ACOUSTIC TOMOGRAPHY BY ORTHOGONALITY SAMPLING

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INTRODUCTION

Inverse scattering problems are of importance for many applications, for example for medical imaging, nondestructive testing, remote exploration, geophysical prospecting or radar. Usually, a wave is sent into a region of space which is to be investigated. Then, due to the structure of the unknown area or the existence of obstacles a scattered wave is generated which is measured far away from the objects under consideration. The task of inverse scattering theory is to reconstruct properties of the unknown scatterers from these remote measurements. Inverse scattering has developed into an important part of applied mathematics with a growing number of interesting and promising new mathematical techniques.

Inverse scattering theory has a long history with classical contributions for example by Lax and Phillips [6]. An introduction into the theory of acoustic and electromagnetic inverse scattering can be found in the work of Colton and Kress [1]. More recently, new classes of methods have been introduced with sampling and probe methods, see [2] - [5], Their main idea is to formulate an indicator function μ defined either in the space R^m or on a set of test domains. This function characterizes the unknown scatterers, their physical properties or their shape.

Here, our goal is to describe a recent method for the reconstruction of the location and shape of an unknown number of scattering objects from measurements of the far field pattern of the scattered acoustic fields. We will introduce the main idea of *orthogonality sampling* and study its properties. Then, we relate it to other schemes like the linear sampling method, the MUSIC algorithm or probe methods. The relation of orthogonality sampling to the point source method is investigated. Finally, we provide numerical examples of reconstructions in a simple setting.

SCATTERING OF WAVES AND ORTHOGONALITY SAMPLING

The goal of this section is to define the forward problem for scattering of a time-harmonic incident wave or of a time-dependent pulse by some obstacle. Here, for simplicity we will restrict our attention to the case of the Dirichlet boundary condition.

We consider the scattering of some time-harmonic acoustic wave $u^i(x) = e^{ikx \bullet d}$ by an impenetrable scatterer D with Dirichlet (sound-soft) boundary condition

$$u\big|_{\partial D} = 0 \qquad (1)$$

in two dimensions. The scattered field is denoted by u^s and the total field

$$u = u^i + u^s \qquad (2)$$

is a solution to the *Helmholtz equation*

$$\Delta u + k^2 u = 0 \qquad (3)$$

with wave number k in the exterior D^e of D. We also assume that u^s satisfies the Sommerfeld radiation condition

$$\frac{\partial u^s}{\partial r} - iku^s \to 0, \; |x| = r \to \infty \qquad (4)$$

uniformly in all directions. It is well known [1], [2] that this scattering problem has a unique solution which depends continuously on the incident field. Also, the scattered field u^s has the asymptotic behaviour of the form

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u^\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\} \tag{5}$$

for $|x| \to \infty$ uniformly in all directions. The field u^∞ is known as far field pattern and serves as our data for the inverse problem. In principle, it can be measured if the phase of the sound field is fully recorded. Here, we think of either measurements carried out by a synchronized array of microphones or by a sweep technique which controls the time behavior of an incident pulse and by which we can measure the full phase information of the scattered field or its far field pattern, respectively, by Fourier transform of the measurements in the time domain.
Simulations of the far field pattern have been carried out by integral equation methods [1] via the Brackhage-Werner approach by a combined single- and double-layer potential on the boundary ∂D.

The basic idea of orthogonality sampling is to study the function

$$\mu(z) = \left|\int_\Lambda e^{ik\hat{x}\bullet z}\, u^\infty(\hat{x})\, ds(\hat{x})\right| \tag{6}$$

It can be seen as orthogonality product between the full far field pattern and the function

$$f_z(\hat{x}) := e^{ik\hat{x}\bullet z}, \quad \hat{x} \in \Lambda \tag{7}$$

or as superposition of plane waves where the wheight function is the data u^∞ on some measurement surface Λ. In this sense (6) is just a part of the inverse Fourier transform of the far field pattern, as used by the Born approximation inversion for scattering by inhomogeneous media, compare [1]. However, in contrast to the Fourier transform here we extend this approach in a different way by the following multi-frequency and multi-direction indicator functions.

For data $u^\infty(\hat{x}, k)$ given for an interval $[k_1, k_2]$ of wave numbers we define

$$\mu_{MF}(z) = \int_{k_1}^{k_2} \left|\int_\Lambda e^{ik\hat{x}\bullet z}\, u^\infty(\hat{x}, k)\, ds(\hat{x})\right| dk \tag{8}$$

i.e. we calculate the integral over the modulus of the term (6) with respect to the wave numbers. Similarly, if the far field pattern $u^\infty(\hat{x}, d, k)$ is given for a set Γ of directions d of incidence, we define

$$\mu_{MD}(z) = \int_\Gamma \left|\int_\Lambda e^{ik\hat{x}\bullet z}\, u^\infty(\hat{x}, \theta, k)\, ds(\hat{x})\right| ds(\theta) \tag{9}$$

Finally, for multi-frequency multi-wave data we define the orthogonality functional by

$$\mu_{MDMF}(z) = \int_\Gamma \int_{k_1}^{k_2} \left|\int_\Lambda e^{ik\hat{x}\bullet z}\, u^\infty(\hat{x}, \theta, k)\, ds(\hat{x})\right| dk\, ds(\theta) \tag{10}$$

COMPARISON WITH OTHER METHODS

There are many sampling and probe methods which have been developed since 1996, compare [1]-[5]. Here, first we would like to describe the fundamental differences in the concept of the approach of the linear sampling method/ the MUSIC algorithm versus orthogonality sampling. Then, we discuss the relation of orthogonality sampling and the point source method [2].

Both the *linear sampling method* and the *MUSIC algorithm* are based on the far field operator defined by

$$(Fg)(\hat{x}) = \int_{\Gamma} u^{\infty}(\hat{x},\theta) g(\theta)\, ds(\theta), \quad \hat{x} \in \Lambda \qquad (11)$$

Which describes the far field pattern for a superposition of incident plane waves with strength $g(\theta), \theta \in \Gamma$. The key idea of the linear sampling method (and the same applies to the MUSIC algorithm) is to test if the function f_z defined in (7) is in the range of F. To this end the equation

$$Fg = f_z \qquad (12)$$

is solved by a regularization scheme. The method shows that the norm of g, depending on the point z, is large if z is not in the scatterer D and it is small if z is in D. In mathematical terms we investigate the range of the operator F.

We observe that for orthogonality sampling we do not study F at all. The relation between f_z and the range of the operator F is not exploited. However, we directly study the scalar product of f_z with the data u^{∞}. Though both methods use f_z the way in which the data is employed is based on a different concept.

Next, we need to compare orthogonality sampling with the *point source method* [2]. The point source method introduced by the author in 1996 has some resemblance to orthogonality sampling. It constructs a kernel function g_z such that the scattered field u^s can be reconstructed from the far field pattern u^{∞} by

$$u^s(z) \approx \int_{\Lambda} g_z(\hat{x})\, u^{\infty}(\hat{x})\, ds(\hat{x}) \qquad (13)$$

where g_z solves some ill-posed integral equation of the first kind. The point source method reconstructs the full scattered field and then looks for the unknown scatterer via the boundary condition for the total field. Thus, the similarity to orthogonality sampling is that the data are multiplied by some kernel and integrated over the measurement set Γ. But the role and construction of the kernel is far beyond the simple exponential exploited by the orthogonality sampling method. The point source method works well for single-frequency one-wave data, where orthogonality sampling basically needs multi-frequency or high-frequency data to achieve good reconstructions, though the case of single-frequency multi-direction data is feasible. We can interpret the point source method and the orthogonality sampling scheme as two different schemes based on the joint idea of constructing an appropriate backprojection kernel g_z to exploit the measurements given for reconstruction.

NUMERICAL RESULTS

The task of this part is to describe the numerical realization of the methods. We first discuss the simulation of the far field pattern. Then, we will provide a numerical study of the above functionals which visualize the scatterers under consideration.

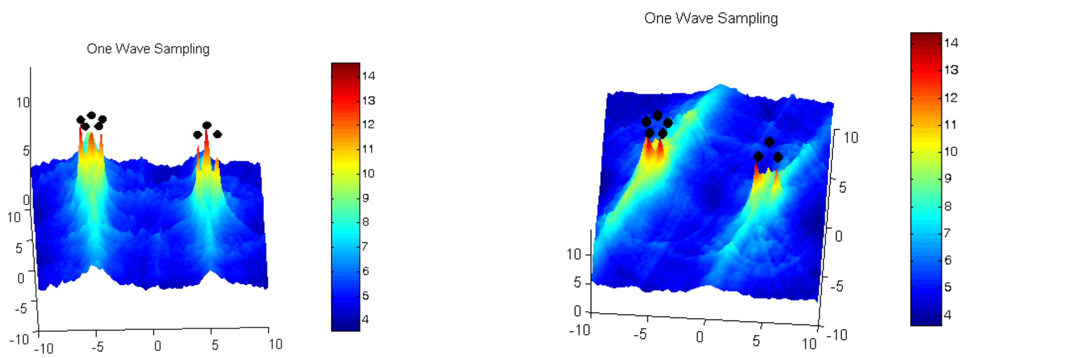
For the calculation of the far field patterns as well as for simulations of the reduced scattered field (\ref{us red2}) we have used the Nystr\"om method as described in Colton and Kress [1]. For multiple scatterers we have not implemented the split of the weak singularity in the integrals, but ignored the singularity. This leads to very flexible code, for which scattering by various objects can be easily implemented, though having low order convergence. An example for a simulated total field for scattering by eight separate scatterers which basically consist out of two groups is shown in Figure 2.

One-Wave Multi-Frequency

We now show results for *one-wave multi-frequency* reconstructions via orthogonoality sampling. We will show results for some generic settings:

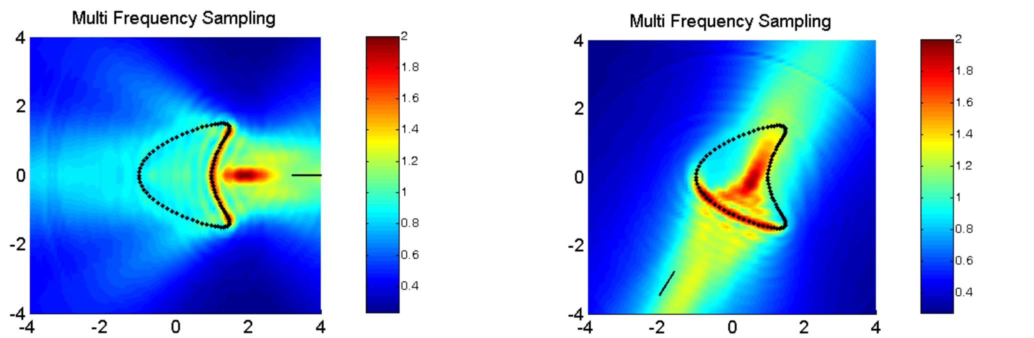
- For several small scatterers
- For a kite shaped scatterer

The location of an unknown number of scatterers can be clearly seen in the Figure 1. It is an expected phenomenon that scatterers which are in the shadow of other scatterers cannot be seen when data for one direction of incidence only is used. In the two images of Figure 1 we can find only the three out of 5 objects which are enlighted by the incoming plane wave, depending on the direction of incidence. The reconstruction of the shape of a kite is shown in figure Figure 2. The method finds the enlighted side of the object very well. It does not find the shadow regions.



(a) (b)

Figure 1: Orthogonality Samping indicator function for one-wave multi-frequency scattering by a scatterer with 8 small components. The location of the components can be recognized by peaks of the indicator function. We show two different directions of incidence of a puls in (a) and (b).

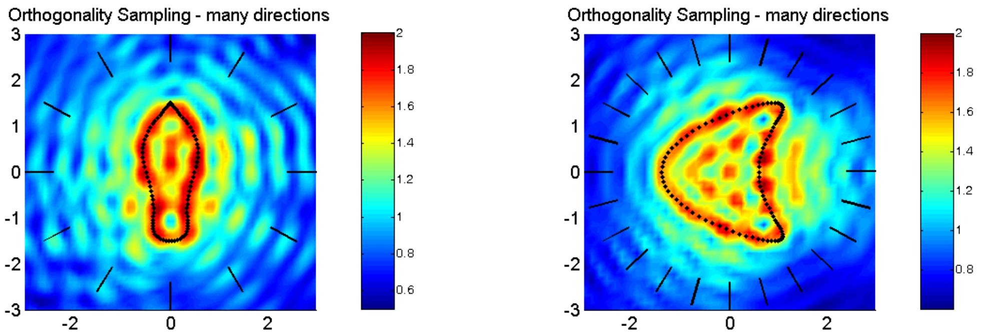


(a) (b)

Figure 2: Reconstruction of the kite shaped scatterer by orthogonality sampling from a pulse coming from the right (a) and from bottom-left (b). The method reconstructs mainly the shape of the side of the scatterer which is enlighted by the pulse.

Multiple directions of Incidence

Next, we would like to consider the fixed-frequency case where we use multiple directions of incidence. This case compares to the setting of the linear sampling method, the factorization method and the singular sources and probe methods. We will prove the the orthogonality sampling can generate images comparable to those of the linear sampling or factorization method, but with a well-posed sampling functional.



(a) (b)

Figure 3: Orthogonality sampling for fixed frequency and several directions of incidence as described in (9). We observe that the location and shape of the scatterers can be well reconstructed.

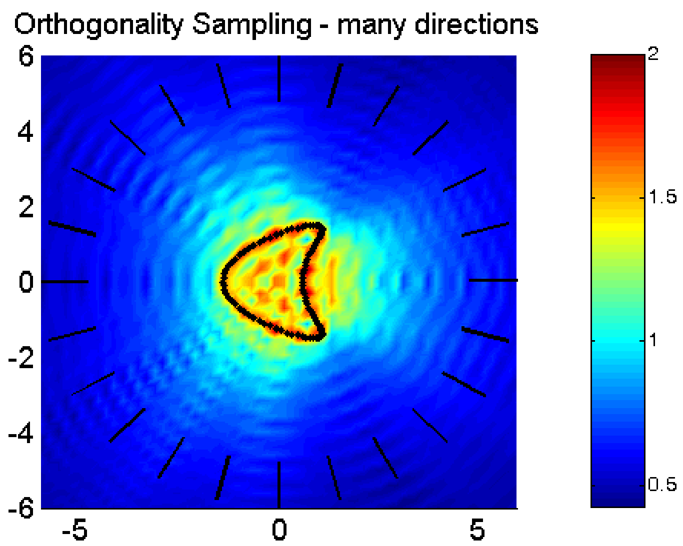
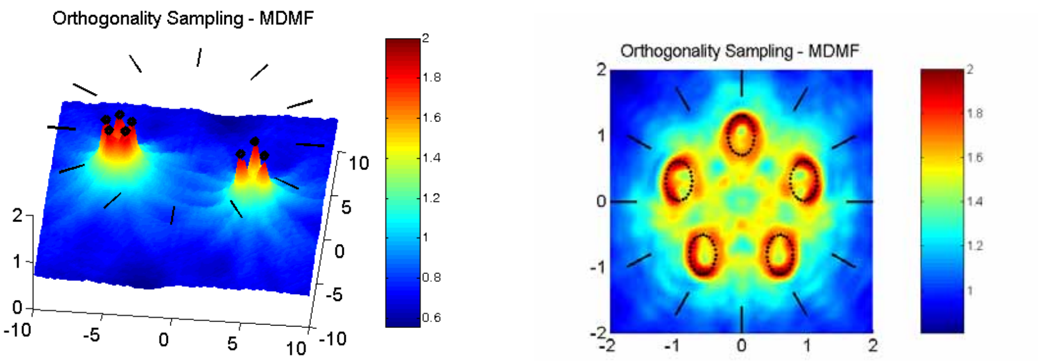


Figure 4: Orthogonality sampling for fixed frequency and several directions of incidence as described in (9), now with more directions of incidence. The reconstruction is getting better as compared to the the results shown in Figure 3.

Multi-Direction Multi-Frequency

Finally, we show results for multi-direction multi-frequency (MDMF). We have generated far field data where several objects have been present. Then, we calculated the functional in a neighbourhood of each of the objects in a higher resolution, which is shown in figures \ref{MDMF}, \ref{MDMF2}. This proves that detailed reconstruction of objects can be achieved even when other objects are present in space. In general, we have added 2-3\% stochastical error to the data before carrying out the reconstructions.



(a) (b)

Figure 5: Orthogonality sampling with the MDMF functional, i.e. multi-direction multi-frequency. Here, detailed reconstructions of the location and shape of multiple obstacles are possible.

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