Proceedings of the Institute of Acoustics

MEASUREMENT OF BOW FORCE

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ABSTRACT

By using signals from force transducers at the terminations of a bowed string, it is possible to reconstruct the velocity of the string at the bowing point and the force exerted on the string by the bow in the plane of the string's motion. The theory of the reconstruction is presented, as well as some methods of realizing the theory in practice with sampled data. Examples using simulated data and data from a bowed E string are given.

1. INTRODUCTION

From the beginning of serious analytical studies of the bowed string, the frictional force between bow hair and string has been assumed to be a function only of the relative velocity of bow and hair, with coefficients of friction representing the relation between the normal force and the frictional force. The only empirical justification for this assumption has been a measurement Lazarus [1], referenced by Cremer [2], that was not made under the dynamical conditions of a bowed string. In this report, we describe a technique for inferring the transverse force the bow exerts on the string during bowed string motion. We describe the requirements for practical implementation of the theory; for verification, we apply the technique to a simulation. We also show an application to real data, and discuss the experimental requirements for a reliable extraction of the frictional bow force.

2. THEORY

In our experiment we attach force transducers to both ends of the string of a violin E string mounted on a monochord. The transducers measure the AC component of the transverse force exerted on them by the string. The force the nut exerts on the string is given by

$$\mathbf{f}_{nut}(t) = Z\,[-\mathbf{u}_+(N,t) + \mathbf{r}_{nut}(t) * \mathbf{u}_+(N,t)] = Z\,[-\delta(t) + \mathbf{r}_{nut}(t)] * \mathbf{u}_+(N,t) \qquad (1)$$

where the bow is assumed to be at x=0, and $u_+(x,t)$ is the velocity of a disturbance at (x,t) traveling *away* from the bow, towards the nut at x=N. Z is the string's characteristic or wave impedance; its reciprocal, Y=1/Z, is the string's characteristic admittance. The notation $r * u$ means the convolution of the function r(t) with u(t). The quantity $r_{nut}(t)$ is the reflection function of the nut termination, and the reflected wave is the second term in eq. (1). $\delta(t)$ is the Dirac delta function, and $\delta(t)*u_+(N,t) = u_+(N,t)$. We will henceforth use the notation $\mathbf{r}'$ for $r(t)-\delta(t)$. A similar expression to eq. (1) descibes the force exerted by the bridge.

Our objective is to write the force the bow exerts on the string in terms of the force given in eq. (1) and its equivalent at the bridge. That force $\mathbf{f}$ for waves on the nut side of the bow is

$$\mathbf{f}(0,t) = Z[\,\mathbf{u}_+(0,t) - \mathbf{u}_-(0,t)] = Z[\mathbf{u}_+(0,t) - \mathbf{u}_-(N, t-\tfrac{N}{c})] \qquad (2)$$

where in eq. (2) $\mathbf{u}_+$ and $\mathbf{u}_-$ are waves travelling away from and towards the bow, respectively. The speed of a transverse disturance on the string is taken to be c. A similar expression of course applies on the bridge side. Equation (1) can be solved for $\mathbf{u}_+(N,t)$:

$$u_+(N,t) = Y\,\mathrm{Inv}\;(\mathbf{r}) * \mathbf{f}_{nut}(t) \tag{3}$$

where the expression $\mathrm{Inv}(\mathbf{r})$ indicates deconvolution: $(\mathrm{Inv}(\mathbf{r}) * \mathbf{r})(t) = \delta(t)$. By using (3) and its equivalent at the bridge, one may write the total force as the sum of the contributions from waves on each side of the bow as:

$$Y\,\mathbf{F}_{bow}(t) = \mathrm{Inv}\;(\tilde{\mathbf{r}}_{nut}) * [\mathbf{f}_{nut}(t+\tfrac{(1-\beta)L}{c}) - \mathbf{r}_{nut} * \mathbf{f}_{nut}(t-\tfrac{(1-\beta)L}{c})]$$

$$+ \mathrm{Inv}\;(\tilde{\mathbf{r}}_{brg}) * [\mathbf{f}_{brg}(t+\tfrac{\beta L}{c}) - \mathbf{r}_{brg} * \mathbf{f}_{brg}(t-\tfrac{\beta L}{c})] \tag{4}$$

where $\mathbf{r}_{brg}$ is the reflection function at the bridge, B and N distances from bow to bridge and nut, respectively, with $L=N+B$, $N=(1-\beta)L$, and $B=\beta L$. By a similar calculation the velocity of the string at the bow, $v_{bow}(t) = u_+(t) + u_-(t)$, from continuity, an equation that holds separately for the string on each side of the bow:

$$Z\,\mathbf{v}_{bow}(t) = \mathrm{Inv}\;(\tilde{\mathbf{r}}_{nut}) * [\mathbf{f}_{nut}(t+\tfrac{(1-\beta)L}{c}) + \mathbf{r}_{nut} * \mathbf{f}_{nut}(t-\tfrac{(1-\beta)L}{c})]$$
$$= \mathrm{Inv}\;(\tilde{\mathbf{r}}_{brg}) * [\mathbf{f}_{brg}(t+\tfrac{\beta L}{c}) + \mathbf{r}_{brg} * \mathbf{f}_{brg}(t-\tfrac{\beta L}{c})] \tag{5}$$

Equations (4) and (5) are the principal theoretical results of this investigation. They express one hitherto inaccessible quantity, $\mathbf{f}_{bow}(t)$, in terms of easily measurable quantities, the forces on transducers at the string terminations. The string velocity at the bow, eq. (5), has long been measured by the standard magnetic induction technique. In the next section we explore three methods for experimental reconstruction of the bow force and velocity from the termination transducer data.

3. EXPERIMENTAL REALIZATION

3.1 Basic Requirements.

The fundamental difference between the formulations of equations (4) and (5) and the actual attainment of a bow force measurement is that the data from the force transducers is necessarily sampled. The accuracy of delay and advance times of the nut and bridge forces, as expressed by their arguments, is subject to the quantization of time intervals as determined by the sampling rate. In units of the sampling period, the expression $\beta L/c$ is necessarily an integer; it is half the round trip time from the bow, here assumed to be applied at a point, to the bridge and back. Thus $\beta\,2L/c$ must be an even integer, as must $(1-\beta)\,2L/c$. The validity of the expressions (4) and (5) to represent bow force and velocity does in fact depend with great sensitivity on the precision of the relative phase of reflection signals from the bridge and nut as given by the arguments of the forces on the right hand sides of these expressions. In general one cannot ever expect, given normal commercial sampling rates, that the timing conditions can both be satisfied. One possibility is the use of a continuously variable sampling rate, with the rate chosen to satisfy the timing criteria. Another possibility is resampling, taking advantage of Nyquist's theorem that if certain bandwidth criteria are satisfied the initial analog signal can be reconstructed from the sampled data at any time whatsoever during the duration of the data set. We have used resampling in an exploration of force and velociy reconstruction presented below.

A second barrier to solution of equations (4) and (5) for real data lies in the evaluation of the deconvolution of $\mathbf{r}$. There are at least three approaches:

(1) If the data are taken on a monochord, e. g., with an E string as in the glass bow investigations that stimulated the present work [3], then it is tempting to assume that the reflection functions at bridge and nut are delta functions: $r(t)=-\rho\,\delta(t)$, where $\rho \le 1$. Then equations (4) and (5) become algebraic, since $\mathrm{Inv}\,(\mathbf{r}) = (\rho - 1)^{-1}$.

(2) The deconvolution can be done by the standard Fourier transform technique. Using the convolution theorem, $\mathrm{FT}[\mathrm{Inv}(\mathbf{r}) * f](t) = F(\omega)/R(\omega)$, where FT [.] indicates Fourier transform of the argument, and the capitol letters are the

(2) The deconvolution can be done by the standard Fourier transform technique. Using the convolution theorem, $FT[Inv(\tilde{r}) * f](t) = F(\omega)/\tilde{R}(\omega)$, where FT [.] indicates Fourier transform of the argument, and the capitol letters are the Fourier transforms of their lower case equivalents. The inverse transform of the RHS then gives the required result in the time domain.

(3) The circulant matrix method. The convolution, y, of two M-periodic lists, say p(i) and q(i), where q(i+M)=q(i) and p(i+M)=p(i), can be written

$$y(n)=\sum_{i=0}^{M-1} p(i)\, q(n-i), \quad n=0,1,...,M-1. \qquad (6)$$

or, in vector notation, **y**=**P**.**q**, where **P** is a "circulant" matrix, constructed as follows: The first row is {p(0), p(M-1), p(M-2),...,p(1)}. Subsequent rows are manufactured by shifting each element one place to the right with wrap-around, so that the first element of the second row, for example, is p(1). (See reference [4)]). The inverse of **P** is readily calculated numerically. All that is necessary to implement method (3) is to choose a section of the data of length such that the last entry ties seamlessly onto the first, thus imposing a pseudo-periodicity on the bridge and nut transducer data, although they may in fact not be really periodic at all. Obviously, the more nearly periodic it is, the better the results will approximate the true (but unknown) bow force. That goal is better attained for a segment of the real data if the data is resampled so that each the timing criteria discussed above are met. The goal here is that each period have an integral number of samples, and particularly that a selection of several periods have an integral number of samples. A practical way to achieve this is to shift the selected subset of the data by half the data's length so that any discontinuities from failure to impose periodic boundary conditions show up more visibly in the middle of the plot of the shifted data set. The required delays must be accomplished by shifting to left or right with wrap-around.

In fact, methods (2) and (3) are basically the same, since the Discrete Fourier transform method implicitly imposes a periodicity M on a data set excerpt of length M. If the data set is carefully constructed with periodic boundary conditions, methods (2) and (3) yield identical results. In both cases, if the various requirements of timing and pseudo-periodicity are not adhered to, the results are completely spurious.

3.2 Reconstruction of a simulation.

Figure 1 shows the bow force (dark lines) and the velocity at the bow of a simple simulation, done without consideration of rotational motion. The units of the forces are the same as velocity because the simulation used reduced units: 2Z=1. The bow and bridge reflection functions were identical Gaussian functions with their maxima four samples beyond their inception, so the period of oscillation was longer than the ostensible length, N+B, of the string. Figure 2 shows the bridge force and nut force waveforms - waveforms that are not normally generated in simulations. The simulation had achieved Helmholtz motion, but not complete periodicity. Figure 3 is the reconstructed version of Fig. 1, using the waveforms of Fig. 2. Figure 4 shows superimposed three quantities: the curve is the F(v) function, a hyperbola, used for the simulation. The light points are (v,F(v)) obtained from the simulation. Note they fall exactly of F(v). (The complete force curve of course would include a vertical line at abscissa position v_{bow} =1, the bowing velocity). The dark points are from the reconstruction, as described above, using the circulant matrix method.

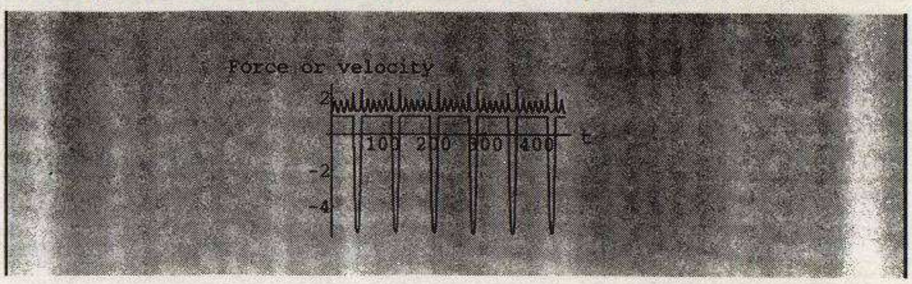

Figure 1: Bow force (dark) and bow velocity (light) from a simulation.

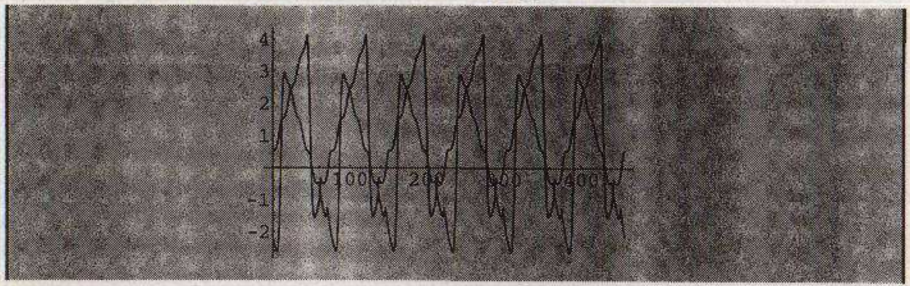

Figure 2. Bridge (dark) and nut (light) waveforms from simulation. Horizontal axis in units of sample period, vertical axis units same as figure 1.

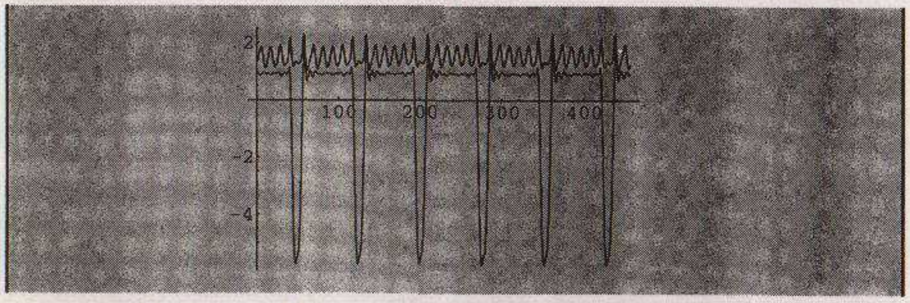

Figure 3: Reconstructed force and velocity waveforms. Same axes as in figure 1.

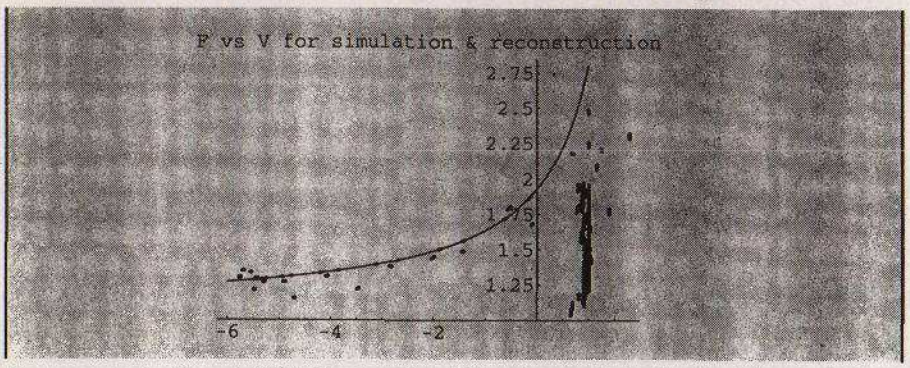

Figure 4. Force vs. velocity. Curve is F(v) in slipping region. Light points as calculated by simulation. Dark points as calculated by reconstruction, from Fig. 2.

We note that the reconstruction does not duplicate the waveforms exactly (compare figures 1 and 3), and does not exactly reproduce the assumed frictional force during slipping (figure 4). We believe these errors are the inevitable result of imposing periodicity on waveforms, in this case on six periods of the oscillation, that are not exactly periodic.

Other lessons are learned by trying to reproduce a simulated waveform. If the timing requirements are not met in the reconstruction, the bow force during slipping can vary from decreasing with time to increasing with time, depending on the direction of the timing error. The result is an F(v) vs. v plot that is an open loop, with the time-ordering of the points traversing the loop clockwise or counterclockwise, depending on the sign of the timing error. An error of a single sampling period in the specification of the travel time from bridge to bow, or nut to bow, results in an open loop.

Figure 5 shows an attempt to reconstruct bridge force and velocity from real data, taken with a glass bow on an E string on a monochord, in order to measure the frictional force during slipping as a function of velocity. Bowing fractional distance β was about 1/6, the bowing velocity was 22 cm/s, and vertical bow force about 0.5 N. The reflection functions at nut and bridge and transducer calibrations were obtained by a separate experiment, in which the string was plucked at midpoint by looping a very thin copper wire around the string and pulling it to the side until it breaks.. The approximate axes scales in the figure are derived from these measurements. The delays correspond to the correct value of β for these data and their sum adds to half the resampled period of 72 samples.

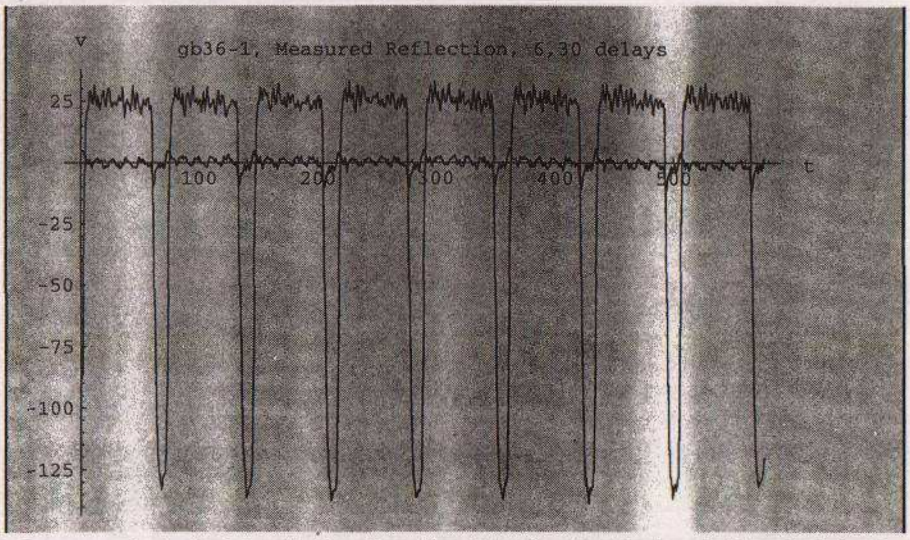

Figure 5. Reconstructed velocity at bow (light) and bow force (dark) from E string bowed with rosined glass bow. Abscissa is in units of the (re)sampling period, 21 μs, the ordinate in cm/s for string velocity, and grams force for the bow force.

Other combinations of delays adding to the correct period gave similar bow forces, but rather implausible velocity during sticking. It would be very desirable to measure the velocity at the bow as well as the force transducer signals, but the ability to measure three signals is not yet available to us. Figure 6 shows F(v) vs. v. Note that since the transducers record only the AC component, the average reconstructed bow force is zero. The loop during slipping is trasversed clockwise in time.

Lacking the necessary additional velocity data, we cannot set the delay between the transducers with confidence. Thus the openness of the loop in the F(v) vs. v plot may be an artifact. Later studies will include measurement of the velocity of the string, a quantity that may also reconstructed (as in figures 1 and 3), using the same functions of the bridge and nut forces that are used in generating the force at the bow (compare equations (4) and (5). The additional experimental information will help remove the uncertainty in setting the delays.

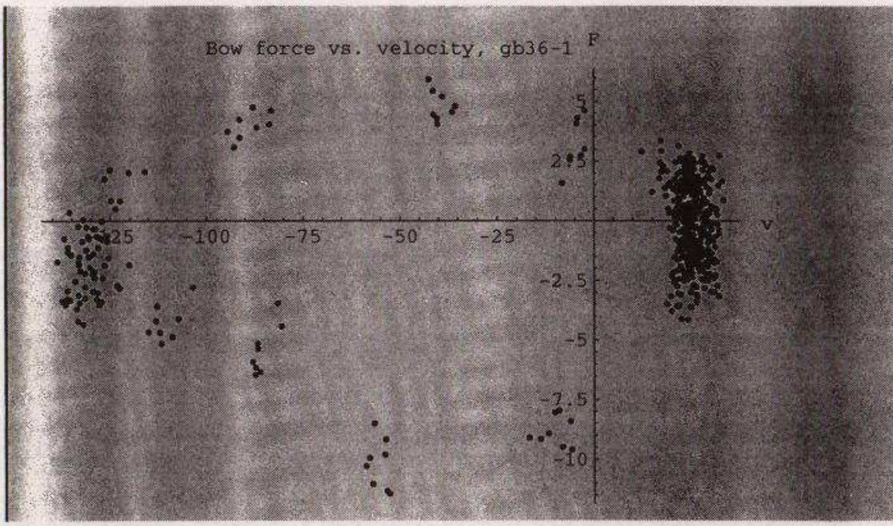

Figure 6. F(v) vs. v, data same as fig. 5. Velocity units in cm/s, force units in grams force, AC component only.

4. CONCLUSION

We have developed a method to reconstruct the force exerted on the string by the bow and the velocity of the string at the bow, using force transducers at the string terminations. The method requires knowledge of the propagation delays from bow to bridge and nut. With the aid of a measurement of the string velocity at the bowing point we expect that we can reconstruct the friction force as a function of relative bow-string velocity, a property essential to proper modelling of violin acoustics.

ACKNOWLEDGEMENT

Partial funding of this research was provided through the generosity of Carnegie Mellon University alumnus, S. F. Lybarger, Physics 1930.

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