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ACOUSTICAL PREDICTION

P A Nelson

Institute of Sound and Vibration Research, University of Southampton, UK

1. INTRODUCTION

It is appropriate at a conference whose theme is "The Next Twenty-Five Years" to spend some time contemplating the future. The future, however, is unpredictable. The notion of a deterministic universe was long ago abandoned by physicists when they realised that they could not even identify the present velocity and position of particles with certainty [1]. From the dynamicists point of view, the recognition in modern times of the complexity of the behaviour exhibited by non-linear dynamical systems makes the identification of their governing equations, and the prediction of their future response, a daunting task [2]. The world in which we live is undoubtedly "non-linear" and predicting the future with certainty is impossible.

Sound, however, is for the most part, governed by linear equations. Acoustical time series, of the type that constantly confront noise control engineers, have a degree of predictability that is relatively easy to evaluate. Well-established classical theories can be used to make estimates, of quantifiable accuracy, of the future behaviour of many of the acoustical waveforms of practical interest. In this paper we shall make use of the theory of linear prediction to help us speculate upon the future of noise control. In particular we shall mostly be concerned with the prospects for "active" techniques. It is probably reasonable to assume that the technology that is currently at our disposal will continue to improve. The improvements in microelectronics during the last fifteen years have been astonishing, and methods for controlling noise are now in production that were barely conceivable at the start of this period. The performance of active methods is most often constrained by the physical behaviour of sound fields; the number of transducers for creating a field to destructively interfere with the unwanted field and for detecting the success of the destructive interference is almost always dictated by the acoustical wavelength. The future of active noise control is thus likely to be determined by the success of advanced technology in overcoming these intrinsic physical limitations. Thus provided technology improves (becoming smaller, lighter, cheaper, more widely available etc) then the battle with the physical limitations will become easier.

Below we first introduce the theory of linear prediction and illustrate the dependence of the predictability of acoustical time series on the bandwidth of the signal. We also discuss the "inversion" of linear systems, since this has often to be accomplished when control is applied and is particularly relevant to control systems whose objective is the production of "virtual acoustic reality". Both of these topics will be dealt with within the analytical framework usually adopted by control engineers, who are mostly faced with the control of dynamical systems of considerably lower order than those found in acoustics and vibrations [3, 4, 5]. We also proceed to discuss feedforward techniques within this framework, even though the successful applications of feedforward control have been undertaken using adaptive signal processing techniques, and only a few authors [e.g. 6, 7, 8] have previously studied the active control of sound from a control engineering viewpoint. However it is interesting to analyse the problem in this way and some further insight into the performance of such systems is given, although the author's use of this theory should not be mistaken for necessarily advocating its direct application in practice. Feedback systems are also dealt with, and we draw on some recent work [9, 10] which demonstrates that the "adaptive signal processing" techniques of feedforward control can also be applied to the design of feedback systems. Here we further illustrate the correspondence between the classical techniques of discrete time feedback design and the more recently developed "signal processing" techniques.

The results derived here may prove useful to those involved in developing adaptive signal processing techniques, since the solutions derived are "optimal" and not subject to the properties of adaptive algorithms designed to minimise a given cost function. In both feedforward and feedback control, the "predictability" of the acoustical time series to be dealt with is shown to be important, particularly so in feedback control. Finally, we again consider the "where does the energy go" problem of active control, the control theory used being capable of clearly illustrating the "absorption" and "prediction" mechanisms of control involved in dealing with stationary random time series, and engage in some further speculation on the possibilities for "power absorbing controllers" whose potential for use in the active control of sound and vibration remains unclear.

2. LINEAR PREDICTION

The theory of linear prediction has its origins in the early work of Yule [11], Kolmogorov [12], and Wiener [13]. It was the latter who showed how one could design a linear filter whose output provided an optimal estimate of future values of the signal input to the filter. Wiener developed the theory for random signals which had stationary statistical properties and which could be assumed to be generated by the output of a linear time invariant "shaping filter" whose input is white noise. Wiener's theory was based on an analysis of the continuous time problem using Laplace transforms. Later Kalman [14] used a state space analysis to show how a similar optimal filter could be derived but which is also capable of dealing with time varying process models and non-stationary signal sources. Here we will make use of Wiener's theory and restrict attention to signals that can be assumed to be generated by linear time invariant shaping filters excited by stationary white noise.

Figure 1 illustrates the basic optimal filtering problem. The signal $r(t)$ (the "received signal" in classical terms) is the signal whose future values we wish to predict. The signal $d(t)$ denotes the "desired signal", and in the case of the pure prediction problem, $d(t) = r(t + \Delta)$ where Δ is the number of time steps in the future for which a prediction is required. Here we have used t to denote the discrete time variable and we use q^{-1} to denote the delay operator such that, for example, $q^{-n}\, r(t) = r(t - n)$. The optimal filter that operates on $r(t)$ in order to produce an optimal estimate of $r(t + \Delta)$ is denoted by $H(q^{-1})$ which defines a polynomial in the delay operator q^{-1}. We define the error sequence $e(t)$ to be the difference between the desired sequence $d(t)$ and the estimate $\hat{d}(t)$ produced by the output of the filter. We seek the filter $H(q^{-1})$ that ensures the minimisation of the expected value (time average) of the squared error sequence and define the quadratic cost function

$$J = E\left[e^2(t)\right] \tag{1}$$

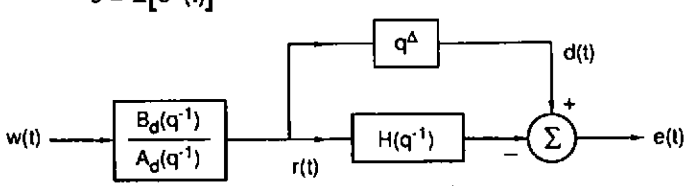

Figure 1 The pure prediction problem when the signal r(t) can be assumed to be generated by passing white noise w(t) through a shaping filter $B_d(q^{-1})/A_d(q^{-1})$. The desired signal d(t) is given by $r(t+\Delta)$.

Substitution of $e(t) = d(t) - H(q^{-1})\, r(t)$ and subsequent expansion shows that

$$J = R_{dd}(0) - 2H(q)\, R_{rd}(\tau) + H(q)\, H(q^{-1})\, R_{rr}(\tau) \tag{2}$$

where we have defined the correlation functions by, for example, $R_{rd}(\tau) = E\left[r(t)\, r(t+\tau)\right]$ and where $H(q)$ denotes the anti-causal counterpart of $H(q^{-1})$ such that $H(q) = h_0 + h_1\, q + h_2\, q^2 + h_3\, q^3 \ldots$ etc. In order to find the filter $H_0(q^{-1})$ that minimises J we assume that $H(q^{-1}) = H_0(q^{-1}) + \varepsilon\, H_\varepsilon(q^{-1})$ where $H_\varepsilon(q^{-1})$ is a realisable (causal and stable) departure from the optimal filter and ε is a small parameter. Substitution of this expression for $H(q)$ into equation (2) soon shows that for J to be minimised we require that

$$H_\varepsilon(q)\left(R_{rd}(\tau) - H_0(q^{-1})\, R_{rr}(\tau)\right) = 0 \tag{3}$$

Since we have chosen $H_\varepsilon(q^{-1})$ to be causal and stable, then $H_\varepsilon(q)$ must be purely anti-causal. Thus if equation (3) is to be satisfied the term in brackets must be equal to zero for positive τ which requires that

$$R_{rd}(\tau) - H_0(q^{-1})\, R_{rr}(\tau) = 0 \ , \ \tau \geq 0 \tag{4}$$

This is the discrete time form of the Wiener-Hopf equation which provides a necessary and sufficient condition for the optimality of the filter $H_0(q^{-1})$. A straightforward technique for solving this equation is to assume that $H_0(q^{-1})$ is a polynomial of finite duration (i.e. has only a finite number I of terms in the delay operator such that $H_0(q^{-1}) = h_0 + h_1\, q^{-1} + h_2\, q^{-2} \ldots h_{I-1}\, q^{-(I-1)}$ where $(I-1)$ is the "degree" of the polynomial.) Then, given the values of $R_{rd}(\tau)$ and $R_{rr}(\tau)$ at a number I of successive values of the discrete time lag τ, a number I of linear simultaneous equations can be written down and solved by matrix inversion. These equations are the *normal equations* as described in detail, for example, by Nelson and Elliott [15, Chapter 4] and yield an optimal solution for the coefficients $h_0\, h_1 \ldots h_{I-1}$ which of course, are simply the coefficients of an FIR digital filter.

Another approach to the solution of the Wiener-Hopf equation is to use classical methods of spectral factorisation. Following the approach presented by Grimble and Johnson [4, Chapter 11], we first define $X(\tau)$ to be the function given by

$$X(\tau) = R_{rd}(\tau) - H_0(q^{-1})\, R_{rr}(\tau) \quad (X(\tau) = 0 \ , \ \tau \geq 0) \tag{5}$$

and then take the z-transform of this relationship to give

$$X(z^{-1}) = S_{rd}(z^{-1}) - H_0(z^{-1})\, S_{rr}(z^{-1}) \tag{6}$$

where, for example, the cross power spectral density $S_{rd}(z^{-1})$ and auto-power spectral density $S_{rr}(z^{-1})$ are defined by

$$S_{rd}(z^{-1}) = \sum_{t=-\infty}^{\infty} R_{rd}(t)\, z^{-t}\ ,\quad S_{rr}(z^{-1}) = \sum_{t=-\infty}^{\infty} R_{rr}(t)\, z^{-t} \tag{7a,b}$$

Since the power spectrum $S_{rr}(e^{-j\omega})$ corresponding to $S_{rr}(z^{-1})$ may be a real, even and non-negative function of frequency, then $S_{rr}(z^{-1})$ can be factored into the product of two terms, one of which has all its poles and zeros inside the unit circle in the z-plane and one of which has all its poles and zeros at "mirror image" locations outside the unit circle in the z-plane. We therefore define the spectral factors $S(z^{-1})$ and $S(z)$ to be given by $S_{rr}(z^{-1}) = S(z^{-1})\, S(z)$. Equation (6) above can then be written as

$$\frac{X(z^{-1})}{S(z)} = \frac{S_{rd}(z^{-1})}{S(z)} - H_0(z^{-1})\, S(z^{-1}) \tag{8}$$

Since $S(z)$ has all its poles and zeros outside the unit circle then $1/S(z)$ also has all its poles and zeros outside the unit circle. The sequence whose z-transform is $1/S(z)$ must therefore be purely anti-causal and thus the sequence whose z-transform is $X(z^{-1})\, /\, S(z)$ must also be anti-causal, since $X(\tau)$ is non-zero only for negative values of the discrete time lag τ. We can therefore assert that $\left\{X(z^{-1})\, /\, S(z)\right\}_+ = 0$ where the braces denote the causal part of the corresponding sequence. It therefore follows from equation (8) that $\left\{\left(S_{rd}(z^{-1})\, /\, S(z)\right) - H_0(z^{-1})\, S(z^{-1})\right\}_+ = 0$ and thus since $H_0(z^{-1})\, S(z^{-1})$ is known to be purely causal it follows that

$$H_0(z^{-1}) = \frac{1}{S(z^{-1})}\left\{\frac{S_{rd}(z^{-1})}{S(z)}\right\}_+ \tag{9}$$

We now make the assumption that the sequence $r(t)$ is derived from the output of a shaping filter whose input is a white noise sequence $w(t)$ such that

$$r(t) = \frac{B_d(q^{-1})}{A_d(q^{-1})}\, w(t) \tag{10}$$

where $B_d(q^{-1})$ and $A_d(q^{-1})$ are respectively polynomials of the form

$$B_d(q^{-1}) = b_{d0} + b_{d1}\, q^{-1} + b_{d2}\, q^{-2}\ ..\ b_{dn_{bd}}\ q^{-n_{bd}} \tag{11}$$

$$A_d(q^{-1}) = 1 + a_{d1}\, q^{-1} + a_{d2}\, q^{-2}\ ..\ a_{dn_{ad}}\, q^{-n_{ad}} \tag{12}$$

The roots of $B_d(q^{-1})$ and $A_d(q^{-1})$ respectively define the zeros and poles of the shaping filter from which $r(t)$ is assumed to originate. This shaping filter is also assumed to be both stable and *minimum phase*, such that all of its poles and zeros are inside the unit circle. If the white noise sequence is assumed to have unit variance the spectral density $S_{rr}(z^{-1})$ is given by $B_d(z^{-1})\, B_d(z)\, /\, A_d(z^{-1})\, A_d(z)$. Thus the spectral factor $S(z^{-1})$ is simply $B_d(z^{-1})\ /\ A_d(z^{-1})$. Furthermore, for the pure prediction problem, since $d(t) = r(t + \Delta)$, then $S_{rd}(z^{-1}) = z^{\Delta}\, S_{rr}(z^{-1})$. It therefore follows that the solution for the optimal predictor is given by

$$H_0(z^{-1}) = \frac{A_d(z^{-1})}{B_d(z^{-1})}\left\{\frac{z^{\Delta}\, B_d(z^{-1})}{A_d(z^{-1})}\right\}_+ \tag{13}$$

This result confirms the well known solution for the optimal predictor which consists of a cascade of the inverse of the shaping filter with a filter whose impulse response is equal to that of the shaping filter but *advanced* by Δ samples.

It is worth emphasising at this stage that given a time series $r(t)$, there are well established procedures which enable the identification of a good model for the shaping filter $B_d(q^{-1})\ /\ A_d(q^{-1})$. A fundamental approach to system identification is that based on the recursive least squares method as described in detail by, for example, Ljung and Söderström [16]. These methods are readily adapted in practice to enable, when given a time series, a good model of a shaping filter to be identified up to reasonably high orders. Once given a model for $B_d(q^{-1})$ and $A_d(q^{-1})$, one is still faced formally with the extraction of the causal part of the term in braces in equation (13).

In order to isolate the causal part of the term in braces we first undertake a partial fraction expansion and write

$$\frac{B_d(z^{-1})}{z^{-\Delta}\, A_d(z^{-1})} = \frac{F(z^{-1})}{z^{-\Delta}} + \frac{G(z^{-1})}{A_d(z^{-1})} \tag{14}$$

where $F(z^{-1})$ and $G(z^{-1})$ are both polynomials in z^{-1} (and therefore represent causal transfer functions). The first term, which can be written as $z^{\Delta}\, F(z^{-1})$, will be entirely non-causal provided that the degree of the polynomial $F(z^{-1})$ is less than Δ. The second term, on the other hand, will be entirely causal if the degree of $G(z^{-1})$ is less than or equal to that

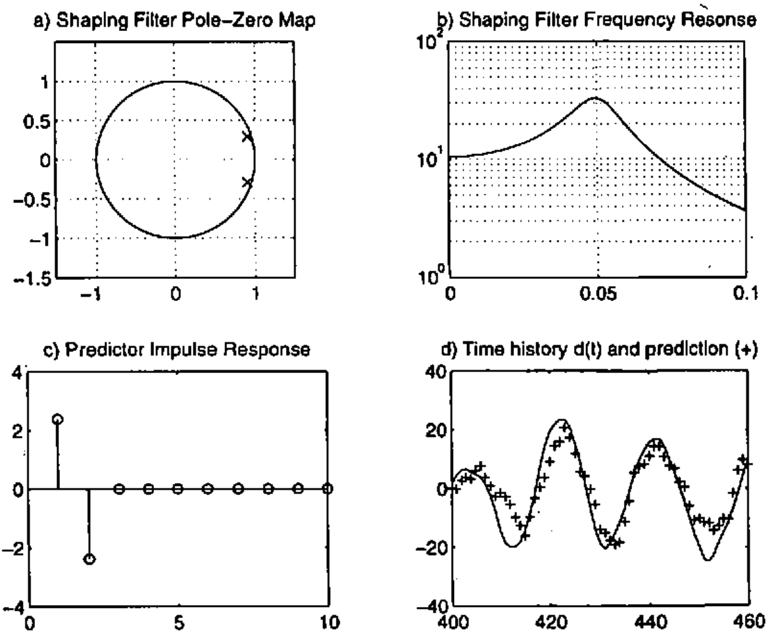

Figure 2 The performance of an optimal predictor when $\Delta = 5$ samples and the shaping filter poles are given by $r = 0.95$ and $\omega_0 = \pi/10$. The predicted values of the time series are shown as crosses in d).

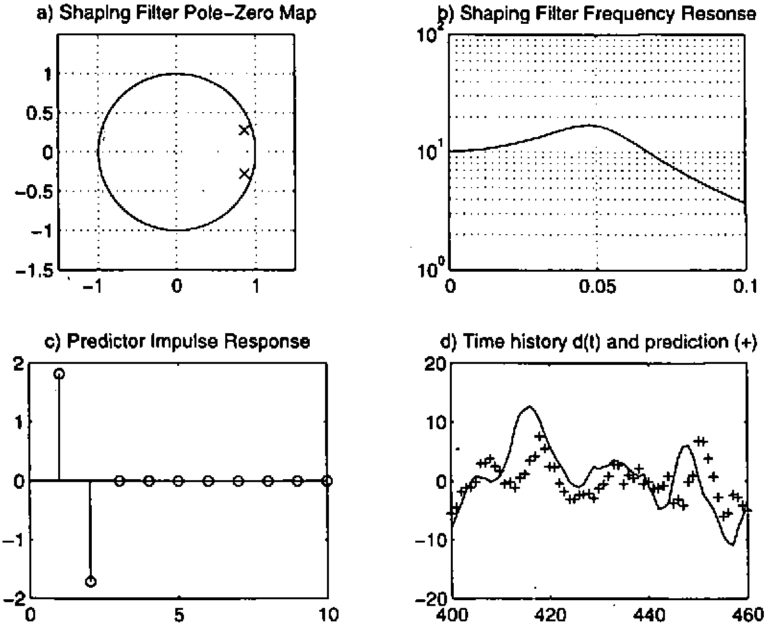

Figure 3 The performance of an optimal predictor when $\Delta = 5$ samples and the shaping filter poles are given by $r = 0.9$ and $\omega_0 = \pi/10$. The predicted values of the time series are shown as crosses in d).

of $A_d(z^{-1})$ (i.e. deg $(G) \leq$ deg (A_d)) [17, Chapter 10]. Equation (14) can also be written as

$$z^{-\Delta} G(z^{-1}) + A_d(z^{-1}) F(z^{-1}) = B_d(z^{-1}) \tag{15}$$

which is a *Diophantine equation* (named after the Greek mathematician Diophantus ~ 300 AD). As described, for example, by Aström and Wittenmark [3, Chapter 10] this is a polynomial equation of the form $PX + QY = R$ where the polynomials P, Q, R are given and X and Y are to be determined. Such an equation has a solution if and only if the greatest common factor of P and Q divides R; if this is satisfied there are infinitely many solutions, but always a unique solution satisfying deg $(X) <$ deg (Q) and deg $(Y) <$ deg (P). Since $z^{-\Delta}$ cannot be a factor of $A_d(z^{-1})$ (since the shaping filter is causal) we can be sure that there exist polynomials $F(z^{-1})$ and $G(z^{-1})$ such that deg $(G) <$ deg (A) and deg $(F) < \Delta$. These conditions respectively ensure that $G(z^{-1})/A_d(z^{-1})$ is causal and that $z^{\Delta} F(z^{-1})$ is entirely non-causal. Thus we can substitute the Diophantine equation (15) into the term in braces in equation (13) to show that

$$\left\{\frac{z^{\Delta} B_d(z^{-1})}{A_d(z^{-1})}\right\}_+ = \left\{\frac{G(z^{-1})}{A_d(z^{-1})} + z^{\Delta} F(z^{-1})\right\}_+ = \frac{G(z^{-1})}{A_d(z^{-1})} \tag{16}$$

Note that a number of methods are available in order to find the polynomials $F(z^{-1})$ and $G(z^{-1})$ that constitute the "minimal degree" solution to the Diophantine equation. One procedure is to expand equation (15) in powers of z^{-1} and equate coefficients. This leads to a set of linear equations in the coefficients of $G(z^{-1})$ and $F(z^{-1})$, these equations being solved by matrix inversion. Once $G(z^{-1})$ is found by using such a technique, the solution for the optimal predictor that follows from equations (13) and (16) is given by

$$H_0(z^{-1}) = \frac{G(z^{-1})}{B_d(z^{-1})} \tag{17}$$

Two simple examples of the performance of an optimal predictor are shown in Figures 2 and 3. The results illustrated have been computed for a second order shaping filter having $B_d(z^{-1}) = 1$ and the denominator polynomial $A_d(z^{-1}) = 1 - 2r\cos\omega_0\, z^{-1} + r^2 z^{-2}$ where ω_0 is the natural frequency and the poles of the system lie at $z = r\, e^{\pm j\omega_0}$. The time histories shown compare the output of the shaping filter with the predicted output at $\Delta = 5$ samples in "the future". The examples given clearly show the decrease in the "predictability" of the signal as the bandwidth of the shaping filter increases and the poles of the system move away from the unit circle. The ability to predict the future of acoustical time series will be shown below to have a profound influence on the ultimate performance of a number of different control strategies.

3. INVERSION OF LINEAR SYSTEMS

An important class of problem in modern acoustics which can be treated using the theory described above is the design of "inverse" or "equalising" filters. Such filters can be designed in order to ensure, for example, the very accurate reproduction of desired acoustical signals at a point (or points) in a sound field, even when the transmission path (or paths) has an imperfect frequency response function. The paths to be equalised are to a good approximation, linear, and therefore readily treated using classical methods. It will also be useful in what follows to describe some important aspects of this problem, in particular the matter of the inversion of non-minimum phase systems.

The problem considered is illustrated in block diagram form in Figure 4. Here we assume that we wish to design a filter $H(q^{-1})$ in order to best approximate (in the least squares sense) the desired signal $d(t)$ which we will here assume to be simply a delayed version of the white noise signal $w(t)$. The delay of Δ samples (the "modelling delay") will be shown to be crucially important in determining the effectiveness of the inversion. The system or "plant" to be inverted is described by the ratio of polynomials $q^{-k} B_S(q^{-1}) / A_S(q^{-1})$ where q^{-k} denotes a delay of k samples through the system. This delay is of course particularly relevant to acoustical problems which are usually characterised by a bulk propagation delay. The other important feature of such plants is that they are almost always stable (so the roots of the polynomial $A_S(z^{-1})$ lie inside the unit circle in the z-plane) but they are also very often non-minimum phase (so some of the roots of the polynomial $B_S(z^{-1})$ lie *outside* the unit circle in the z-plane). In the case of stable minimum phase plants, the design of $H(q^{-1})$ is, in principle, trivial. Thus if the modelling delay Δ is chosen to be equal to k, the bulk delay through the plant, then we can choose $H(q^{-1})$ to be given by $A_S(q^{-1}) / B_S(q^{-1})$ in order to provide "perfect" equalisation of the plant. However, for non-minimum phase plants, this inverse filter will be unstable since there are roots of $B_S(z^{-1})$, and thus poles of $H(z^{-1})$, outside the unit circle in the z-plane.

It is possible however, even in the case of non-minimum phase plants, to design inverse filters that minimise the mean squared value of the error sequence $e(t)$ depicted in Figure 4. Note that in so far as the design of $H(q^{-1})$ is concerned, such an optimal filter must satisfy the Wiener-Hopf equation given by equation (4) and as a consequence, the optimal filter is given by the solution described by equation (9).

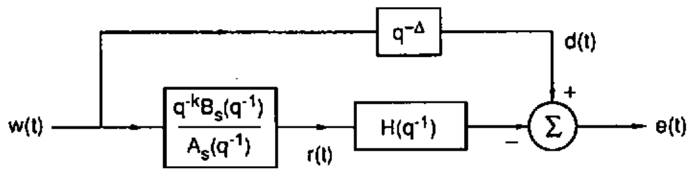

Figure 4 The inversion problem; we wish to find $H(q^{-1})$ which ensures that the desired signal $d(t)$ is reproduced with minimum mean square error, where the desired signal is simply a delayed version of the white noise signal $w(t)$.

The power spectral density of the sequence $r(t)$ will now be given by $B_S(z^{-1})\, B_S(z) \,/\, A_S(z^{-1})\, A_S(z)$ but since $B_S(z^{-1})$ may have zeros outside the unit circle we define the spectral factor $S(z^{-1})$ to be given by

$$S(z^{-1}) = P(z^{-1}) \,/\, A_S(z^{-1}) \tag{18}$$

where the polynomial $P(z^{-1})$ has all its zeros inside the unit circle and satisfies

$$P(z^{-1})\, P(z) = B_S(z^{-1})\, B_S(z) \tag{19}$$

Note that this equation *does not* imply that $P(z^{-1}) = B_S(z^{-1})$. The procedure for finding $P(z^{-1})$ is to form the polynomial $B_S(z^{-1})\, B_S(z)$, find the roots lying within the unit circle and attribute only those to $P(z^{-1})$. That this can be achieved in the case of non-minimum phase plants can be understood by assuming $B_S(z^{-1}) = B_S^+(z^{-1})\, B_S^-(z^{-1})$ where $B_S^+(z^{-1})$ and $B_S^-(z^{-1})$ have roots respectively inside and outside the unit circle. The product $B_S(z^{-1})\, B_S(z)$ can then be written as

$$B_S(z^{-1})\, B_S(z) = B_S^+(z^{-1})\, B_S^-(z^{-1})\, B_S^+(z)\, B_S^-(z) \tag{20}$$

Now note that since $B_S^-(z^{-1})$ has all its zeros outside the unit circle, then $B_S^-(z)$ will have all its zeros *inside* the unit circle. Furthermore, if the degree of the polynomial $B_S^-(z^{-1})$ is n say, then $z^{-n} B_S^-(z)$ will be a *causal* polynomial with all its zeros inside the unit circle. In addition, we can also write the above identity as

$$B_s(z^{-1})\, B_s(z) = B_s^+(z^{-1})\, z^n B_s^-(z^{-1})\, B_s^+(z)\, z^{-n} B_s^-(z) \tag{21}$$

where $z^n B_s^-(z^{-1})$ is now an *anti-causal* polynomial with its zeros outside the unit circle. We can therefore identify the spectral factor $P(z^{-1})$ as being given by

$$P(z^{-1}) = B_s^+(z^{-1})\, z^{-n} B_s^-(z) \tag{22}$$

Similarly of course, the anti-causal spectral factor $P(z)$ is given by $B_s^+(z)$ $z^n B_s^-(z^{-1})$. This decomposition amounts to the "reflection" of the zeros of $B_s^+(z^{-1})$ that are outside the unit circle to equivalent "mirror image" locations inside the unit circle that are given by the roots of the causal polynomial $z^{-n} B_s^-(z)$. As a simple example, if $B_s^-(z^{-1}) = 1 + 2\, z^{-1}$ then there is a zero at $z = -2$. However, $z^{-n} B_s^-(z) = z^{-1}\, (1 + 2z)$ which in turn can be written as $2(1 + 0.5\, z^{-1})$ showing that this polynomial has a zero at $z = -0.5$.

Returning to the solution for the optimal filter given by equation (9), for the problem depicted in Figure 4 the cross power spectral density between the signals $r(t)$ and $d(t)$ is given by $S_{rd}(z^{-1}) = S_{dr}(z) = z^{k-\Delta} B_s(z)/A_s(z)$, where it is assumed that the white noise sequence $w(t)$ has unit variance. The solution for the optimal filter thus becomes

$$H_0(z^{-1}) = \frac{A_s(z^{-1})}{P(z^{-1})}\left\{\frac{z^{k-\Delta} B_s(z)}{P(z)}\right\}_+ \tag{23}$$

First note the form of solution when the plant to be inverted is minimum phase, where $P(z^{-1}) = B_s(z^{-1}) = B_s^+(z^{-1})$. Under these conditions

$$H_0(z^{-1}) = \begin{cases} 0 & \Delta < k \\ \dfrac{A_s(z^{-1})}{B_s(z^{-1})}\, z^{-(\Delta - k)} & \Delta \geq k \end{cases} \tag{24}$$

Thus when the modelling delay Δ is less than the delay k through the plant, the solution for the optimal filter is zero since the causal part of $z^{k-\Delta}$ is zero; any non-zero filter applied to the signal $r(t)$ will only serve to *increase* the mean squared value of the error signal. For a value of $\Delta = k$, then the optimal filter is simply the inverse of the minimum phase plant and for $\Delta > k$, the inverse filter must include a delay $(\Delta - k)$ to compensate for the excess modelling delay.

When the plant to be inverted is non-minimum phase, it is still necessary to ensure that $\Delta \geq k$ to give a non-zero optimal filter since the term in braces in equation (23) is entirely anti-causal for $\Delta < k$. Generally speaking, the greater the margin by which Δ exceeds k, the greater will be the success of the inverse filter in reducing the mean-square error. A simple example demonstrates this point. Assuming $B_s(z^{-1}) = (1 + a z^{-1}), |a| > 1$, then $B_s^+(z^{-1}) = 1$ and $B_s^-(z^{-1})$ is given by $(1 + a z^{-1})$. Thus $P(z^{-1}) = a(1 + a^{-1} z^{-1})$ and the expression for the optimal filter (assuming $A(z^{-1}) = 1$) reduces to

$$H_0(z^{-1}) = \frac{1}{a(1 + a^{-1} z^{-1})} \left\{ \frac{z^{k-\Delta}(1 + az)}{a(1 + a^{-1} z)} \right\}_+ \tag{25}$$

The denominator polynomial can be rewritten as a numerator polynomial by using the Binomial expansion $(1 + x)^{-1} = 1 - x + x^2 \ldots$ thus showing that

$$H_0(z^{-1}) = \frac{1}{a^2(1 + a^{-1} z^{-1})} \left\{ z^{k-\Delta}(1 + az)(1 - a^{-1} z + a^{-2} z^2 - a^{-3} z^3 \ldots) \right\}_+ \tag{26}$$

The polynomial in braces is thus shown to be a series that decays in reverse time (i.e. with increasing powers of z), the rate of decay being controlled by the position in the z-plane of the zero at $z = -(1/a)$. The closer that this zero is to the unit circle ($|z| = 1$) then the slower the rate of decay of the polynomial in braces. As the modelling delay Δ is increased, then the number of terms is increased that comprise the causal part of the term in braces. The quality of the approximation to an exact inverse is therefore improved as both the modelling delay is increased and as the zeros of the system to be inverted lie further from the unit circle in the z-plane. This is further illustrated in Figure 5 which shows three examples of systems to be inverted and the corresponding impulse response of the optimal inverse filter. The results shown were deduced from equation (23) by using a Diophantine equation in order to extract the causal part of the term in braces.

The inversion of non-minimum phase systems has particular relevance to the signal processing problems associated with virtual acoustic imaging systems. The objective of such systems is the reproduction of a sound field in the region of a listeners head that results in the listener having the impression that a "virtual" source of sound exists at a spatial position that is other than that of the real sources used to generate the field. Considerable progress is currently being made in this area [e.g. 18, 19, 20] largely because filter design methods have been developed that enable the accurate inversion of multi-channel systems (typically consisting of 16 to 25 transmission paths from reproduction sources to error sensors in the reproduced field). A particularly important contribution in this regard is that described by

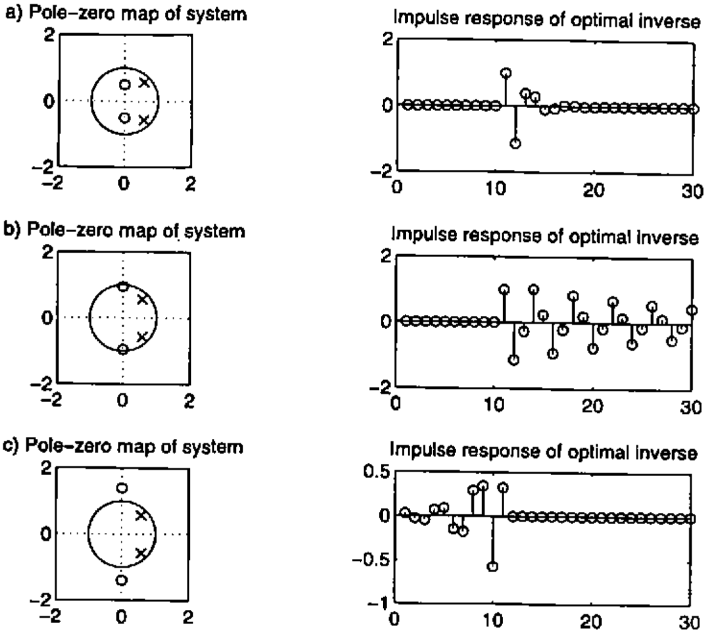

Figure 5 Three pole-zero maps of systems to be inverted and the impulse responses of their optimal inverse filters. In each case the system was also assumed to have a delay k = 5 samples and the modelling delay Δ was chosen to be 15 samples. The systems in a) and b) are minimum phase whilst the system in c) is non-minimum phase. Note the long duration of the impulse response of the inverse filter for the system in b) which has zeros close to the unit circle.

Kirkeby *et al* [21] who have shown that such large scale systems can be successfully inverted by using frequency domain techniques that rely on the use of a properly chosen modelling delay together with a "regularisation parameter" that has the effect of ensuring that non-minimum phase zeros very close to the unit circle do not result in inverse filters having an excessively long duration. The regularisation of the inversion stems from the penalisation of the mean square filter output signals in addition to the mean square error signals. This will be returned to below. Another important contribution that offers a useful technique for the inversion of non-minimum phase "acoustical" systems is that described by Miyoshi and Kaneda [22]. It is interesting to note that the latter technique, which involves the use of more reproduction sources that there are error sensors, relies on the Bezout identity, a particular form of the Diophantine equation $PX + QY = R$ described in Section 2 with R set equal to unity. To summarise, our understanding of the problem of inverting acoustical systems is now well advanced and opens up many possibilities for future manipulation of the acoustical environment.

4. FEEDFORWARD CONTROL

The development of "active" systems for the control of unwanted noise has relied on the use of feedforward control, wherein an unwanted disturbance is detected prior to its arrival at the position in space where control is required. The relevant block diagram is shown in Figure 6, and in Figure 7 in the form of the equivalent optimal filtering problem. The solution for the optimal filter that minimises the mean square error in this case is again given by equation (9) but with a reversal of sign since the signals involved are added at the summing junction. Thus

$$H_0(z^{-1}) = \frac{-1}{S(z^{-1})}\left\{\frac{S_{rd}(z^{-1})}{S(z)}\right\}_+ \tag{27}$$

where the signal $r(t)$ is now defined as shown in Figure 7. Again assuming $B_d(z^{-1})$ to be minimum phase, but allowing for the possibility that $B_s(z^{-1})$ is again non-minimum phase enables the expression for the spectral factor $S(z^{-1})$ to be written as

$$S(z^{-1}) = \frac{P(z^{-1})\,B_d(z^{-1})}{A_s(z^{-1})\,A_d(z^{-1})} \tag{28}$$

where it is assumed that the system to be controlled is stable. It also follows from the block diagram of Figure 7 that the cross spectrum $S_{rd}(z^{-1})$ is given by

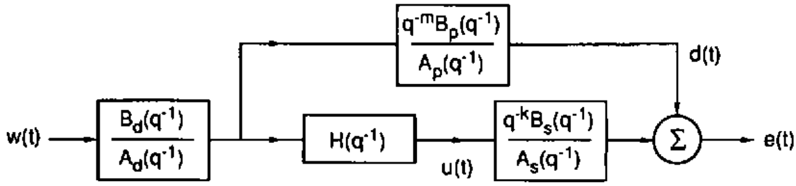

Figure 6 Feedforward control; a primary disturbance (assumed to be generated by passing white noise through a shaping filter) is detected prior to passing via a "primary path" $q^{-m}B_p(q^{-1})/A_p(q^{-1})$.

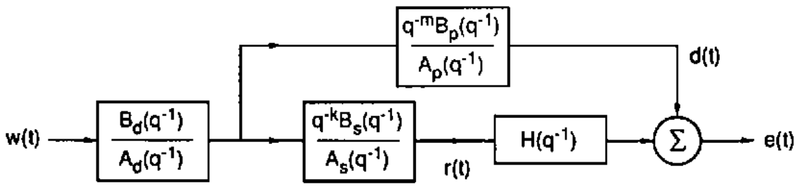

Figure 7 Feedforward control in the form of an optimal filtering problem. The reversal of operation of the filter $H(q^{-1})$ and the "secondary path" $q^{-k}B_s(q^{-1})/A_s(q^{-1})$ is permitted when the systems are linear and time invariant.

$$S_{rd}(z^{-1}) = \frac{z^{-m}\,B_p(z^{-1})}{A_p(z^{-1})}\,\frac{z^{k}\,B_s(z)}{A_s(z)}\,\frac{B_d(z^{-1})\,B_d(z)}{A_d(z^{-1})\,A_d(z)} \tag{29}$$

where it is again assumed that the white noise input has unit variance. Substitution of these results into the expression for the optimal filter shows that

$$H_0(z^{-1}) = -\frac{A_s(z^{-1})\,A_d(z^{-1})}{P(z^{-1})\,B_d(z^{-1})}\left\{\frac{z^{k-m}\,B_s(z)\,B_p(z^{-1})\,B_d(z^{-1})}{P(z)\,A_p(z^{-1})\,A_d(z^{-1})}\right\}_+ \tag{30}$$

Now consider some special cases in order to better understand the implications of this solution. Assume firstly that the system to be controlled is minimum phase, then we can write $P(z) = B_s(z)$ and then provided $m \geq k$, the term in braces will be entirely causal and the solution reduces to

$$H_0(z^{-1}) = -z^{-(m-k)} \frac{A_s(z^{-1})}{B_d(z^{-1})} \frac{B_p(z^{-1})}{A_p(z^{-1})} \tag{31}$$

The optimal filter is thus seen to consist of an inversion of the system (or "secondary path") to be controlled $\left(A_s(z^{-1}) / B_s(z^{-1})\right)$ cascaded with a model of the "primary path" $B_p(z^{-1}) / A_p(z^{-1})$. The delay $(m - k)$ simply compensates for the excess delay in the primary path over that in the secondary path. Perfect control is in principle possible. Note that the properties of the disturbance spectrum (as quantified by $B_d(z^{-1}) / A_d(z^{-1})$) do not appear in the solution.

Another interesting case is to again consider a minimum phase secondary path and assume that the primary path is simply a delay of m samples $\left(B_p(z^{-1}) = A_p(z^{-1}) = 1\right)$. In this case the solution reduces to

$$H_0(z^{-1}) = -\frac{A_s(z^{-1})}{B_s(z^{-1})} \frac{A_d(z^{-1})}{B_d(z^{-1})} \left\{ \frac{z^{k-m} B_d(z^{-1})}{A_d(z^{-1})} \right\}_+ \tag{32}$$

It is now clear that if the delay k in the secondary path exceeds the delay m in the primary path, the optimal filter is a cascade of the secondary path inverse $\left(A_s(z^{-1}) / B_s(z^{-1})\right)$ and an *optimal predictor* of the disturbance signal as given by equation (13).

The full solution given by equation (30) thus reflects the need to invert the secondary path, model the primary path and to predict the disturbance. In the general case it is possible to extract the causal part of the term in braces by first undertaking the partial fraction expansion

$$\frac{z^{k-m} B_s(z) B_p(z^{-1}) B_d(z^{-1})}{P(z) A_p(z^{-1}) A_d(z^{-1})} = \frac{z^g F(z^{-1})}{P(z)} + \frac{G(z^{-1})}{A_p(z^{-1}) A_d(z^{-1})} \tag{33}$$

where $z^g F(z^{-1})/P(z)$ will be entirely non-causal provided that g is chosen appropriately and $G(z^{-1})/A_p(z^{-1}) A_d(z^{-1})$ will be causal if deg (G) < deg $(A_p A_d)$. We thus seek the minimal degree solution of the Diophantine equation.

$$G(z^{-1}) P(z) + z^g F(z^{-1}) A_p(z^{-1}) A_d(z^{-1}) = z^{k-m} B_s(z) B_p(z^{-1}) B_d(z^{-1}) \tag{34}$$

and having determined $G(z^{-1})$ and $F(z^{-1})$, the solution for the optimal filter given by equation (30) becomes

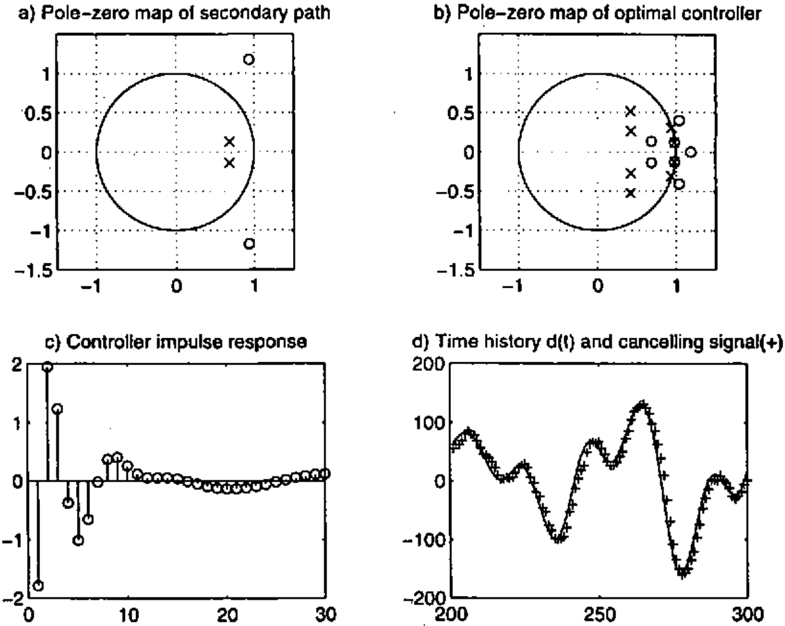

Figure 8 An example of a feedforward optimal controller when the secondary path is non-minimum phase with a delay k of 6 samples. The primary path includes a delay m of 17 samples and has zeros at $r = 0.5$, $\theta = \pm 2\pi/15$ and poles at $r = 0.99$, $\theta = \pm 2\pi/20$. The shaping filter (Figure 7) generating the disturbance has zeros at $r = 0.5$, $\theta = \pm 2\pi/11$ and poles at $r = 0.99$, $\theta = \pm 2\pi/50$.

$$H_0(z^{-1}) = -\frac{A_s(z^{-1})\, G(z^{-1})}{P(z^{-1})\, B_d(z^{-1})\, A_p(z^{-1})} \tag{35}$$

An example of a feedforward controller designed by using this method is shown in Figure 8. Of course the practical design of real feedforward controllers has not relied on this solution technique. A much simpler and highly effective approach in practice is to solve the Wiener-Hopf equation by constraining the controller to be FIR and making use of, for example, the filtered-x LMS algorithm [23, 24] in order to design the controller adaptively such that the mean square error is minimised. Such adaptive approaches have proven their worth in the control of disturbances such as the periodic sound produced by aircraft propellers, [25, 26, 27], and automobile engines [28]. In none of these cases is the bandwidth of the disturbance signal infinitesimal, but it is usually sufficiently narrow for the disturbance to be highly predictable, thus enabling good results to be obtained even when the delay in the secondary path exceeds that in the primary path. In the case of less predictable signals, such as the noise generated in automobiles by the motion relative to the road surface, a secondary path delay that is much less than that in the primary path is of course an important pre-requisite [29, 30]. Similar comments apply to other problems involving the feedforward control of broadband disturbances such as those found in air conditioning ducts [31, 32] or those generated by gas turbine exhausts [33]. However, the physical behaviour of sound fields limits the performance of such controllers to low frequencies, but active systems for the control of propeller noise are now in production and the use of active control for the suppression of fan tones from the next generation of high by-pass ratio jet engines is receiving detailed attention [34, 35]. The future of feedforward control systems for the suppression of unwanted sound is now largely a matter of economics. The physical processes involved are well understood and our mastery of the technology continues to improve. In the short term it will mostly be "high technology" applications which prove economic, but more widespread use of active techniques seem probable in the long term. For more discussion, the reader is referred to Eriksson's recent interesting article on product development in active control [36].

5. FEEDBACK CONTROL

The use of feedback to control unwanted noise has long been considered a possibility [37] and is now in commercial production in the form of "active headsets". Feedback control is the only control strategy available when there is no realistic possibility of detecting a disturbance signal prior to its arrival at the position at which control is desired. The use of feedback for the suppression of unwanted vibrations or the modification of the dynamic response of structures has also been considered by many authors, see, for example, [38]. Here we consider some basic features of the feedback control problem, again with attention restricted to the single channel case. The block diagram of relevance is shown in Figure 9, where we denote $G_0(q^{-1})$ as the feedback controller and $q^{-k}\, B_s(q^{-1}) \,/\, A_s(q^{-1})$ as the plant to be controlled. The disturbance $d(t)$ at the output of the plant is assumed to be generated by passing white noise through a (minimum phase) shaping filter $B_d(q^{-1}) \,/\, A_d(q^{-1})$. A very simple technique for designing a controller that minimises the mean square "error signal" at the output of the plant is to re-interpret the design of the feedback controller as an optimal filtering problem. The block diagram manipulations involved are illustrated in Figures 10 and 11. In Figure 10, the controller is assumed to consist of two parts; one of which provides a perfect model of the plant to be controlled and a further part $H(q^{-1})$ which can be designed as an optimal filter as illustrated in Figure 11. It is of course possible to implement controllers having precisely the structure indicated in Figure 10 where the controller includes a model of the plant response $\left(q^{-k}\, \hat{B}_s(q^{-1}) \,/\, \hat{A}_s(q^{-1}) \text{ say}\right)$. Such a strategy is termed "internal model control". An essentially equivalent control strategy has been adopted recently by a number of workers and has been discussed at length and analysed thoroughly by Elliott *et al* [9]. Elliott *et al* also discuss the historical origins of the controller design procedure which dates back to Newton *et al* [39] who built on the classical work of Wiener. One important assumption that is necessary in such a controller design procedure is that the plant to be controlled is *stable*. The need for this assumption was later removed by Youla *et al* [40] who showed how the controller design method could be extended in order to deal with unstable plants.

Here we continue to consider the case of stable plants and use this controller design procedure to describe some fundamental limitations of feedback control. It follows from the block diagram manipulations illustrated in Figures 9, 10 and 11 that the controller $G_0(z^{-1})$ can be expressed in terms of the "optimal cascade compensator" $H_0(z^{-1})$ by

$$G_0(z^{-1}) = \frac{H_0(z^{-1})}{1 + H_0(z^{-1})\, W_s(z^{-1})} \qquad (36)$$

where $W_s(z^{-1}) = z^{-k}\, B_s(z^{-1}) \,/\, A_s(z^{-1})$ is the transfer function of the plant. Thus provided $W_s(z^{-1})$ is known, or can be identified accurately, then $G_0(z^{-1})$ can be computed after finding $H_0(z^{-1})$. The latter is deduced from the solution of the equivalent feedforward problem as suggested by Figure 11. Thus the solution given by equation (30) above is directly applicable but with the primary path transfer function set to unity. Therefore we compute $H_0(z^{-1})$ from

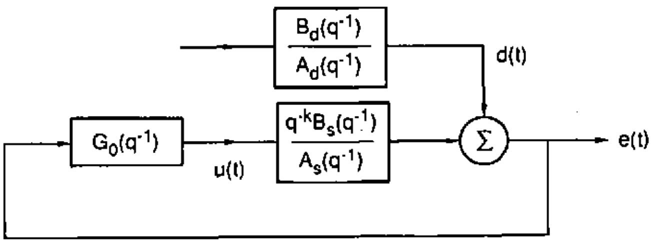

Figure 9 Feedback control, where the disturbance can only be detected at the output of the system to be controlled.

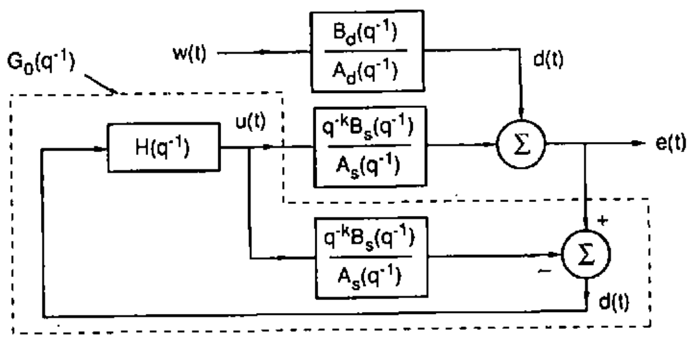

Figure 10 The feedback control problem when the controller $G_0(q^{-1})$ is assumed to consist of a filter $H(q^{-1})$ plus a model of the system to be controlled. (after Elliott *et al* [9])

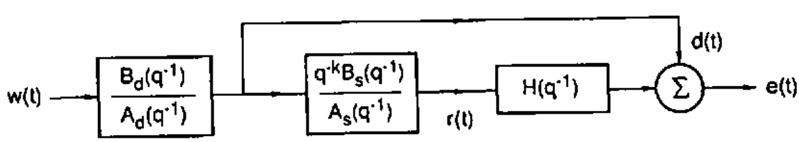

Figure 11 The design of the optimal feedback controller by treating the design of $H(q^{-1})$ as an optimal filtering problem. Note the correspondence with figure 7.

$$H_0(z^{-1}) = -\frac{A_s(z^{-1})\,A_d(z^{-1})}{P(z^{-1})\,B_d(z^{-1})}\left\{\frac{z^k\,B_s(z)\,B_d(z^{-1})}{P(z)\,A_d(z^{-1})}\right\}_+ \tag{37}$$

The function of $H_0(z^{-1})$ is thus shown to involve the inversion of the plant to be controlled and the prediction of the disturbance. For a minimum phase plant $P(z^{-1}) = B_s(z^{-1})$ the solution reduces to

$$H_0(z^{-1}) = -\frac{A_s(z^{-1})\,A_d(z^{-1})}{B_s(z^{-1})\,B_d(z^{-1})}\left\{\frac{z^k\,B_d(z^{-1})}{A_d(z^{-1})}\right\}_+ \tag{38}$$

which clearly shows that $H_0(z^{-1})$ is a cascade of the inverse of the plant $A_s(z^{-1})\,/\,B_s(z^{-1})$ and a k-step ahead predictor of the disturbance signal i.e. the optimal predictor of the disturbance signal when subject to the delay through the plant. In the case $k = 0$ the term in braces is purely causal and the solution reduces to $H_0(z^{-1}) = -A_s(z^{-1})\,/\,B_s(z^{-1})$, i.e. a pure inversion of the plant. Thus for zero delay minimum phase plants, perfect suppression of the disturbance is in principle possible, irrespective of the spectrum of the disturbance. However, for a white disturbance *any* non-zero delay will result in no possibility of disturbance suppression.

These results give some insight into the operation of optimal feedback controllers and suggest some important notions; feedback control is likely to be much easier to realise for minimum phase plants and for those non-minimum phase plants whose zeros are not close to the unit circle and are therefore easier to invert successfully. The results also suggest that more success can be achieved with small plant delays and *predictable disturbances*. However, a vitally important issue in feedback design is robustness, and in particular robust stability. Naturally in any real control system the plant to be controlled will have dynamics that are to a greater or lesser extent uncertain and it is important that the controller design remains stable given small perturbations in the plant dynamics.

Elliott and Sutton [41] consider the issue of robustness in some detail within the context of the feedback control of sound and demonstrate clearly that a simple method of increasing robustness is to introduce a term into the cost function to be minimised that weights the controller "effort". This of course is the classical approach to the definition of the cost function minimised in linear quadratic Gaussian (LQG) optimal control when the signals involved may be considered to be stationary [3]. In the single channel case, the cost function minimised has the form

$$J = E\left[e^2(t) + \beta u^2(t)\right] \tag{39}$$

where β is the (generally small) effort weighting parameter and $u(t)$ is the controller output sequence.

It is straightforward to demonstrate [42], by using the argument outlined in Section 2 above, that the filter $H(q^{-1})$ illustrated in Figure 10 that ensures the minimisation of this cost function must satisfy the Wiener-Hopf equation

$$R_{rd}(\tau) + H_0(q^{-1})\left[R_{rr}(\tau) + \beta\, R_{dd}(\tau)\right] = 0, \quad \tau \geq 0 \qquad (40)$$

where $R_{dd}(\tau)$ is the autocorrelation function of the disturbance signal. Also by analogy with the reasoning presented in Section 1, it can be shown [42] that the solution of this equation is given by

$$H_0(z^{-1}) = \frac{1}{S_\beta(z^{-1})}\left\{\frac{S_{rd}(z^{-1})}{S_\beta(z)}\right\}_+ \qquad (41)$$

where the spectral factors $S_\beta(z^{-1})$ and $S_\beta(z)$ are defined by

$$S_\beta(z^{-1})\, S_\beta(z) = S_{rr}(z^{-1}) + \beta\, S_{dd}(z^{-1}) \qquad (42)$$

(A clear proof that such a spectral factorisation is possible is given by, for example, Aström and Wittenmark [3]). It is usual in the frequency domain analysis of the LQG optimal control problem to work with an ARMAX model of the system [4] and when such a model is adopted, it can also be shown [42] that the solution given here reduces exactly to that given by Grimble and Johnson [4, Chapter 12]. These authors also show that this solution reduces to the solutions for "minimum variance" and "generalised minimum variance" control when given the appropriate assumptions. Some examples of the performance of a feedback controller are shown in Figures 12 and 13. These examples clearly show the dependence of the performance of the controller on the predictability of the disturbance signal. These controllers were designed by using a Diophantine equation to extract the causal part of the solution given by equation (41).

The design of feedback controllers for the suppression of unwanted sound has mostly used analogue electronics [43, 44] (in active ear defenders) but recently a number of workers have adopted digital feedback control systems based on the architecture illustrated in Figure 10 in assessing the possibilities for active control of sound in ducts [45], for local control of sound fields [46], in active vibration isolation systems [47], for the suppression of sun-roof flow oscillations [48], and for the rejection of periodic disturbances in a pulsed flow [49]. Such an approach is also referred to as "secondary path neutralisation" in the recent text by Kuo and Morgan [50]. One of the other interesting

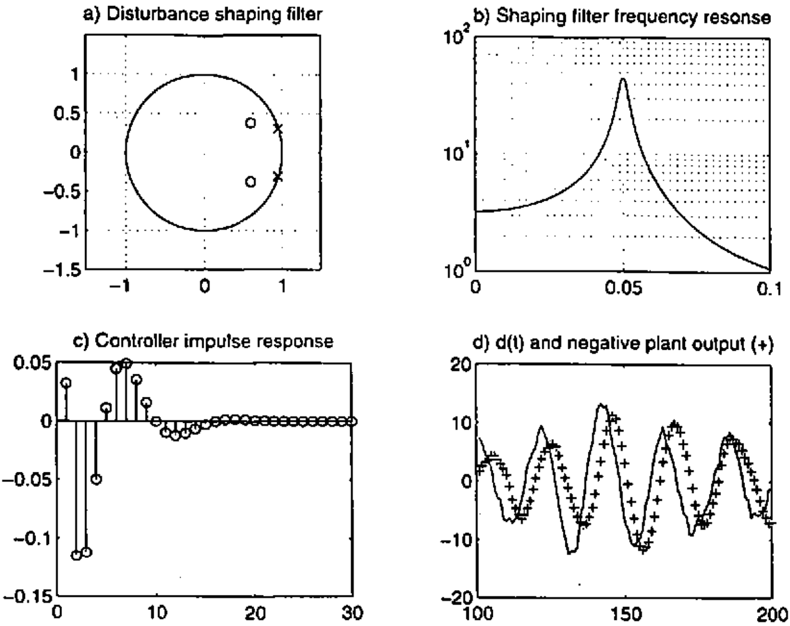

Figure 12 Feedback control of a non-minimum phase second order plant (having poles at $r = 0.8$, $\theta = \pm 2\pi/32$ and zeros at $r = 1.5$, $\theta = \pm 2\pi/17$) including a delay k of 4 samples. The disturbance shaping filter has poles at $r = 0.99$. The controller impulse shown in c) is that of $H_0(q^{-1})$. An effort weighting $\beta = 0.1$ was included in the controller design.

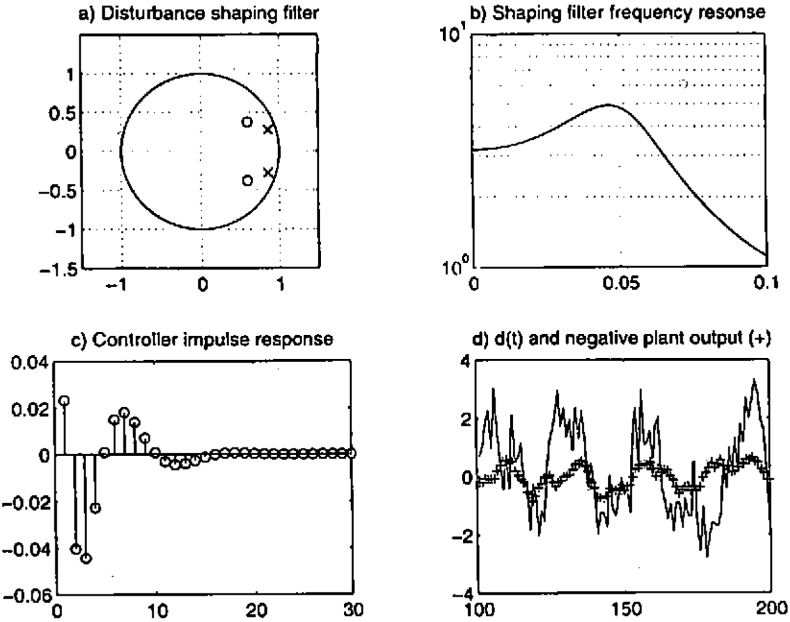

Figure 13 Feedback control of a non-minimum phase second order plant as in Figure 12. In this case the disturbance shaping filter has poles at r = 0.9.

possibilities is the use of feedback control for the modification of the response of structures in order to prevent the radiation of sound from vibrating surfaces driven by "unpredictable" disturbances. Such a case is that analysed by Thomas and Nelson [51] who attempted to assess the possibilities for the feedback control of sound radiation from a plate driven by a turbulent boundary layer. It was concluded that reductions in sound power were in principle achievable, given a complete knowledge of the system "states", including those of the filters used to model the disturbance. The potential for reductions in sound power radiated were greatest at the dominant resonance frequencies of the panel, presumably where the disturbance, as measured at the "plant output" was most predictable. Although it may be some time before such "smart structures" find their way into practical application, other work on the active control of double panel structures [52, 53] shows that there may be a more realistic possibility of enhancing their sound transmission loss by feedback control of the basic mass-air-mass resonance frequency which tends to degrade low frequency performance.

6. FEEDFORWARD CONTROL OF POWER INPUT

The analysis of active control systems in acoustics and vibrations has often been assisted by evaluating the effect of active control on global properties such as, for example, the total power input to an acoustical system [54] or the total vibrational energy in a vibrating structure [55]. Here we re-consider the problem of power input minimisation when the time history of the primary source fluctuation (volume velocity in acoustics, force in vibrations) can be modelled as a stationary random signal. The relevant block diagram is shown in Figure 14. Here we use $x_p(t)$ to denote the primary source strength and $y_p(t)$ to denote the total signal produced (pressure in acoustics, velocity in vibrations) at the point of application of the primary source. Similarly $x_s(t)$ denotes the secondary source strength fluctuation (volume velocity or force) and $y_s(t)$ denotes the signal produced (pressure or velocity) at the point of application of the secondary source. The "impedances" or "mobilities" relating these variables are denoted by $Z_{pp}(q^{-1})$, $Z_{ss}(q^{-1})$, $Z_{ps}(q^{-1})$ and $Z_{sp}(q^{-1})$ such that the net signals produced at the points of application of the primary and secondary sources are respectively given by

$$y_p(t) = Z_{pp}(q^{-1})\, x_p(t) + Z_{ps}(q^{-1})\, x_s(t) \tag{43}$$

$$y_s(t) = Z_{ss}(q^{-1})\, x_s(t) + Z_{sp}(q^{-1})\, x_p(t) \tag{44}$$

These equations thus quantify the influence that each source has upon the other. Note that each of the transfer functions $Z(q^{-1})$ are causal and stable

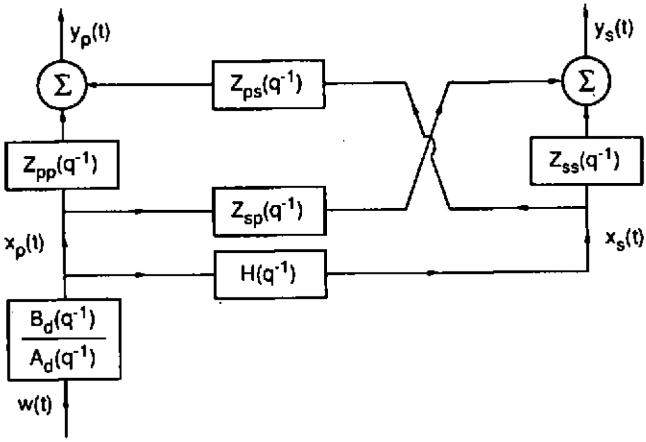

Figure 14 The feedforward control of power input. The signals $x_p(t)$ and $x_s(t)$ respectively represent the strengths of primary and secondary sources (volume velocity or force) and $y_p(t)$ and $y_s(t)$ are the net signals produced (pressure or velocity) at the positions of the sources.

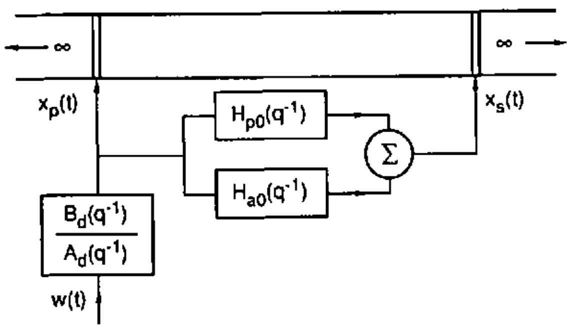

Figure 15 A doubly infinite duct containing a single primary and a single secondary plane monopole source.

Now, consider the form of the secondary source time history that is necessary to ensure that the total time averaged power input to the acoustical medium or vibrating structure is minimised. The primary source strength fluctuation $x_p(t)$ is assumed to be derived from the output of a white noise excited shaping filter $B_d(q^{-1})/A_d(q^{-1})$. We wish to determine the optimal causal filter $H_0(q^{-1})$ that operates on this signal in order to give the secondary source strength fluctuation $x_s(t)$ that ensures the minimisation of the total power input. We thus seek to minimise the cost function

$$W = E\left[x_p(t)\, y_p(t)\right] + E\left[x_s(t)\, y_s(t)\right] \tag{45}$$

when it is assumed that $x_s(t) = H(q^{-1})\, x_p(t)$. Substitution of this relationship together with equations (43) and (44) into equation (45) shows, after expansion and taking the expectation operator, that the expression for the cost function can be written as

$$W=\left[Z_{pp}(q^{-1}) + \left(Z_{ps}(q) + Z_{sp}(q^{-1})\right) H(q) + H(q)\, H(q^{-1})\, Z_{ss}(q^{-1})\right] R_{pp}(\tau) \tag{46}$$

where the autocorrelation function $R_{pp}(\tau) = E\left[x_p(t)\, x_p(t+\tau)\right]$. One can then proceed to find the causal stable filter $H(q^{-1})$ that minimises this cost by using the usual technique [42] of assuming that $H(q^{-1}) = H_0(q^{-1}) + \varepsilon H_\varepsilon(q^{-1})$ as described in Section 2 above. This shows that the condition for W to be minimised can be written as

$$H_\varepsilon(q)\left[H_0(q^{-1})\left(Z_{ss}(q) + Z_{ss}(q^{-1})\right) + \left(Z_{ps}(q) + Z_{sp}(q^{-1})\right)\right] R_{pp}(\tau) = 0 \tag{47}$$

We therefore deduce, that since $H_\varepsilon(q)$ is anti-causal, that the Wiener-Hopf equation that must be satisfied by the optimal causal filter is given by

$$\left[H_0(q^{-1})\left(Z_{ss}(q) + Z_{ss}(q^{-1})\right) + \left(Z_{ps}(q) + Z_{sp}(q^{-1})\right)\right] R_{pp}(\tau) = 0,\ \tau \geq 0 \tag{48}$$

Before discussing the solution of this equation, note that we can undertake an identical procedure in order to find the optimal value of the causal filter $H_a(q^{-1})$ that ensures the minimisation of *only* the secondary source power output. That is, we seek to minimise

$$W_s = E\left[x_s(t)\, y_s(t)\right] \tag{49}$$

where $x_s(t)$ is assumed to be given by $x_s(t) = H_a(q^{-1})\, x_p(t)$. As shown in a similar analysis presented previously [56], the minimisation of W_s ensures the maximisation of the *power absorbed* by the secondary source. By following a procedure exactly analogous to that presented above, it follows that the Wiener-Hopf equation that must be satisfied by the optimal value $H_{a0}(q^{-1})$ of $H_a(q^{-1})$ is given by

$$\left[H_{a0}(q^{-1})\left(Z_{ss}(q) + Z_{ss}(q^{-1})\right) + Z_{sp}(q^{-1})\right] R_{pp}(\tau) = 0,\ \tau \geq 0 \tag{50}$$

This suggests that we can write the optimal filter $H_0(q^{-1})$ in equation (48) as the superposition of two terms such that

$$H_0(q^{-1}) = H_{a0}(q^{-1}) + H_{p0}(q^{-1}) \tag{51}$$

where $H_{a0}(q^{-1})$ must satisfy equation (50). It then follows from substitution of this expression into equation (48) that $H_{p0}(q^{-1})$ must satisfy

$$\left[H_{p0}(q^{-1})\left(Z_{ss}(q) + Z_{ss}(q^{-1})\right) + Z_{ps}(q)\right] R_{pp}(\tau) = 0,\ \tau \geq 0 \tag{52}$$

It is sometimes possible to solve each of the Wiener-Hopf equations (50) and (52) by using the spectral factorisation technique outlined in Section 2 above. We thus take z-transforms and define the spectral factors

$$S_z(z^{-1})\, S_z(z) = \left(Z_{ss}(z^{-1}) + Z_{ss}(z)\right) S_{pp}(z^{-1}) \tag{53}$$

Note that $Z_{ss}(z^{-1}) + Z_{ss}(z)$ will produce a polynomial in z^{-1} that is symmetric. However, this spectral factorisation may not always be possible, although is certainly possible in several simple cases. The conditions under which we can take the step of writing equation (53) are not year clear. However on the assumption that equation (53) is valid, by analogy with the argument present in Section 2, we can write the solutions to equations (50) and (52) in the form

$$H_{a0}(z^{-1}) = \frac{-1}{S_z(z^{-1})}\left\{\frac{Z_{sp}(z^{-1})\, S_{pp}(z^{-1})}{S_z(z)}\right\}_+ \tag{54}$$

$$H_{p0}(z^{-1}) = \frac{-1}{S_z(z^{-1})}\left\{\frac{Z_{ps}(z)\, S_{pp}(z^{-1})}{S_z(z)}\right\}_+ \tag{55}$$

A better idea of the physical interpretation of these solutions can be gleaned by considering a simple model problem for which the spectral factorisation is trivially simple. Figure 15 shows a doubly infinite rigid walled rectangular duct containing a single primary and a single secondary "plane monopole" source [15]. The impedance $Z(q^{-1})$ relating pressure to volume velocity in such a duct can be written as $Z(q^{-1}) = Z_0\, q^{-k}$ where k is the number of samples used to represent the acoustic propagation delay from the source to a given axial position in the duct, and $Z_0 = \rho_0 c_0/2S$ where $\rho_0 c_0$ is the characteristic acoustic impedance of the medium and S the area of duct cross-section. Thus if we use I samples to represent the propagation delay between primary and secondary sources we can write $Z_{sp}(q^{-1}) = Z_0 q^{-I}$, $Z_{ps}(q^{-1}) = Z_0\, q^{-I}$ and $Z_{ss}(q^{-1}) = Z_0$. Assuming that the primary source strength $x_p(t) = \left[B_d(q^{-1})/A_d(q^{-1})\right]\, w(t)$ where $w(t)$ is white noise, then shows that the spectral factor $S_z(z^{-1}) = \sqrt{2Z_0}\, B_d(z^{-1})\, /A_d(z^{-1})$. Substitution of these results into equation (54) then shows that the term in braces is entirely causal and the solution for $H_{a0}(z^{-1})$ reduces to

$$H_{a0}(z^{-1}) = -\,(1/2)\, z^{-I} \tag{56}$$

This is an expression of the well-known fact [15] that in order to maximise its absorption, the secondary plane monopole source strength must be a delayed, inverted replica of the primary source fluctuation but with one half its amplitude.

The solution for $H_{p0}(z^{-1})$ is similarly illuminating. Noting that $Z_{ps}(z) = Z_0\, z^I$ and substituting into equation (55) shows, after some cancellation of terms, that

$$H_{p0}(z^{-1}) = -\frac{A_d(z^{-1})}{2\, B_d(z^{-1})}\left\{\frac{z^I\, B_d(z^{-1})}{A_d(z^{-1})}\right\}_+ \tag{57}$$

Comparison with equation (13) shows that this filter is a factor (–1/2) times an *optimal predictor* of the primary source strength fluctuation at k samples in the future, this corresponding to the time of propagation from the secondary to the primary source. The secondary source must therefore do its best to prevent radiation from the primary source by sending out a signal which, having propagated for a duration of I samples, will do its best to "load" the primary source with, in this case, the appropriate pressure fluctuation.

These results have been presented before [56] in the case of a pair of free field point monopole sources, but from the analysis presented above, the interpretation of $H_0(q^{-1})$ as a pair of filters $H_{a0}(q^{-1})$ and $H_{p0}(q^{-1})$, each with the physically identifiable functions of "absorption" and "prediction", would appear to have broader applicability. This conjecture awaits a more rigorous investigation.

7. FEEDBACK CONTROL OF POWER INPUT : POWER ABSORBING CONTROLLERS

Having discussed the basic mechanisms relating to the control of power input, we now re-consider the possibility of extracting energy from vibrating systems by using controllers designed only to absorb energy. The extraction of energy through active control has been considered by a number of authors [57, 58, 59, 60]. It has consistently been found, when using analyses conducted at a single frequency, that the minimisation of the power output of (or maximisation of the power absorbed by) the secondary source can lead to an increase in the energy in the system, additional energy being drawn from the primary source. However, it has also been found (from an analysis of a one-dimensional acoustical system [56]) that if the primary source has a white noise output, the maximisation of the energy absorbed by the secondary source does lead to the minimisation of the energy in the system. Here we make a further attempt to clarify the position with regard to the potential for "power absorbing controllers".

The block diagram of relevance is shown in Figure 16. The controller $G_0(q^{-1})$ is designed in order to detect the output variable (pressure in acoustics, velocity in vibrations) and feedback a signal via an actuator having transfer function $W_s(q^{-1}) = q^{-k}\, B_s(q^{-1})/A_s(q^{-1})$ that applies an input to the system (volume velocity in acoustics, force in vibrations) to ensure that the power absorbed from the system is maximised. By using the controller parameterisation given in equation (36) with $W_s\,(z^{-1})$ replaced by the product $W_s(z^{-1})\; Z_{ss}(z^{-1})$ we can proceed to design the optimal cascade compensator $H(q^{-1})$ whose presence is shown by the dashed lines in Figure 16. The equivalent filter design problem is also sketched in Figure 17.

Before embarking on the design of the feedback controller, in view of the existing literature on the subject, it would be wise to establish whether such a device would be worthwhile, even if it could be made to function satisfactorily. The problem to be addressed is the possibility of increasing the power output of the primary source when control is applied. With reference to the block diagram of Figure 16, the power output of the primary source can be written as

$$W_p = E\big[x_p(t)\; y_p(t)\big] \qquad (58)$$

Now note that the signal $y_p(t)$ arises from a superposition of two contributions; one due to the primary source given by $Z_{pp}(q^{-1})\; x_p(t)$ and one due to the secondary source that is given by $Z_{ps}(q^{-1})\; x_s(t)$.

It is the latter contribution that is of concern here. With a power absorbing controller operating, with reference to Figure 16, the secondary source input signal $x_s(t)$ will be given by

$$x_s(t) = W_s(q^{-1})\, H(q^{-1})\, Z_{sp}(q^{-1})\, x_p(t) \tag{59}$$

Thus the *change* in total power output of the primary source due to the operation of the secondary source is given by

$$\Delta W_p = E\left[x_p(t)\, Z_{ps}(q^{-1})\, x_s(t)\right] \tag{60}$$

which can therefore be written as

$$\Delta W_p = E\left[x_p(t)\, Z_{ps}(q^{-1})\, W_s(q^{-1}) H(q^{-1})\, Z_{sp}(q^{-1})\, x_p(t)\right] \tag{61}$$

Writing the autocorrelation function of the primary source strength as $R_{pp}(\tau) = E\left[x_p(t)\, x_p(t+\tau)\right]$ then shows that

$$\Delta W_p = Z_{ps}(q^{-1})\, Z_{sp}(q^{-1})\, H(q^{-1})\, W_s(q^{-1}) R_{pp}(\tau) \tag{62}$$

This expression is in the form of a polynomial in q^{-1} that operates on the autocorrelation function $R_{pp}(\tau)$.

In order to understand the implications of this expression, we write the polynomial in q^{-1} as $q^{-m}\,(1 + r_1\, q^{-1} + r_2\, q^{-2} \ldots)$ where m represents the accumulation of all the bulk delays in the transfer functions appearing in equation (62). Thus

$$\Delta W_p = q^{-m}\,(1 + r_1\, q^{-1} + r_2\, q^{-2} \ldots)\, R_{pp}(\tau)$$

$$= R_{pp}(-m) + r_1\, R_{pp}(-m-1) + r_2\, R_{pp}(-m-2) \ldots \tag{63}$$

Now we see that if $R_{pp}(\tau) = 0$ for $\tau \leq -m$, then ΔW_p will also be zero. In simple terms, if the autocorrelation function of the primary source strength time history has decayed to zero within the time taken for a unit pulse to propagate to the secondary source, via the controller and secondary actuator, and then back to the primary source, then the power output of the primary source will not be modified by the action of the secondary source. Clearly, the total power output of a white noise primary source can never be modified by the action of a secondary source placed away from the secondary source (although re-distribution of energy in the spectrum may occur), but if the primary source emits a pure tone, the possibility of increasing the primary source power output always exists, since the autocorrelation $R_{pp}(\tau)$ extends indefinitely.

The design of a power absorbing controller can be tackled using the same type of analysis as that presented in the last section, but here we work with the optimal cascade compensator as illustrated in Figure 16,

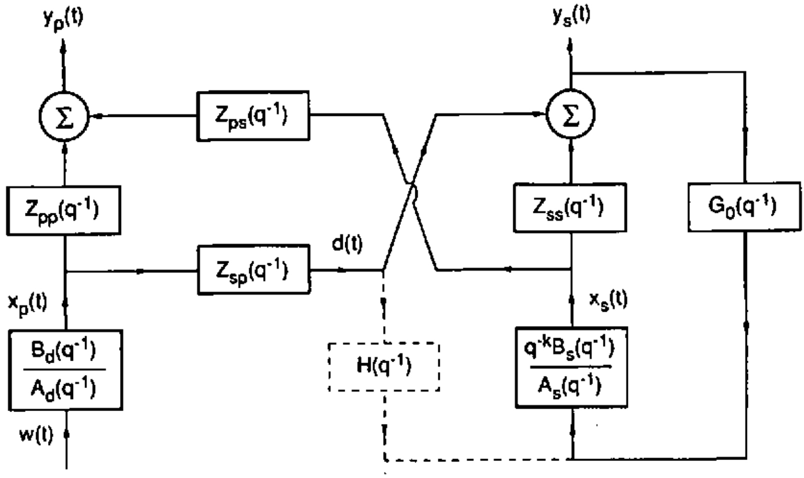

Figure 16 The feedback control of power input. The output of the controller (pressure or velocity) is fed back via a compensator $G_0(q^{-1})$ and via an actuator in order to produce an input to the system (volume velocity or force). The solution for $G_0(q^{-1})$ requires calculation of the optimal cascade compensator $H(q^{-1})$.

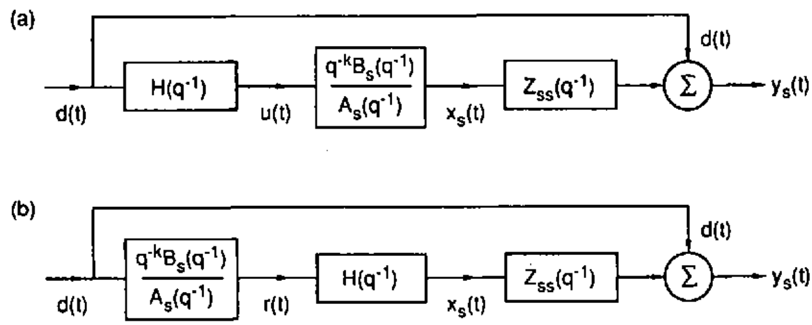

Figure 17 The filter design problem of Figure 16 shown in two equivalent forms.

include the actuator dynamics in the analysis and also penalise the mean square effort associated with the controller. We therefore seek to minimise

$$W_s = E\left[x_s(t)\, y_s(t) + \beta\, u^2(t)\right] \tag{64}$$

where the signals in this expression are defined in Figure 17. Noting that $y_s(t) = d(t) + Z_{ss}(q^{-1})\, H(q^{-1})\, r(t)$ and that $x_s(t) = H(q^{-1})\, r(t)$ shows that this cost function can be written as

$$W_s = H(q)\, R_{rd}(\tau) + Z_{ss}(q)\, H(q)\, H(q^{-1})\, R_{rr}(\tau) + \beta H(q)\, H(q^{-1})\, R_{dd}(\tau) \tag{65}$$

As usual we find the minimum of this cost function by assuming a small realisable departure from the optimal filter $H_0(q^{-1})$. The Wiener-Hopf equation that results is given by

$$R_{rd}(\tau) + H_0(q^{-1})\left[\left(Z_{ss}(q^{-1}) + Z_{ss}(q)\right) R_{rr}(\tau) + 2\beta\, R_{dd}(\tau)\right] = 0,\ \tau \geq 0 \tag{66}$$

The solution for $H_0(q^{-1})$ that satisfies this equation can in principle be found by spectral factorisation techniques. The solution is given by

$$H_0(z^{-1}) = \frac{-1}{S_{z\beta}(z^{-1})}\left\{\frac{S_{rd}(z^{-1})}{S_{z\beta}(z)}\right\}_+ \tag{67}$$

where we define the spectral factors $S_{z\beta}(z^{-1})$ and $S_{z\beta}(z)$ by

$$S_{z\beta}(z^{-1})\, S_{z\beta}(z) = \left(Z_{ss}(z^{-1}) + Z_{ss}(z)\right) S_{rr}(z^{-1}) + 2\beta\, S_{dd}(z^{-1}) \tag{68}$$

Similar comments apply to this solution to those given above regarding the validity of the spectral factorisation; it may not always be possible to undertake this for a given polynomial $Z_{ss}(z^{-1})$. It is clear however that effective acoustic absorbers can be designed by using adaptive signal processing techniques [e.g. 61, 62]. For example, the recent work of Ise and Tachibana [62] uses a similar controller architecture to that illustrated by the dashed lines in Figure 16; a direct measure of the primary disturbance impinging on a loudspeaker is first obtained by "cancelling the feedback" from the loudspeaker to a detection microphone. Having recovered the disturbance signal, this is passed via a control filter whose characteristics are designed to maximise the acoustic intensity flowing into the loudspeaker, several algorithms being available to accomplish this [e.g. 62, 63, 64]. The results presented are impressive, with an absorption coefficient close to unity being achieved over a broad frequency range when the device is used to terminate a rigid tube supporting only plane wave propagation. The effect of the device on the power output of the primary source is not reported, but the argument presented above suggests that this should not present a difficulty when dealing with broadband disturbance signals that have an autocorrelation function of short duration. An interesting application of such devices is in the "flattening" of the low frequency response of rooms as discussed, for example, by Darlington [65]. Other work by Hirami [66] has used power absorption as a cost function in an adaptive controller design which appears to function extremely well in reducing the vibrations of a plate at the point at which control is applied, although no details are given of the full energy balance of the system. MacMartin and Hall [67] also present results of experiments using a control system designed to absorb power from a vibrating beam, and assume in the controller design that the beam is infinite. The design technique is undertaken using a very similar approach to that suggested here, although in continuous time. The results show considerable promise.

In summary, control strategies based on power absorption may yet have a future, but the danger in increasing the energy in a system remains, especially for narrow band primary sources. The controller design procedure outlined here should certainly be regarded as speculative and there are still a number of important issues to be addressed before the potential for these devices becomes completely clear. As we have shown above, feedback systems work well when the disturbance is predictable and yet this is precisely the condition which may lead to increases in the energy in the system. More work in this area is still required; the future of power absorbing controllers is hard to predict!

8. CONCLUSIONS

We have reviewed recent research into active techniques for controlling sound and made an effort to predict their impact on the future of noise control. Both feedforward and feedback techniques have been discussed within a control engineering framework and an attempt has been made to relate this point of view to the adaptive signal processing methods currently in use. The application of the theory helps to understand the performance limits associated with active control systems, but the filter design procedures presented here are highly unlikely to supplant current methods based on the use of adaptive FIR filters for the real-time solution of the Wiener-Hopf equation.

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