TWOPORT MODELS OF THE LOUDSPEAKER: *RELATING THE TWOPORT TO ANALOGOUS CIRCUIT MODELS*

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1 INTRODUCTION

The small signal, low frequency, lumped-parameter behaviour of a direct radiating electrodynamic loudspeaker is described by the interplay of four variables. These are 1) applied voltage *v*, 2) voice-coil current *i*, 3) pressure difference across the diaphragm *p* and 4) diaphragm velocity *u*. Simultaneous knowledge of these four variables (or their equivalent) is required in order to model the loudspeaker, particularly when descriptions of power exchange are sought.

In use, the acoustic variables *p,u* will be related by an impedance, *z*,

$$z = \frac{p}{u} \qquad (1)$$

which is defined by the operating environment. In light of this relationship (1), it is convenient to keep the acoustic variables together in any equation describing the loudspeaker, allowing either one to be eliminated at will by substitution. Bearing this in mind and appealing to nothing more than linearity, we may write the simultaneous frequency-domain equations:

$$\begin{aligned} v &= \alpha p + \beta u \\ i &= \gamma p + \delta u \end{aligned} \qquad (2)$$

in which α,β,γ and δ are fixed, complex functions of frequency completely describing the loudspeaker. Equations 2 are a general model of the loudspeaker in *any* application whilst substitution of a particular load impedance, z, will describe behaviour in *the* application to which that load corresponds.

The parameters α,β,γ and δ may be measured or deduced from experiment. Alternatively, they may be written in terms of objective parameters of the loudspeaker. Such expressions have been presented elsewhere[1,2] but they are reproduced here for completeness

$$\begin{aligned} \alpha &= \frac{S_D z_{EB}}{Bl} \\ \beta &= \frac{z_m z_{EB} + (Bl)^2}{Bl} \\ \gamma &= \frac{S_D}{Bl} \\ \delta &= \frac{z_m}{Bl} \end{aligned} \qquad (3)$$

The simultaneous equations 2 may be written as a single matrix equation, in which vectors of electrical and acoustical variables are related by a square "TwoPort" matrix:

$$\begin{Bmatrix} v \\ i \end{Bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{Bmatrix} p \\ u \end{Bmatrix} \qquad (4)$$

The TwoPort matrix (which, in this configuration, is also called a "transmission matrix", "chain matrix" or "ABCD matrix") is named from analogy with a standard method of electrical network analysis[3], which relates activity at two pairs of connections (two "ports") in an electrical circuit.

The TwoPort matrix of the functions *α,β,γ* and *δ* is a fixed, *general* model of the loudspeaker. Specification of loading conditions, *z*, in a particular application (and elimination of one of the elements of the vector of acoustic variables by substitution from equation 1) allows solution for response *in that application*.

2 A THÉVENIN EQUIVALENT OF THE LOUDSPEAKER

Thévenin equivalents are another method of electrical network analysis, described in standard texts[3] and in a useful summary available on the internet[4]. They allow an electrical network comprised of an arbitrary combination of voltage sources, current sources and impedances to be replaced by an equivalent network consisting of a single voltage source and single series impedance, thus simplifying the representation of the network. The Thévenin equivalent for the loudspeaker described by equation 4 is derived from the TwoPort model below. The same steps allow the Thévenin equivalent of any TwoPort (transmission) network to be identified.

Taking the first of equations 2 and dividing both sides by α gives….

$$\frac{v}{\alpha} = p + \frac{\beta}{\alpha} u \qquad (5)$$

A heuristic approach to the derivation of the Thévenin equivalent directly from equation 5 is possible[2]. However, a more rigorous approach is pursued below.

Taking the second of equations 2 and eliminating *p* (by substitution from equation 1) gives…

$$i = \gamma z u + \delta u \qquad (6)$$

Although the RHS of equation 6 is now independent of pressure *p*, we may still write…

$$i = 0.p + (\gamma z + \delta).u \qquad (7)$$

Dividing both sides of equation 7 by *(γz + δ)* gives….

$$\frac{i}{(\gamma z + \delta)} = 0.p + u \qquad (8)$$

Equations 5 & 8 now form a simultaneous pair, analogous to equations 2, which may be combined into a single matrix equation (c.f. equation 4)…

$$\begin{Bmatrix} \dfrac{v}{\alpha} \\ \dfrac{i}{(\gamma z + \delta)} \end{Bmatrix} = \begin{bmatrix} 1 & \dfrac{\beta}{\alpha} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} p \\ u \end{Bmatrix} \qquad (9)$$

This expression relates a vector of modified electrical variables to the vector of acoustical variables via a "TwoPort" matrix. This matrix has the special form of the TwoPort representation of a series impedance, having value *β/α*. [TwoPort descriptors of standard network configurations are not readily found in the literature – they are appended to this paper for convenience.]

Under general drive conditions equation 9 is a rather poor model of the loudspeaker, as the modified current, *i/(γz+δ),* is a function of the (application-specific) load, z. However, under the special case of pure voltage drive, equation 9 reveals a useful, general model of the loudspeaker, which is application-independent. This is the Thévenin equivalent.

Under pure voltage drive, the series impedance *β/α* is the Thévenin impedance of the loudspeaker and the pure voltage source *v/α* is the Thévenin voltage. The Thévenin equivalent circuit is shown in Figure 1 below.

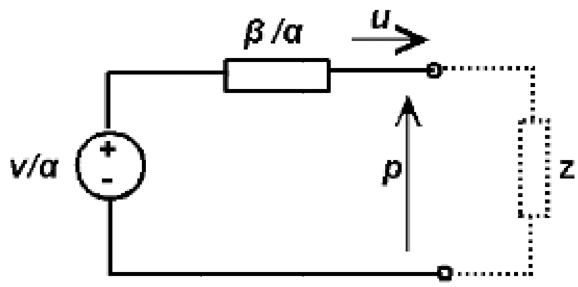


Figure 1 Thévenin Equivalent of the Loudspeaker

3 INTERPRETING THE THÉVENIN EQUIVALENT

The Thévenin impedance *β/α* can be studied by substitution from equations 3….

$$\frac{\beta}{\alpha} = \frac{z_m z_{EB} + [Bl]^2}{S_D z_{EB}} = \frac{z_m}{S_D} + \frac{[Bl]^2}{S_D z_{EB}} \tag{10}$$

Equation 10 reveals that the Thévenin impedance of the loudspeaker is formed of two series components. First, a term associated with the mechanical dynamics of the diaphragm and its suspension, z_m / S_D. Second, a term associated with the motor, $[Bl]^2/S_D z_{EB}$. Writing the impedances in terms of their usual resistive and reactive components can further expand both terms.

Substituting for elements of the mechanical impedance of the moving parts of the loudspeaker gives …

$$\frac{z_m}{S_D} = \frac{j\omega m_m + r_m - \frac{jk_m}{\omega}}{S_D} \tag{11}$$

Expressing the voice coil's blocked impedance in terms of resistive and inductive components gives

$$\frac{[Bl]^2}{S_D z_{EB}} = \frac{[Bl]^2}{S_D\left(R_{EB} + j\omega L_{EB}\right)} \tag{12}$$

This can further be expanded into a standard form….

$$\frac{[Bl]^2}{S_D\left(R_{EB}+j\omega L_{EB}\right)}=\frac{1}{\frac{S_D}{[Bl]^2}\left(R_{EB}+j\omega L_{EB}\right)}$$

$$=\frac{1}{\frac{1}{\frac{[Bl]^2}{S_D R_{EB}}}+\frac{1}{\frac{[Bl]^2}{S_D j\omega L_{EB}}}} \qquad (13)$$

$$\left(=\frac{1}{\frac{1}{z_{p1}}+\frac{1}{z_{p2}}}\right)$$

which is the impedance of the parallel connection of two impedances, z_{p1} and z_{p2}. From equation 13, z_{p1} is seen to equal the real (i.e. resistive) impedance $[Bl]^2/S_D R_{EB}$ whilst z_{p2} is the negative imaginary (i.e. capacitive) impedance $[Bl]^2/j\omega S_D L_{EB}$

Equations 10,11 and 13 allow the Thévenin impedance of the loudspeaker to be visualized as a network having three series elements describing the mechanical dynamics (equation 11) in series with the parallel combination of two elements describing the motor (equation 13). This is shown as Figure 2.

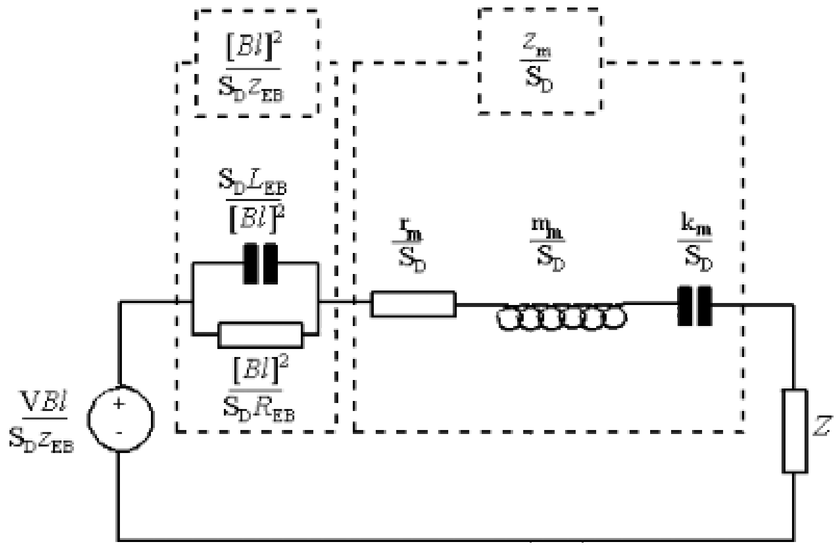


Figure 2 Low Frequency Analogous Circuit of the Loudspeaker (*under pure voltage drive*)

This is the familiar analogous circuit of the loudspeaker found in electro-acoustics texts[5], where it is derived using a rather less direct argument.

4 EFFECTS OF NON-ZERO GENERATOR SOURCE IMPEDANCE

Practical power amplifiers do not operate as voltage sources. Their (Thévenin) source impedance, z_g, is never identically zero, is always band-limited and may be frequency-dependant and reactive even within the intended operating bandwidth. In the case of amplifiers intended for current drive, the source impedance is intentionally high. Under these practical conditions, the model of Figures 1 & 2 is incorrect (although the generator source impedance may be so low in some situations that the model remains a useful approximation).

To model the effects of non-zero source impedance, equation 4 might be developed to include the effects of the series source impedance as an additional explicit TwoPort (see Appendix), giving….

$$\begin{aligned}\begin{Bmatrix} v \\ i \end{Bmatrix} &= \begin{bmatrix} 1 & z_g \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{Bmatrix} p \\ u \end{Bmatrix} \\ &= \begin{bmatrix} \alpha + z_g\gamma & \beta + z_g\delta \\ \gamma & \delta \end{bmatrix}\begin{Bmatrix} p \\ u \end{Bmatrix}\end{aligned} \qquad (14)$$

Following the steps used in section 2, the Thévenin equivalent of the system described by equation 14 has Thévenin equivalent voltage of $v/(\alpha+z_g\gamma)$ and Thévenin equivalent impedance of $(\beta+z_g\delta) / (\alpha+z_g\gamma)$.

Whilst these results look very different to those for pure voltage drive (when $z_g = 0$), it can be verified that $(\alpha+z_g\gamma)$ is the same as α (in equation 4) evaluated with $z_{EB}+z_g$ rather than z_{EB}. Similarly, $(\beta+z_g\delta)$ is the same as β (in equation 4) evaluated with $z_{EB}+z_g$ rather than z_{EB}. In other words, the model presented above is valid for a practical power amplifier having Thévenin source impedance z_g as long as the substitution $z_{EB} \rightarrow z_{EB}+z_g$ is made (as is well known[5]).

5 A NORTON EQUIVALENT OF THE LOUDSPEAKER

Whilst the Thévenin equivalent circuit replaces an arbitrary network with an equivalent voltage source and series resistor, the Norton equivalent[3] generates an equivalent comprised of a current source and parallel impedance. Central to the duality between the two methods is the fact that the Thévenin and Norton impedances are equal. The Norton current source is the Thévenin voltage source divided by this impedance.

Given the definitions above, a Norton equivalent of the loudspeaker can be produced directly from the Thévenin equivalent (Figure 1). This Norton equivalent is shown as Figure 3. Note that, whilst the equivalent network of Figure 3 includes a current source, the network models behaviour of a loudspeaker driven by a pure voltage source - indeed the current source in Figure 3 has magnitude proportional to the applied voltage, v.

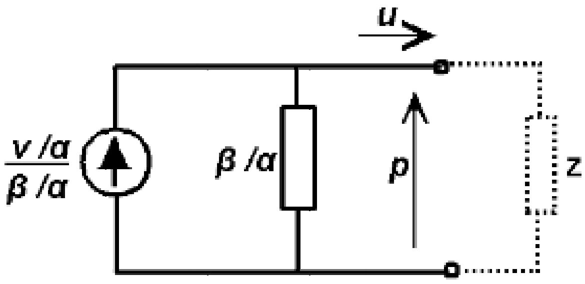


Figure 3 Norton Equivalent of a Loudspeaker (*under pure Voltage Drive*)

6 A NORTON EQUIVALENT OF THE LOUDSPEAKER DRIVEN BY AN IDEAL CURRENT SOURCE

The Norton equivalent, with its current source, might be considered a natural model of a loudspeaker under pure current drive. Under such drive conditions, the Thévenin or Norton impedance differs from to the value β/α appropriate for voltage drive. Analysis of the loudspeaker under current drive proceeds along similar lines to those of section 2.

Taking the second of equations 2 and dividing both sides by *δ* gives….

$$\frac{i}{\delta} = \frac{\gamma}{\delta} p + u \qquad (15)$$

Then taking the first of equations 2, eliminating *u* by substitution from equation 1 and re-introducing a mute term *0.u* gives …

$$\frac{v}{\left(\alpha + \frac{\beta}{z}\right)} = p + 0.u \qquad (16)$$

The simultaneous equations 15 & 16 can be re-written as a matrix equation…

$$\begin{Bmatrix} \dfrac{v}{\left(\alpha + \frac{\beta}{z}\right)} \\ \dfrac{i}{\delta} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{\gamma}{\delta} & 1 \end{bmatrix} \begin{Bmatrix} p \\ u \end{Bmatrix} \qquad (17)$$

which relates a vector of modified electrical variables to the vector of acoustical variables through a TwoPort matrix. Reference to the Appendix shows this to be the TwoPort matrix of a parallel impedance of value δ/γ, which is the Norton equivalent impedance we seek.

The Norton equivalent of the loudspeaker driven by a pure current source is shown as Figure 4. Of course, this model is vastly simplified with reference to the model for voltage drive, as the current drive is insensitive to motional impedance effects of the electro-dynamic motor.

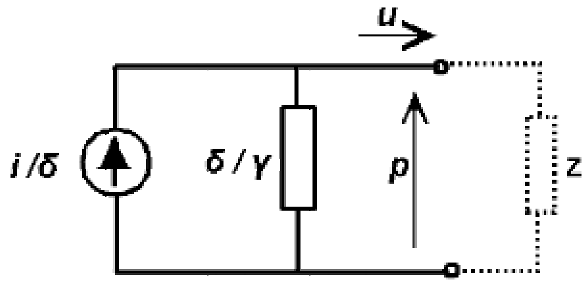


Figure 4 Norton Equivalent of a Loudspeaker (*under pure current drive*)

7 CONCLUDING REMARKS

A method for constructing Thévenin and Norton equivalents of TwoPort (transmission) networks has been presented and applied to the TwoPort model of a conventional electrodynamic loudspeaker. The Thévenin equivalent forms a natural model for the conventional mode of operation of the loudspeaker: voltage drive. This readily can be extended to include the effects of non-zero generator source impedance.

The Thévenin equivalent model of the loudspeaker was demonstrated to be identically the conventional low-frequency analogous circuit (impedance form) for a loudspeaker, familiar to students of electro-acoustics. This identity serves two important functions. Firstly, it demonstrates a means for "translating" between TwoPort formulations and more conventional descriptors. Secondly, and more importantly, it has pedagogical significance. The conventional derivation of the analogous circuit is difficult, particularly for those without secure understanding of the methods of electrical circuit analysis. It is suggested that the approach taken in the present paper (without the formal rigor) allows the analogous circuit to be introduced to a wider audience.

8 REFERENCES

1 P Darlington, Low Frequency Modelling Using 2 Port Methods, Proceedings of the AES UK Conference: Microphone and Loudspeakers: *the ins and outs of audio,* LONDON, UK, 1998.

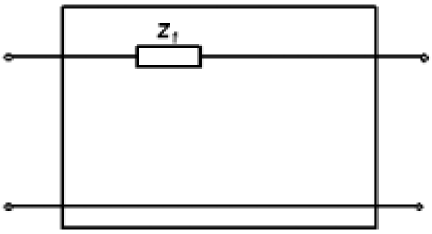
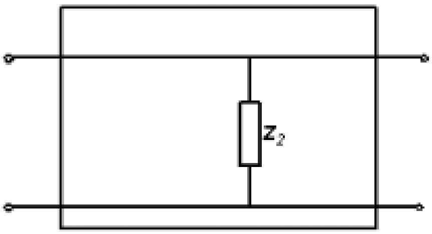
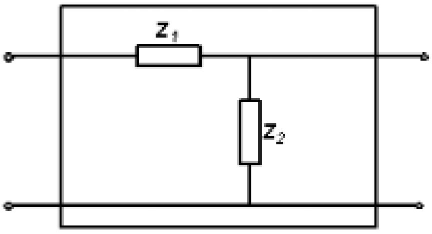
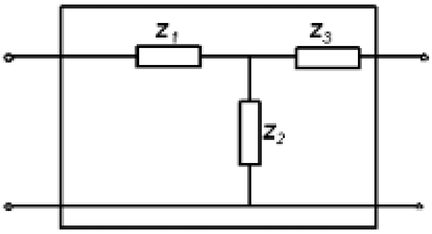
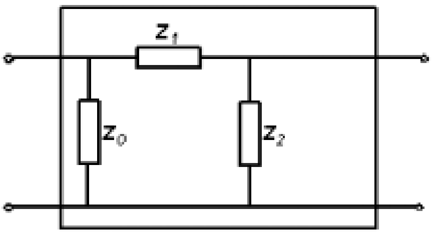
2 M. Colloms and P. Darlington, High Performance Loudspeakers, 6th ed (2005)

3 M. E. Van Valkenberg, Network Analysis 3rd ed., Prentice Hall (1974).

4 C.R. Nave, Thevenin's Theorem (1999). Retrieved from The HyperPhysics Web Site: http://hyperphysics.phy-astr.gsu.edu/hbase/electric/Thevenin.html

5 L.L. Beranek, Acoustics, McGraw-Hill (1954)

APPENDIX – TwoPort Matrices of some Standard Networks

Name	Circuit Representation	TwoPort Matrix
Series Impedance		$\begin{bmatrix} 1 & z_1 \\ 0 & 1 \end{bmatrix}$
Parallel Impedance		$\begin{bmatrix} 1 & 0 \\ \dfrac{1}{z_2} & 1 \end{bmatrix}$
Potential Divider		$\begin{bmatrix} 1+\dfrac{z_1}{z_2} & z_1 \\ \dfrac{1}{z_2} & 1 \end{bmatrix}$
T Network		$\begin{bmatrix} 1+\dfrac{z_1}{z_2} & z_3\left[1+\dfrac{z_1}{z_2}\right]+z_1 \\ \dfrac{1}{z_2} & \dfrac{z_3}{z_2}+1 \end{bmatrix}$
π Network		$\begin{bmatrix} 1+\dfrac{z_1}{z_2} & z_1 \\ \dfrac{1}{z_0}\left[1+\dfrac{z_1}{z_2}\right]+\dfrac{1}{z_2} & \dfrac{z_1}{z_0}+1 \end{bmatrix}$