the frequency response function can be made:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \tag{10}$$

Unfortunately, equation (10) is completely useless for most types of signal as it is impossible to calculate the Fourier Transforms. The difficulty stems from the doubly infinite time limits of the integration in the definition of the Fourier transform (equation (9)). It is clearly impossible to measure both input and output of a system for ever before using the resulting data in a calculation - all sense of urgency in performing the experiment would be lost before the result was available !

If Fourier Transforms are taken of finite length segments of the signals in an attempt to produce estimates of the true Fourier Transforms, equation (10) will still be found to be useless ! This is because the Fourier Transform is not a statistically consistent estimator - increasing the length of the Transform does not increase its accuracy ! An alternative approach has to be found for the practical estimation of transfer functions.

6.1 Power Spectral Density

Although the Fourier Transform itself displays statistical inconsistency, it is possible to produce useful statistics in the frequency domain. Consider the truncated Fourier Transform, $F_T(\omega)$:

$$F_T(\omega) = \int_{\frac{-T}{2}}^{\frac{T}{2}} f(t).e^{-j\omega t}\,dt \tag{11}$$

(notice that this is defined over a finite interval of time). If a number of different truncated Fourier Transforms $X_{T,i}$ of the same signal x(t) are calculated and the square of their magnitudes averaged, the result:

$$S_{xx}(\omega) = \frac{1}{NT} \sum_{i=1}^{N} |X_{T,i}(\omega)|^2 \qquad (12)$$

is called the power spectral density of x. The power spectral density is a consistent statistic - the quality of the estimate improves with N, the number of truncated Fourier transforms averaged together.

6.2 Practical estimation of System Gain

If the power spectral density of both input and output of a system are estimated, it is possible to form a useful estimate of the gain component of the frequency response function :

$$H(\omega) \approx \sqrt{\frac{S_{yy}(\omega)}{S_{xx}(\omega)}} \qquad (13)$$

The approach embodied by equation (13) is used in contemporary instruments and analyzers to measure the gain component of the frequency response of an LTI system. It has the benefit that only single channel instrumentation is required (as it is possible to estimate first the output power spectrum and then the input power spectrum). Unfortunately, the ratio of output to input power spectra cannot resolve the time relationship between components of the input and output of a system. This not only means that phase response cannot be measured on conventional single channel instruments (see section 6.4) but also that it is difficult to verify that measurements are free from corruption by noise or distortion (see section 7).

6.3 Cross Power spectral density

The power spectral density function introduced in section 6.1 destroyed all the phase information in the truncated fourier transforms by averaging the squared magnitude. If, instead of averaging the squared magnitude, the average of the product of two truncated fourier transforms derived from synchronously measured segments of the input and output of a system is computed, a new statistic results:

$$S_{x,y}(\omega) \approx \frac{1}{NT} \sum_{i=1}^{N} X_{T,i}(\omega)\, Y^{*}_{T,i}(\omega) \qquad (14)$$

$S_{x,y}(\omega)$ is the cross power spectral density between x and y. Being complex valued it preserves the phase relationship between frequency components of x and y introduced by the phase response of the system which relates them. Since the data used in calculating the truncated fourier transforms of input and output must be synchronously measured, the cross spectrum must be measured on instrumentation with two input channels (or the equivalent).

6.4 Practical estimation of Frequency Response

Armed with the cross power spectral density function defined in section 6.3, it is possible to define an expression for an estimate of the frequency response functions of an LTI system:

$$H(\omega) = \left|\frac{S_{x,y}(\omega)}{S_{xx}(\omega)}\right| \qquad \Phi(\omega) = \angle\left[\frac{S_{x,y}(\omega)}{S_{xx}(\omega)}\right] \tag{15}$$

These equations form the basis of practical transfer function measurements in many contemporary frequency analyzers.

7 COHERENCE - are my measurements worth anything ?

When sinewave testing was discussed in section 4.1, the importance of avoiding distortion and background noise contamination was noted. This is reasonably easy to detect in a sinewave test, but becomes very difficult to detect when broadband test signals (or even program material) are used as the input. Fortunately, those analysis methods which use twin channel synchronous methods (or their equivalent) can use the data employed in the estimation of the system transfer function to make a test for validity.

If a measurement of the gain component of the frequency response of a system is made in the presence of noise or system distortion, different results will be produced by the method of equations (13) and (15). This is because the output power spectrum estimate in equation (13) will include the power due to the noise and the distortion products, whereas the cross power spectrum in equation (15) discriminates against the noise and the distortion products, since they have no consistent statistical relationship to the input. Any difference between the results of gain estimates based on the two methods would be evidence of a bad measurement - at least one in which noise was present or the system was not linear. This difference is expressed by a single statistic, found on all reputable systems capable of measuring transfer functions; coherence.

The ordinary coherence function:

$$\gamma^2_{x,y} = \frac{|S_{x,y}(\omega)|^2}{S_{xx}(\omega) \,.\, S_{yy}(\omega)} \tag{16}$$

has value between 0 and 1 for any frequency. Its value describes the fraction of the power in the signal y due to linear operations on the signal x at the frequency ω. If x and y are derived from a noise free measurement of an LTI system, γ^2 would have unit value. If the coherence is less than 1, noise or distortion are present.

The validity of transfer function measurements should always be confirmed by measurement of coherence.

8 PRACTICAL PROBLEMS

The measurement of frequency response functions using sinewave testing (section 4.1) is capable of exact results but only in the hopelessly impractical situation where an infinite (or, at least, very large) number of different test frequencies are applied sequentially at the input. In the measurement of acoustic systems, which can have long impulse responses, this would take a very long time, as each measurement takes longer than the impulse response length. The practical broadband methods described in sections 6.2 and 6.4 are attractive because they avoid this lengthy measurement. However, this time saving is achieved at some considerable expense - the frequency response function measured is only an estimate of the true result. Some of the factors which make practical frequency response measurement using power spectral estimates ***approximations*** are described below.

8.1 Frequency resolution and spectral smearing

A (lengthy) evaluation of the frequency response of an LTI system using sinewave tests could correctly evaluate the function at any frequency. Practical estimation techniques using power and cross power spectral estimates cannot do this - they only estimate the frequency response function with a fixed frequency resolution, such that fine detail in the function cannot be resolved.

This limitation on frequency resolution is imposed by the use of the truncated Fourier transform (equation 11) in the estimates. The truncated Fourier transform $F_T(\omega)$ of a signal f(t) does not generally equal the true Fourier transform:

$$F_T(\omega) \neq F(\omega) \tag{17}$$

as it is calculated using only a limited amount of the information contained in f(t).

Specifically, the truncated fourier transform of length T is unable to resolve components of $F(\omega)$ which are closer than order 1/T Hz apart in frequency - the components appear in $F_T(\omega)$ smeared together, such that this effect is called "spectral smearing". This means that

> *the frequency resolution of frequency domain measurements using (cross) power spectral estimates is practically limited to 1/T Hz, where T is the length of the constituent Fourier Transforms.*

As well as smearing together the fine detail of an exact Fourier transform, truncated Fourier transforms have significant spectral components at frequencies where the true Fourier transform has none. This "sidelobe generation", which is part of the spectral smearing process, can be minimised if the time domain data is "windowed" before applied to the truncated Fourier transform operation.

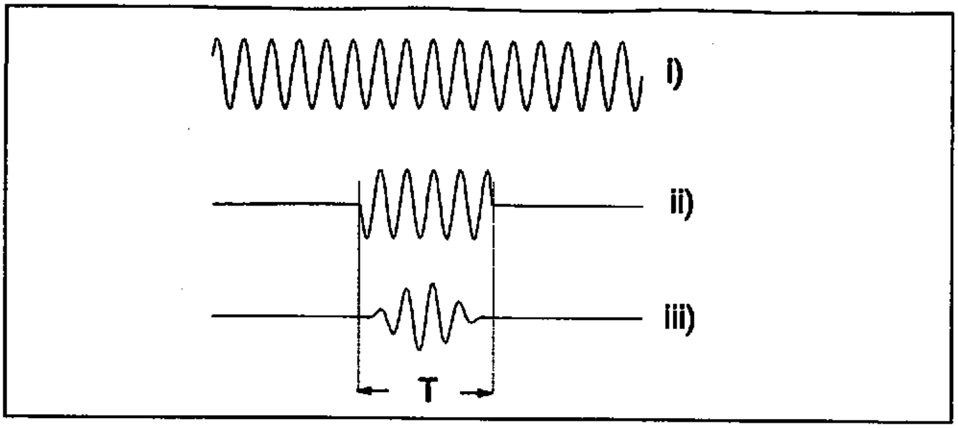

Figure 3 Time windowing a sinusoid (trace i)) with a rectangular window (trace ii)) and a raised cosine window (trace iii)).

Windowing, as illustrated in Figure 3, consists of multiplying the time domain data by a windowing function. The truncated Fourier transform of a signal implies a rectangular window shape, as the data is included within the window and excluded without it. Rectangular windows generate a significant amount of spectral smearing. Better time windows avoid the sudden steps at the beginning and ending of a rectangular window.

The application of a time window can cause a significant change in the spectral smearing of a truncated Fourier Transform due to sidelobe generation, but

> *the effective frequency resolution of a frequency response estimate made with truncated Fourier transform remains of order 1/T Hz., whatever time windowing is applied.*

8.2 SAMPLING and Shannon's theorem

The truncated Fourier transforms introduced in section 6.4 as being central to contemporary transfer function measurement are most effectively implemented via the Fast Fourier Transform (FFT) algorithm on a digital computer. The FFT is so computationally efficient that frequency domain measurement techniques are (usually) much faster than time domain alternatives, explaining why section 5 on impulse response measurement was approached via the frequency domain. In order that the FFT can be applied, the data must be in digital form - that is to say the analog voltage (representing whatever electronic, mechanical or acoustical parameter is being recorded) must be sampled.

Sampling is a two stage process; time discretisation and amplitude quantization. Amplitude quantization is simply a noise generating process, which can be ignored if the data converters are of high quality and the signal to be quantized has reasonably flat power spectrum. The effects of time discretisation are more significant.

In time discretisation the signal f(t) is converted to a sequence of numbers, f_k, each representing the value of f(t) at a particular instant of time:

$$f_k = f(kT_s) \qquad k \text{ integer} \tag{18}$$

where T_s is the sample period. The time discretisation element of the sampling process introduces a bandwidth limitation on the information content of f(t) that can be correctly represented in the number sequence f_k, as a result of Shannon's sampling theorem.

Shannon's theorem considers one of the sinusoidal Fourier components of f(t). This component can be correctly sampled as long as its period is greater than $2T_s$. If f(t) contains any frequency components which have period shorter than $2T_s$, then these high frequency components will not only be inaccurately represented, they will actually appear in the number stream f_k as components having longer periods, greater than $2T_s$. As frequency shifting is a non-linear operation, this "aliasing" of one frequency to another must be avoided in the context of the measurement and analysis of LTI systems. Shannon's sampling theorem means that

the analysis bandwidth of digital systems designed to measure the transfer functions of LTI systems is lowpass limited to frequencies lower than $(2T_s)^{-1}$.

SUMMARY

Reproduced sound systems and their constituent parts are dynamics systems, controlled by differential equations. Most systems (excepting processing elements such as compressors, etc.) are designed to operate linearly. Within the linear operating envelope, the systems are Linear, time-invariant and a body of mathematical analysis tools can be applied. The frequency response function, or transfer function, is a complete descriptor of LTI system performance, and is useful in acoustics and audio, both as primary data and to derive other indicators of system behaviour.

Although the formal basis of the frequency response function results from a mathematical abstraction that is impractical to apply, techniques for the practical estimation of the transfer functions of LTI systems have been presented. These techniques (relying on the "Fourier Transform" method of spectral estimation) allow transfer function estimates of limited (but useful) frequency resolution and bandwidth to be made using contemporary instruments.