inter•noise 96
25th Anniversary Congress – Liverpool

INCE: 24

A NEW FAST PARABOLIC EQUATION (PE) BASED ON A LAGRANGE PADE (LP) EXPANSION OF IMPLICIT FUNCTIONS FOR 2D SOUND PROPAGATION

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1. INTRODUCTION

In the parabolic equation (PE) method for wave problems in near-homogeneous media, the dependence on one variable is changed from second to quasi-first order. Thus for single frequency 2D scalar wave functions Ψ(x, z) the Helmholtz equation

$$\left[D_x^2 + D_z^2 + k(z)^2 \right] \Psi = 0 \; ; \quad D_x = \partial/\partial x, \; etc \qquad (1)$$

is transformed into symbolically first order form [1]

$$\left[D_x - ik_0 \sqrt{(1+q)} \right] \Psi \;\; = \; ; \quad q = \left[k(z)^2 - k_0^2 + D_z^2 \right] / k_0^2. \qquad (2)$$

Here k_0 is a fixed wave number within, or close to, the range of $k(z)$, which may depend on height, but not, in general, on distance. As all terms in the straight brackets in (1) and (2) commute, they may be manipulated like scalar variables. The "narrow angle" (NA) and the "wide angle" (WA) PE's are obtained from the linear approximation of the root in (2) and from its (1, 1) Padé approximant respectively.

The authors prefer an alternative approach where Ψ is factorized into an exponential "carrier" Ψ_0 and a slowly varying "modulator" ϕ, leading to [2]

$$\Psi = \exp(ik_0 x)\, \phi \; ; \quad \left[D_x^2 + 2ik_0 D_x + k_0^2 \, q \right] \phi = 0, \qquad (3)$$

where q and k_0 have the same meanings as in (2). This approach generates the NA and WA PE's by an iterative process; but it can be used even when the terms in brackets depend explicitly on x.

A drawback of either approach is that any error in D_x propagates into the solution of Ψ at larger distances. Taylor's theorem can be written symbolically

$$\Psi(x + h, z) = \exp(h\, D_x)\, \Psi(x, z); \tag{4}$$

hence this error cannot be reduced by taking smaller step lengths. For this reason Collins [3] uses higher order Padé approximants of the square root in (2). However, this solves only part of the problem as the main interest lies not in D_x itself, but in its exponential according to (4). As the operator q involves D_z^2 and thus, numerically, a tridiagonal or even more complicated matrix, an expansion for $\exp(h\, D_x)$ directly is preferable. This can be achieved by means of a theorem due to Lagrange [4], which is discussed below.

2. ADAPTATION OF LAGRANGE'S THEOREM

Lagrange has given an expansion for an analytic function $f(y)$ of a variable y which is defined implicitly by another analytic function $g(y)$ as

$$y = b + \varepsilon\, g(y). \tag{5}$$

The resulting series is [4]

$$f(y) = f(b) + \sum \frac{\varepsilon^n}{n!} \frac{d^{n-1}}{db^{n-1}} \left[g^n(b) \frac{df}{db} \right]. \tag{6}$$

This corresponds exactly to (3) and (4) on equating

$$y = D_x\,;\;\; f(y) = e^{hy}\,;\;\; b = \tfrac{1}{2}\, i\, k_0\, q\,;\;\; g(y) = y^2\,;\;\; \varepsilon = i/2k_0 \tag{7}$$

Use of the expansion for e^{hb} yields the double sum

$$\exp(hy) = \sum_n \sum_m C_{nm}\, \varepsilon^n\, h^m\, b^{n+m} \tag{8}$$

where

$$\left.\begin{aligned} C_{00} &= 1\,;\;\; C_{n0} = 0\,, && (n > 0), \\ C_{nm} &= \frac{(2n + m - 1)!}{n!\,(m-1)!\,(m+n)!}\,, && (m > 0) \end{aligned}\right\} \tag{9}$$

Combining terms with a common power in b yields

$$\exp(hD_z) = \sum W_N(\varepsilon/h)\, B^N \; ; \quad B = hb \tag{10}$$

for which one finds

$$W_0 = 1, \quad W_1 = 1, \quad W_2 = \varepsilon/h + ½ \tag{11}$$

and the higher coefficients satisfy the relation [2]

$$W_N = \frac{2(2N-3)}{N}\frac{\varepsilon}{h} W_{N-1} + \frac{1}{N(N-1)} W_{N-2} \; ; \qquad (N > 2). \tag{12}$$

The next step consists in finding Padé approximants for the power series (10)

$$\sum W_N B^N = \frac{U_{LM}(B)}{V_{LM}(B)} = \frac{1 + \alpha_1 B + \ldots\ldots \alpha_L B^L}{1 + \beta_1 B + \ldots\ldots \beta_M B^M} + O(B^{L+M+1}) \tag{13}$$

The coefficients in the polynomials $U_{LM}(B)$ and $V_{LM}(B)$ are found by standard linear equations, (not reproduced here), by transforming (13) to

$$\left(\sum W_n B^n \right) * V_{LM}(B) = U_{LM}(B), \tag{14}$$

valid for powers of B up to degree $L+M$. To-date we have only considered equal L and M. For $L < M$ the same number M of linear equations involving b or q must be solved for a lower overall accuracy; for $L > M$ the numerical process is likely to become unstable.

In view of the relation $e^{hy} * e^{-hy} = 1$, we have

$$C_{nm}(\varepsilon, -h) = (-)^m C_{nm}(\varepsilon, h) \tag{15}$$

in (9), and hence in (13):

$$V_{LM}(\varepsilon/-h) = U_{ML}(\varepsilon/h) \; ; \quad U_{LM}(\varepsilon/-h) = V_{ML}(\varepsilon/h). \tag{16}$$

When equation (1) is turned into a PE, we have

$$h \text{ real, } \epsilon, b, \text{ and hence } y \text{ pure imaginary} \tag{17}$$

whence odd powers of hb are imaginary, and even powers real. This makes

$$U_{MM} = [\, V_{MM} \,]^* \tag{18}$$

where * denotes the complex conjugate. This simplifies the expressions and also ensures that U and V have the same modulus and their ratio has modulus unity, in agreement with the property of exponentials of pure imaginary argument.

Finally, as only one tridiagonal matrix can be treated at one time, the numerator and denominator in (13) must be decomposed into linear factors (by Newton-Raphson or similar methods):

$$V_{LM} = \prod (1 + \rho_i B), \text{ similarly for } U_{LM} \tag{19}$$

For $L = M$ and the special case (17) this simplifies to

$$\frac{U_{MM}}{V_{MM}} = \prod_{1}^{M} \frac{(1 + \rho_i B)^*}{1 + \rho_i B} \tag{20}$$

3. APPLICATION OF THE LP-PE FOR REALISTIC ATMOSPHERIC PROPAGATION CASES

When applying the Lagrange-Padé (LP) method to the wave problem (1) with step lengths $h = \Delta x$, the expansion parameter ϵ/h becomes

$$\epsilon/h = 1/(2ik_0 \Delta x) = \lambda_0/(4i\pi\Delta x) \tag{21}$$

In [2] the authors have given a specimen list of coefficients ρ_i in (20) for M ranging from 1 to 4 and for h/λ_0 ranging from 2 to 1/32. The case $M = 1$ is equivalent to the standard WA-PE; the only coefficient in (20) has the value $\rho_i = -0.5 - i\lambda_0 / 4\pi h$. For $M > 1$ the real parts of ρ_i are always negative, adding up to -½; this can be interpreted as if the interval h was divided into M unequal sub-intervals; but the imaginary parts vary widely in size and even in sign.

For Ψ varying smoothly with z at a given x, step lengths h of $8\lambda_0$ with $M=4$ gave results identical with those obtained with the WA-PE and $h = \lambda_0/5$. However near a concentrated source, and sometimes near the upper boundary, the strong variation of Ψ with z sometimes carries through to higher values of x, giving spurious spikes in the contour lines, or even complete instability of the results. This we believe to be due to the fact that for $M = L$ the LP procedure is at the limit of numerical stability; we shall investigate this further.

To avoid this difficulty we began the process near the source with the standard WA PE ($M = 1$) and gradually stepped up M and h with increasing distance.

The subrange configuration used was in 4 zones: $M = 1$ for 10 - 800m, $M = 2$ for 800 - 1400m, $M = 3$ for 1400 - 2500m and $M = 4$ for > 2500m, the corresponding range steps being $\lambda/5$, $4\lambda/5$, 3λ and 8λ [2].

We also developed a new algorithm which permits numerical solutions for realistic meteorological profiles in the presence of a flat ground. An artificial absorbing layer (AAL) (see [1]) was used to minimise upper boundary reflections. The AAL includes artificial upward refraction so that its depth can be kept to a minimum. Errors were generated at very large ranges because the AAL errors propagated and eventually corrupted the predictions even close to ground. To date satisfactory predictions out to ranges exceeding 10 km at frequencies up to 100 Hz have been obtained.

The LP-PE predictions have been tested against the published benchmark cases [5] at 10 Hz and 100 Hz (which are identical to our own CN-PE predictions). The agreement is excellent at 10 Hz (Figure 1) downwind. At 100 Hz at the larger ranges the LP-PE uses very large range steps (8 wavelengths). However at the sampled points the agreement is still good even though the detailed behaviour between range samples is lost (Figure 2).

References

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Acknowledgements

The authors wish to thank the Directorate of Health and Safety, UK Ministry of Defence and the US Army Atmospheric Research Laboratory, White Sands for their financial support of this work.

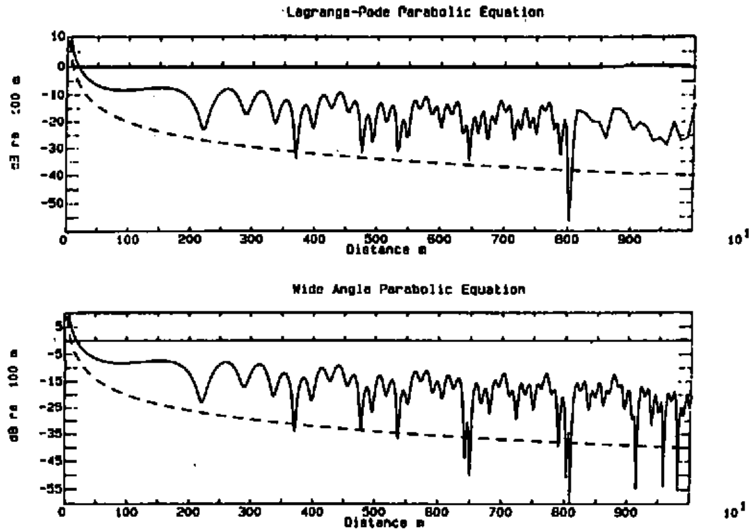

Figure 1 10 Hz Benchmark Test
Sound Speed Gradient g = 0.1

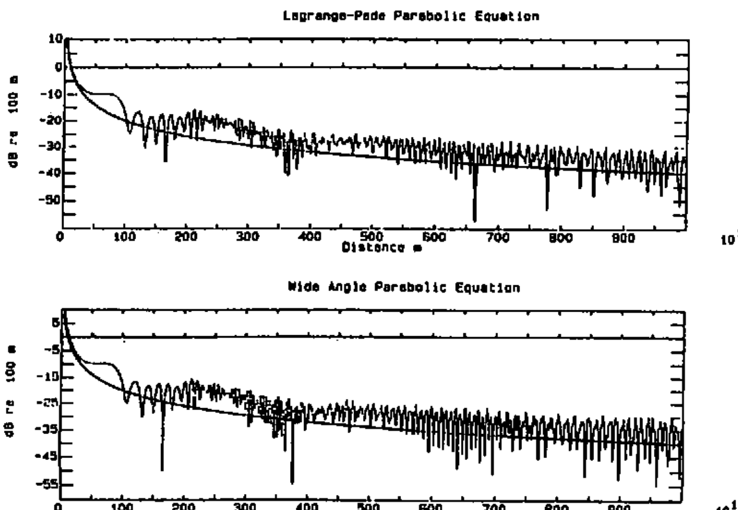

Figure 2 100 Hz Benchmark Test
Sound Speed Gradient g = 0.1