A SIMPLE METHOD FOR MODELLING THE INTERACTION OF A BUBBLE WITH A RIGID STRUCTURE

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1 Introduction

Over the past few years there has been considerable interest in the problem of determining the motion of the bubble which results from an underwater expolsion, especially when there is a submerged structure close to the point of the explosion. Typically the initial radius of such bubbles is relatively small and the high pressure of the gas inside the bubble causes it to expand. Eventually the hydrostatic pressure of the surrounding water will reverse the growth of the bubble and it will collapse until the internal pressure causes it to re-bound and start growing again. However it has been observed that as the bubble collapses it forms a re-entrant jet, the direction of which is dependent on the geometry of the fluid region in which the bubble exists. Often the direction of this jet is of interest since if it is directed towards a near-by structure immersed in the water it is possible that the impact of the jet is a mechanism for causing damage to the structure.

A number of mathematical models have been proposed for solving the problem of determining the bubble's motion. One such model is based on the boundary integral method and an axisymmetric formulation has been successfully used by Blake et al [3, 4] to study the motion of bubbles close to infinite rigid boundaries or free surfaces. Best [1] proposed a modification to the axisymmetric boundary integral method to model the motion of the toroidal bubble which results from the jet penetrating the opposite side of the bubble. Other models, such as those of Harris [6, 7], have employed a fully three-dimensional boundary integral method to predict the motion of a bubble close to a rigid structure submerged in the water. Clearly models such as these can be used to determine the complete motion of the bubble, structure and surrounding fluid and hence the direction of the bubble's jet can be found directly from the results of the simulation. However, the numerical proceedures employed in the solution of this type of problem are usually computationally expensive.

Alternative models, based on certain simplifying assumptions, can be used to quickly determine an estimate of the direction of the bubble's jet. The usual assumptions are that the bubble remains spherical and that its centroid remains stationary. Under these assumptions it is possible to compute the Kelvin impulse of the bubble, the direction of which will be the same as the direction of the bubble jet [1, 5]. This paper will use the simple spherical bubble model to determine the direction of the bubble jet for a bubble close to a submerged rigid structure, and show how the analysis can be adpated to include the effects due to the explosion taking place in shallow water.

2 Mathematical Model

Assume that the fluid is irrotational, incompressible and invicid. Then at any point $\mathbf{p}$ in the fluid the velocity $\mathbf{u}$ can be expressed as the gradient of some scalar potential function ϕ which, in turn, satisfies Laplace's equation. The total potential ϕ is the sum of ϕ_b the potential due to the motion of the bubble, ϕ_s the potential due to the presence of the structure and ϕ_g contributuions to the potential from the geometry of the surrounding fluid region.

By assuming that the bubble remains spherical throughout its life-time and that its centroid is stationary it is possible to represent the velocity potential due to the bubble as a points source located at $\mathbf{p}_b$, the bubble's centroid, in the form

$$\phi_b(\mathbf{p}) = \frac{m(t)}{|\mathbf{p} - \mathbf{p}_b|} \tag{1}$$

where $m(t)$ is the time-dependent source strength. On the surface of the bubble the rate of change of the bubble's radius must equal the normal velocity of the water and so

$$m(t) = -R^2\dot{R} \tag{2}$$

where R is the bubble radius and an over-dot denotes differentiation with respect to time.

Clearly when considering a bubble in an infinite or unbounded fluid

$$\phi_g(\mathbf{q}) = 0 \tag{3}$$

For a fluid region bounded by an infinite rigid plane or an infinite free surface ϕ_g can be found using the method of images. Let

$$\begin{aligned} r_1(\mathbf{p},\mathbf{q},n) &= \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - 2nh - z_q)^2} \\ r_2(\mathbf{p},\mathbf{q},n) &= \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - 2nh + z_q)^2} \end{aligned} \tag{4}$$

where (x_p, y_p, z_p) and (x_q, y_q, z_q) are the cartesian coordinates of the points $\mathbf{p}$ and $\mathbf{q}$ respectively. If the fluid is bounded by either a free surface at $z = 0$ or a rigid boundary at $z = -h$ then

$$\begin{aligned} \phi_g(\mathbf{q}) &= -\frac{m(t)}{r_2(\mathbf{p}_b,\mathbf{q},0)} \quad \text{Free surface} \\ \phi_g(\mathbf{q}) &= \frac{m(t)}{r_2(\mathbf{p}_b,\mathbf{q},1)} \quad \text{Rigid boundary} \end{aligned} \tag{5}$$

respectively, whilst if there is both a free surface at $z = 0$ and rigid boundary at $z = -h$

$$\phi_g(\mathbf{q}) = m(t)\left(\sum_{n=-\infty, n\neq 0}^{\infty}\left[\frac{(-1)^n}{r_1(\mathbf{p}_b,\mathbf{q},n)} - \frac{(-1)^n}{r_2(\mathbf{p}_b,\mathbf{q},n)}\right] - \frac{1}{r_2(\mathbf{p}_b,\mathbf{q},0)}\right) \tag{6}$$

The velocity potential due to the structure can be written in the form [8]

$$\phi_s(\mathbf{p}) = \int_{S_s} G(\mathbf{p},\mathbf{q})\sigma(\mathbf{q})\, dS_{\mathbf{q}} \tag{7}$$

where S_s denotes the surface of the structure, σ is a surface source density function and $G(\mathbf{p},\mathbf{q})$ is the appropriate Green's function for the fluid region under consideration. For a bubble in an infinite fluid

$$G(\mathbf{p},\mathbf{q}) = \frac{1}{4\pi r_1(\mathbf{p},\mathbf{q},0)} \tag{8}$$

whilst for either a free surface at $z = 0$ or a rigid boundary at $z = -h$ the Green's functions are

$$\begin{aligned} G(\mathbf{p},\mathbf{q}) &= \frac{1}{4\pi r_1(\mathbf{p},\mathbf{q},0)} - \frac{1}{4\pi r_2(\mathbf{p},\mathbf{q},0)} \quad \text{Free surface} \\ G(\mathbf{p},\mathbf{q}) &= \frac{1}{4\pi r_1(\mathbf{p},\mathbf{q},0)} + \frac{1}{4\pi r_2(\mathbf{p},\mathbf{q},1)} \quad \text{Rigid Boundary} \end{aligned} \tag{9}$$

respectively. For problems with both a free surface at $z = 0$ and a rigid boundary at $z = -h$ the appropriate Green's function is

$$G(\mathbf{p},\mathbf{q}) = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\left[\frac{(-1)^n}{r_1(\mathbf{p},\mathbf{q},n)} - \frac{(-1)^n}{r_2(\mathbf{p},\mathbf{q},n)}\right] \tag{10}$$

For $\mathbf{p} \in S_s$ it is possible to differentiate (7) along the normal to obtain [8]

$$\frac{\partial \phi_s}{\partial \mathbf{n_p}} = -\frac{1}{2}\sigma(\mathbf{p}) + \int_{S_s}\frac{\partial G}{\partial \mathbf{n_p}}(\mathbf{p},\mathbf{q})\sigma(\mathbf{q})\,dS_{\mathbf{q}} \tag{11}$$

Equations (7) and (11) can be written in operator notation as

$$\phi_s(\mathbf{p}) = A[\mathbf{p}]\sigma \quad \frac{\partial \phi_s}{\partial \mathbf{n}} = \left(-\frac{1}{2}I + B[\mathbf{p}]\right)\sigma \tag{12}$$

where

$$A[\mathbf{p}]\sigma = \int_{S_s} G(\mathbf{p},\mathbf{q})\sigma(\mathbf{q})\,dS_{\mathbf{q}} \quad B[\mathbf{p}]\sigma = \int_{S_s}\frac{\partial G}{\partial \mathbf{n_p}}(\mathbf{p},\mathbf{q})\sigma(\mathbf{q})\,dS_{\mathbf{q}} \tag{13}$$

respectively. Thus ϕ_s can be expressed in terms of its normal derivative as

$$\phi_s(\mathbf{p}) = A[\mathbf{p}]\left(-\frac{1}{2}I + B[\mathbf{p}]\right)^{-1}\frac{\partial \phi_s}{\partial \mathbf{n}} \tag{14}$$

Since the structure is rigid, it follows that on its surface

$$\frac{\partial \phi}{\partial \mathbf{n}} = 0 \tag{15}$$

and hence

$$\frac{\partial o_s}{\partial \mathbf{n}} = -\frac{\partial \phi_b}{\partial \mathbf{n}} - \frac{\partial \phi_g}{\partial \mathbf{n}} \tag{16}$$

For ease of notation let

$$\frac{\partial \phi_b}{\partial \mathbf{n}} + \frac{\partial \phi_g}{\partial \mathbf{n}} = m(t)d_s(\mathbf{p}_b,\mathbf{q}) \tag{17}$$

where the exact definition of the function $d_s(\mathbf{p},\mathbf{q})$ can be easily deduced from the definitions of the potentials ϕ_b and ϕ_g. Substituting (16) and (17) into (14) gives

$$\phi_s(\mathbf{p}) = -m(t)A[\mathbf{p}]\left(-\frac{1}{2}I + B[\mathbf{p}_b]\right)^{-1} d_s(\mathbf{p}_b,\mathbf{q}) \tag{18}$$

In order to obtain a differential equation for the bubble radius assume that both ϕ_s and ϕ_g on the surface of the bubble can be approximated by their repsective values at the bubble centroid. This means that the contributions to the potential on the surface of the bubble can be written in the form

$$\phi_s = m(t)\mu_s \quad \phi_g = m(t)\mu_g \tag{19}$$

where μ_g is obtained by putting $\mathbf{p} = \mathbf{p}_b$ in the appropriate definition of ϕ_g and

$$\mu_s = -A[\mathbf{p}_b]\left(-\frac{1}{2}I + B[\mathbf{p}_b]\right)^{-1} d_s(\mathbf{p}_b, \mathbf{q}) \tag{20}$$

The kinetic energy of the bubble is given by [9]

$$K = \frac{\rho}{2}\int_{S_b} \phi \frac{\partial \phi}{\partial \mathbf{n}}\, dS \tag{21}$$

where S_b denotes the surface of the bubble and ρ is the density of the fluid. Equating the rate of change of the kinetic energy to the rate of work being done leads to the second order ordinary differential equation

$$\left(R + (\mu_s + \mu_g)R^2\right)\ddot{R} + \left(\frac{3}{2} + 2(\mu_s + \mu_g)R\right)\dot{R}^2 = \frac{P_b}{\rho} - \frac{P_\infty}{\rho} + gz_b \tag{22}$$

for the bubble radius R, where g and P_∞ are the acceleration due to gravity and far-field pressure in the $z = 0$ plane respectively. P_b denotes the pressure inside the bubble, and for an explosion bubble it is assumed that the gas inside the bubble obeys a simple gas law of the form

$$P_b = P_0\left(\frac{R(0)}{R(t)}\right)^{3\gamma} \tag{23}$$

where P_0 is the initial pressure inside the bubble and γ is the polytropic index of the gas inside the bubble.

The Kelvin impulse $\mathbf{I}$ is defined as [9]

$$\mathbf{I}(t) = \rho\int_{S_b} \phi\mathbf{n}\, dS \tag{24}$$

It can be shown that [9]

$$\mathbf{I}(t) = \mathbf{I}(0) + \int_0^t \mathbf{F}(\tau)\, d\tau \tag{25}$$

where [2]

$$\mathbf{F}(t) = 4\pi m(t)\nabla\left(\phi_s(\mathbf{p}_b) + \phi_g(\mathbf{p}_b)\right) + \frac{4}{3}\pi R^3 \rho g \qquad (26)$$

All the above equations can be non-dimensionalised using methods described in, for eaxmple, Blake et al [3].

A typical calculation to determine the Kelvin impulse of a bubble proceeds as follows. Initially the boundary element method is used to obtain approximations to the integral operators A and B. It is noted that for a given structure the approximation to B has to computed only once, whilst the approximation to A has to be computed only once for each bubble/structure combination. The values of μ_s and μ_g can now be calculated and the differential equation (22) can now be integrated through time using a fourth-order Runge-Kutta scheme. The Kelvin impulse is computed as the calculation proceeds using the trapezium rule to approximate (25).

3 Numerical Results and Conclusions

This section shows the results of using the Kelvin impulse to determine the direction of the bubble jet for some typical submerged structures. Figure 1 shows the direction of the Kelvin impulse for a single bubble at different locations relative to rigid cylinder in an unbounded fluid. The length of the cylinder is ten times the maximum radius of the bubble and the radius of the cylinder is the same as the maximum radius of the bubble. These results show that there is a region close to the cylinder where the bubble would be attracted towards the cylinder. Further, it can be seen that for bubbles far enough away from the cylinder the jet is directed in the same direction as the bouyancy forces. Figure 2 shows the results for a bubble at different locations relative to a rigid sphere, of radius four times the maximum radius of the bubble, in shallow water. These results clearly show that there is a region close to the sphere where the bubble jet is attracted towards the sphere. This example also illustrates that if the bubble is close to one of the infinite surfaces then jet is directed away from the free surface and towards the rigid boundary.

These results show that this is a practical method for computing the direction of the bubble jet for bubbles close to rigid structures. In terms of computer CPU time this scheme is much quicker than a full boundary integral formulation. For example, the calculations illustrated in Figure 2 for a bubble at 326 different locations relative to a rigid sphere in shallow water requires about 25 minutes on a 100 MHz 486DX4 computer, whilst a single run of the full boundary element method for a bubble near an infinite rigid boundary with no sphere requires over 6 minutes on a 300 MHz Alpha processor [10].

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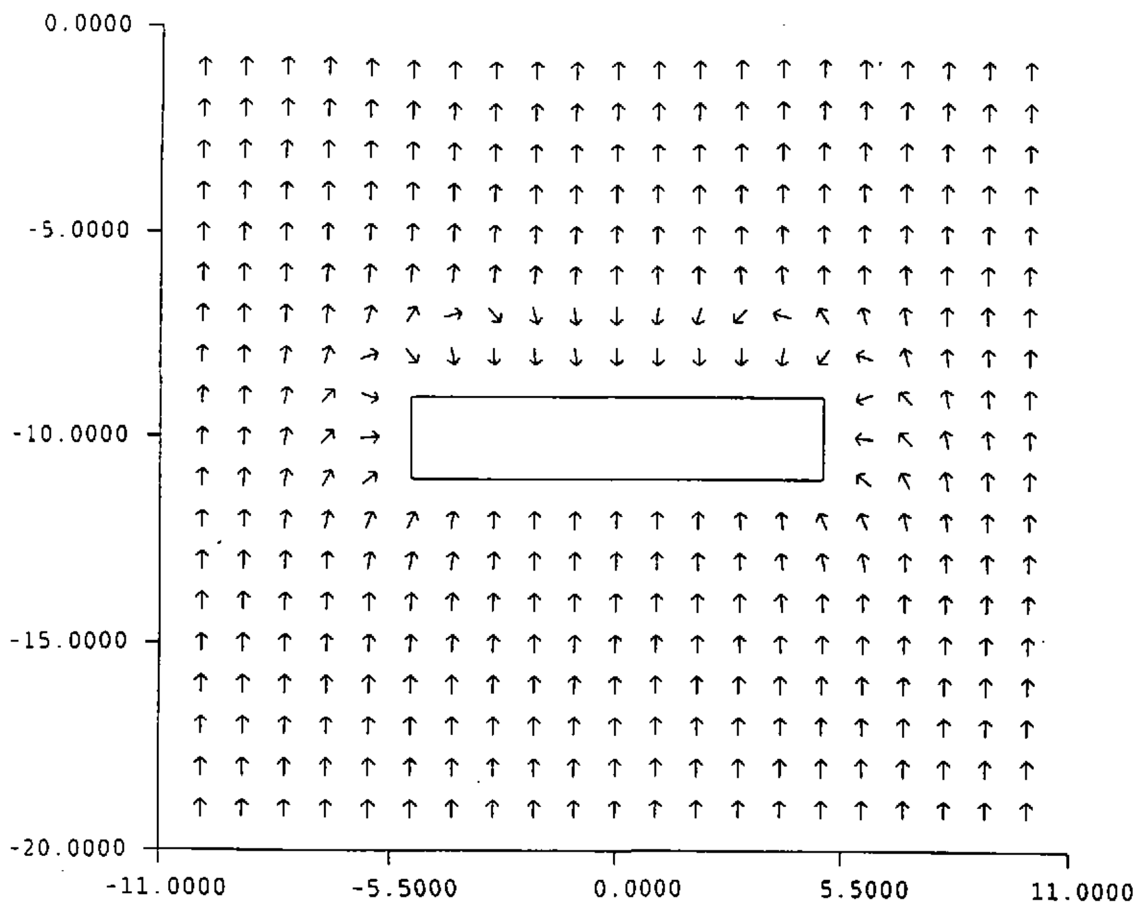

Figure 1: The direction of the Kelvin impulse for a bubble at different locations relative to a rigid cylinder.

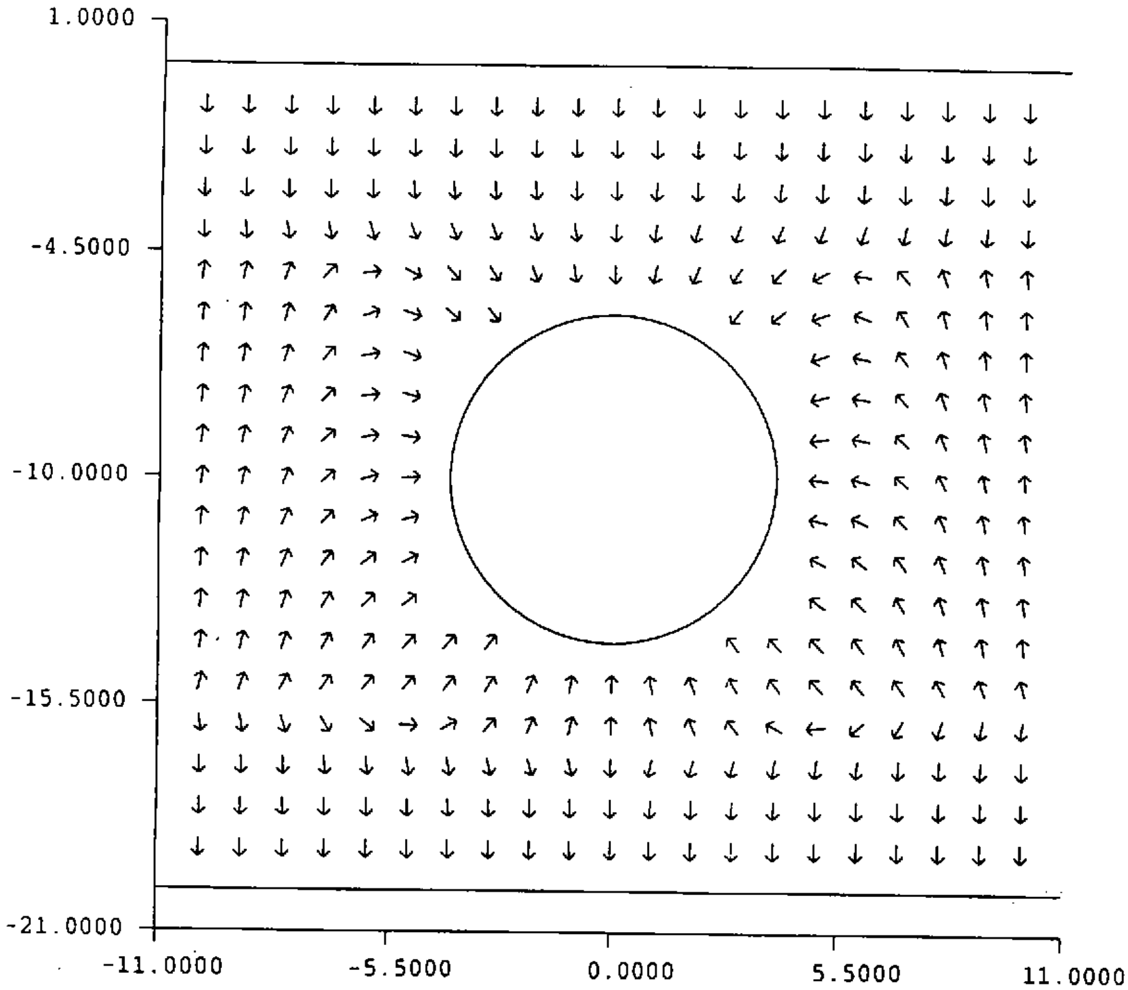

Figure 2: The direction of the Kelvin impulse for a bubble at different locations relative to a rigid sphere in shallow water.