DETECTION OF COMPLEX OBSTACLES USING FEW FARFIELD MEASUREMENTS.

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1 Introduction

Let D be a bounded domain of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus \overline{D}$ is connected. We assume that its boundary ∂D is of class C^2. The propagation of time-harmonic acoustic fields in homogeneous cylinder media can be modelled by the Helmholtz equation

$$\Delta u + \kappa^2 u = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D}, \tag{1}$$

where $\kappa > 0$ is the wave number. At the obstacle boundary, ∂D, we assume the total field u to satisfy the impedance boundary condition. That is,

$$\frac{\partial u}{\partial \nu} + i\kappa\sigma u = 0 \text{ on } \partial D \tag{2}$$

with some impedance function σ where ν is the outward unit normal of ∂D. We assume that σ is a real valued C^1-continuous function of order $\beta \in (0, 1]$ and has a uniform lower bound $\sigma_- > 0$ on ∂D. The boundary ∂D is referred to be coated.

For a given incident plane wave $u^i(x, d) = e^{i\kappa d \cdot x}$ with incident direction $d \in \mathbb{S}^1$, where $\mathbb{S}^1$ is a unit circle in $\mathbb{R}^2$, we look for a solution $u(x, d) := u^i(x, d) + u^s(x, d)$ of (**??**), (**??**), where the scattered field u^s satisfies the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - i\kappa u^s\right) = 0 \tag{3}$$

with $r = |x|$ and the limit is uniform for all directions $\hat{x} \in \mathbb{S}^1$. It is well known [**?**] that the scattered wave has the asymptotic behavior: $u^s(x, d) = \frac{e^{i\kappa r}}{\sqrt{r}} u^\infty(\hat{x}, d) + O(r^{-3/2}), \quad r \to \infty$, where the function $u^\infty(\hat{x}, d)$ with $\hat{x} = x/|x|$ on $\mathbb{S}^1$ is called the far-field of the scattered wave $u^s(x, d)$ corresponding to incident direction d. The problem we are considering is formulated as the following inverse scattering problem.

Complex obstacles reconstruction problem. *Give $u^\infty(\hat{x}, d)$ for every $\hat{x} \in S^1$ and for N incident directions $d = d_1, d_2, , ..., d_N$ reconstruct the complex obstacle $(\partial D, \sigma)$.*

The object of this paper is to design a level set type algorithm incorporated with boundary integral methods to reconstruct $(\partial D, \sigma)$ from few incident directions. Reconstructing shapes by level set methods, introduced in [**?**] has a quite long history, see [**?**] and [**?**] for more details. The level set method [**?**] tracks the motion of an interface by embedding the interface as the zero level set of the signed distance function. The motion of the interface is matched with the evolution of the zero level set. Therefore by working on one dimension higher level set function it is not necessary to track the propagation of the interface, topological changes occur in a natural manner, and the technique extends easily to three dimensions. However in our framework of incorporating level set methods with boundary integral methods we need an explicit boundary representation from the given level set function required by the boundary integral method. Moreover, the novelty of our work lies in that we can reconstruct both the shape D and the boundary term $\sigma(x)$ by using the gradient descent method to minimize a least squares functional related with the given far field data. To do so we need to first compute derivatives of the minimizing functional with respect to the shape and the impedance function, update the level set function via the shape derivative and update the impedance function via the impedance derivative alternatively. This is a non-convex problem and there is no global uniqueness guaranteed. Nevertheless our numerical results surprisingly show very good reconstructions of both even for non-convex shape obstacles and a smooth (non-constant) impedance function. Furthermore our algorithm is shown to be stable in terms of initial guess and noise.

The rest of the paper is organized as follows: in Sec. **??** we propose a least squares functional to minimize and concerns to compute partial derivatives of this minimizing functional with respect to the shape and the impedance functional. In Sec **??** we display numerical results from an algorithm designed as a descent method based on the computed partial derivatives of our functional.

2 The Minimization Problem and the Partial Derivatives

As we mentioned, we are given the far field patterns, $u_j^\infty(\partial D_0, \sigma_0)(\hat{x})$, $j = 1, ..., N$, of the scattered waves $g_j^s(\partial D, \sigma)(x)$, corresponding to the exact shape D_0 and the exact impedance function $\sigma_0(x)$ at the incident direction d_j, we want to reconstruct the shape of the obstacle ∂D and the surface impedance $\sigma(x)$ on the obstacle. To do so, we minimize the following least squares functional

$$F(\partial D, \sigma) = \frac{1}{2} \sum_{j=1}^{N} \int_{\mathbb{S}^1} |u_j^\infty(\partial D, \sigma)(\hat{x}) - g_j^\infty(\hat{x})|^2 \, ds(\hat{x}), \tag{4}$$

where $u_j^\infty(\partial D, \sigma)(\hat{x})$ is the computed far field pattern obtained from the shape D and the impedance function $\sigma(x)$ at the incident direction d_j. To calculate the derivative of these functionals with respect to the shape or the impedance function, we first find the derivative of the far field pattern $u_j^\infty(\partial D, \sigma)(\hat{x})$ with respect to the scattered wave $u_j^s(\partial D, \sigma)(x)$ and then calculate the derivative of $u_j^s(\partial D, \sigma)(x)$ with respect to the shape or the impedance function based on a variational formulation of our original scattering problem, see [**?**] for the details. To simplify the notation, we omit the dependence on d_j and rewrite

$$F(\partial D, \sigma) = \frac{1}{2} \int_{\mathbb{S}^1} |u^\infty(\partial D, \sigma)(\hat{x}) - g^\infty(\hat{x})|^2 \, ds(\hat{x}).$$

We define the map from the scattered wave u^s to the far field pattern u^∞ as

$$A : L^2(\partial D) \to L^2(\mathbb{S}^1) : \quad u^s(\partial D, \sigma)(x)|_{\partial D} := f(x) \to u^\infty(\partial D, \sigma)(\hat{x}),$$

which is the Dirichlet scattering problem solver. We know the explicit representation of A as $(Af)(\hat{x}) = (K^\infty - i\eta S^\infty)(I + K - i\eta S)^{-1}f$, where, $\eta > 0$ is a constant, I is the identity operator, S and K are the single layer operator and the double layer operator respectively, and S^∞ and K^∞ are correspondingly the far field counterpart of the single or double layer operator, see Sec. 3 in [?]. Now, by denoting $u^s_{\partial D}$ and u^s_σ the partial derivatives of $u^s(\partial D, \sigma)$ with respect to ∂D and σ respectively, we obtain derivatives of the minimizing functional $F(\partial D, \sigma)$ from (??) with respect to ∂D and σ as follows:

$$\begin{aligned} F_{\partial D}(\partial D, \sigma) = &\ \mathrm{Re}\left[\textstyle\int_{\mathbb{S}^1} (Au^s_{\partial D})(\hat{x})\, \overline{u^\infty(\partial D, \sigma)(\hat{x}) - g(\hat{x})}\, ds(\hat{x})\right] \\ = &\ \mathrm{Re}\left[\textstyle\int_{\partial D} u^s_{\partial D}(x)\, \overline{(A^*(u^\infty(\partial D, \sigma) - g))\,(x)}\, ds(x)\right] \end{aligned}$$

and

$$\begin{aligned} F_{\sigma}(\partial D, \sigma) = &\ \mathrm{Re}\left[\textstyle\int_{\mathbb{S}^1} (Au^s_{\sigma})(\hat{x})\, \overline{u^\infty(\partial D, \sigma)(\hat{x}) - g(\hat{x})}\, ds(\hat{x})\right] \\ = &\ \mathrm{Re}\left[\textstyle\int_{\partial D} u^s_{\sigma}(x)\, \overline{(A^*(u^\infty(\partial D, \sigma) - g))\,(x)}\, ds(x)\right], \end{aligned}$$

where $A^* : L^2(\mathbb{S}^1) \to L^2(\partial D)$ is the adjoint operator of A. We define $\gamma(x) := (A^*(u^\infty(\partial D, \sigma) - g))(x)$, then the above equations become

$$F_{\partial D}(\partial D, \sigma) = \mathrm{Re}\left[\int_{\partial D} u^s_{\partial D}(x)\, \overline{\gamma(x)}\, ds(x)\right] \tag{5}$$

and

$$F_{\sigma}(\partial D, \sigma) = \mathrm{Re}\left[\int_{\partial D} u^s_{\sigma}(x)\, \overline{\gamma(x)}\, ds(x)\right]. \tag{6}$$

2.1 The Derivative of the Minimizing Functional F with respect to the Shape and to surface impedance

For an objective functional that is the integral on the volume of D or along the boundary of D, the following formula can be easily obtained. If D is a smooth bounded open set, $f(x) \in W^{1,1}(R^N)$, and $\mathcal{F}(D) := \int_D f(x)dx$, the shape derivative is

$$d_S\mathcal{F}(D)(\theta) = \int_D \nabla \cdot (\theta(x) f(x)) = \int_{\partial D} \theta(x) \cdot n(x) f(x)\, ds(x). \tag{7}$$

If D is a smooth bounded open set, $f(x) \in W^{2,1}(R^N)$, and $\mathcal{F}(D) := \int_{\partial D} f(x)\, dx$, the shape derivative is

$$d_S\mathcal{F}(D)(\theta) = \int_{\partial D} \theta(x) \cdot n(x)\left(\frac{\partial f}{\partial n} + Hf\right) ds(x), \tag{8}$$

where H is the mean curvature of ∂D defined by $H = \nabla \cdot n$. These two formulas indicate that the shape derivative depends only on the boundary when the objective functional is a volume integral and the curvature plays a role when the objective functional is a surface integral. Let the map $V(x) \in W^{1,\infty}(R^2, R^2)$ and denote the perturbed shape as $D_t = (I + tV(x))D$ and the solution of (??)-(??)-(??) when replacing D by D_t as u_t, and $u^s_{\partial D} = u_{\partial D} = \lim_{t\to 0} \frac{1}{t}(u_t - u)$. Derivating formally and using the two shape derivatives shown above, we obtain the shape derivative of the least squares functional $F(\partial D, \sigma)$ under the map $V(x)$ as

$$F_{\partial D}(\partial D, \sigma)(V) = -\mathrm{Re}\left[\int_{\partial D} (V.n) W ds(x)\right] \tag{9}$$

where $W := \left\{\nabla u \overline{\nabla w} + (-\kappa^2 + i\kappa\sigma(\nabla.n))u\overline{w} + i\kappa\frac{\partial(\sigma u\overline{w})}{\partial n}\right\}$, with u and w is the solutions for the scattering problem and w is the one of its adjoint problem with the impedance boundary condition of w is given by $\overline{\gamma(x)}$ on ∂D. With the same procedure including the same adjoint equation, we obtain the derivative of $F(\partial D, \sigma)$ with respect to real valued function

$$F_\sigma(\partial D, \sigma)(h) = \int_{\partial D} h\,\mathrm{Im}(\kappa u\overline{w})\,ds(x). \tag{10}$$

3 Numerical Results

All of our numerical results shown here, the red solid line denotes the exact shape or the exact impedance function, the green dash dot line denotes the initial shape or the initial impedance function, and the blue dashed line denotes the reconstructed shape or the reconstructed impedance function.

Fig 1: Reconstruction of ∂D when $\sigma(x)$ is known.

Fig 2 and Fig 3: Reconstructions of ∂D and $\sigma(x)$ without noise.

Fig 4 and Fig 5: Reconstructions of ∂D and $\sigma(x)$ with noise. In this case the noise percentage δ is defined as $\delta := \frac{||g^\delta(\partial D,\sigma) - g^0(\partial D,\sigma)||_{L^2}}{||g^0(\partial D,\sigma)||_{L^2}}$, where $g^0(\partial D, \sigma)$ is the noise free far field data obtained from the exact shape D and the exact impedance function $\sigma(x)$.

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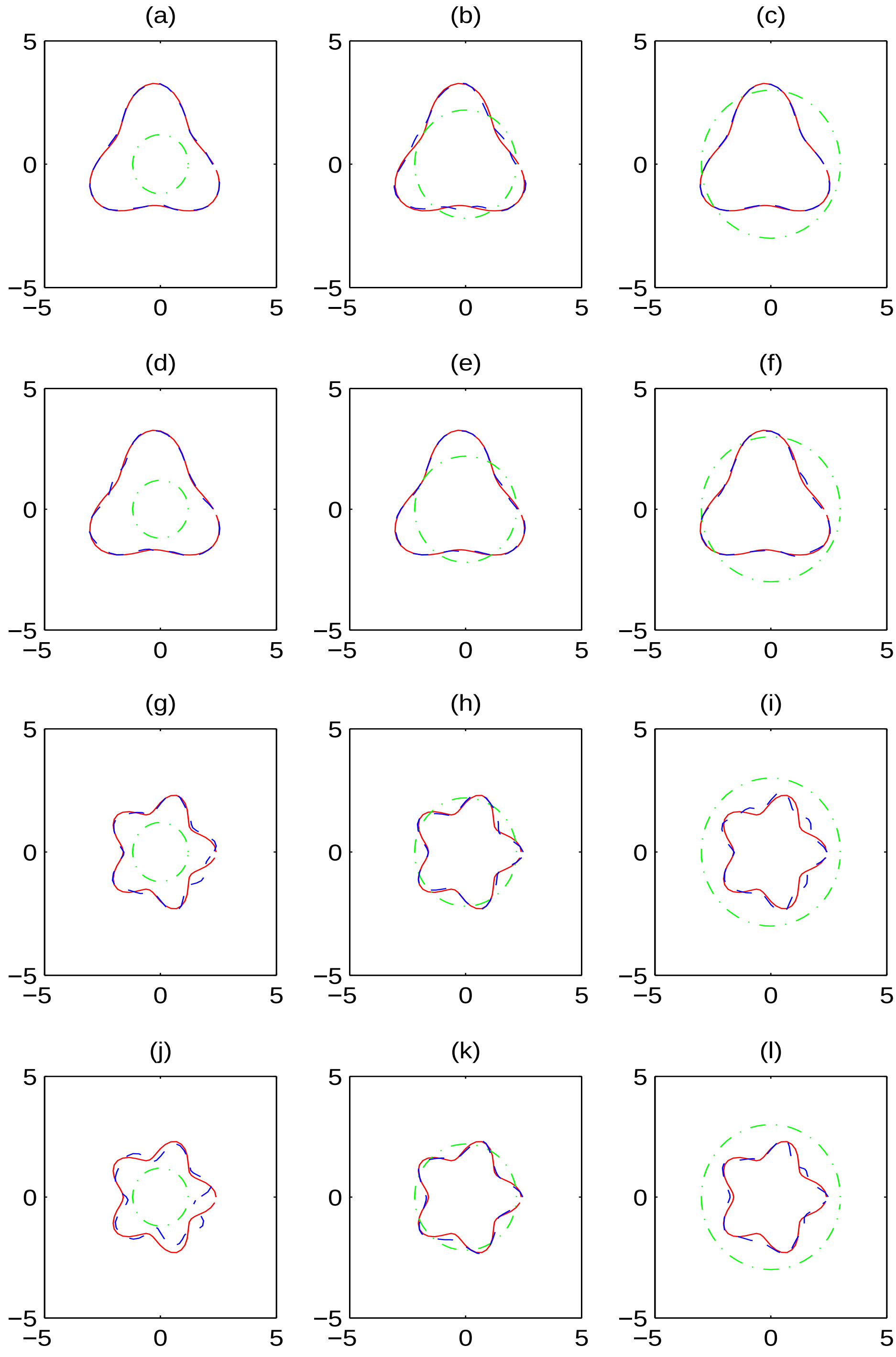

Figure 1: First row: star shape, $\sigma(x) = 1$; Second row: star shape, $\sigma(x) = \frac{2+sin\theta cos\theta}{(3+sin\theta)^2}$; Third row: leave shape, $\sigma(x) = 1$; Fourth row: leave shape, $\sigma(x) = \frac{2+sin\theta cos\theta}{(3+sin\theta)^2}$. First column: initial guess of $r = 1.2$; Second column : $r = 2.2$; Third column: $r = 3.0$.

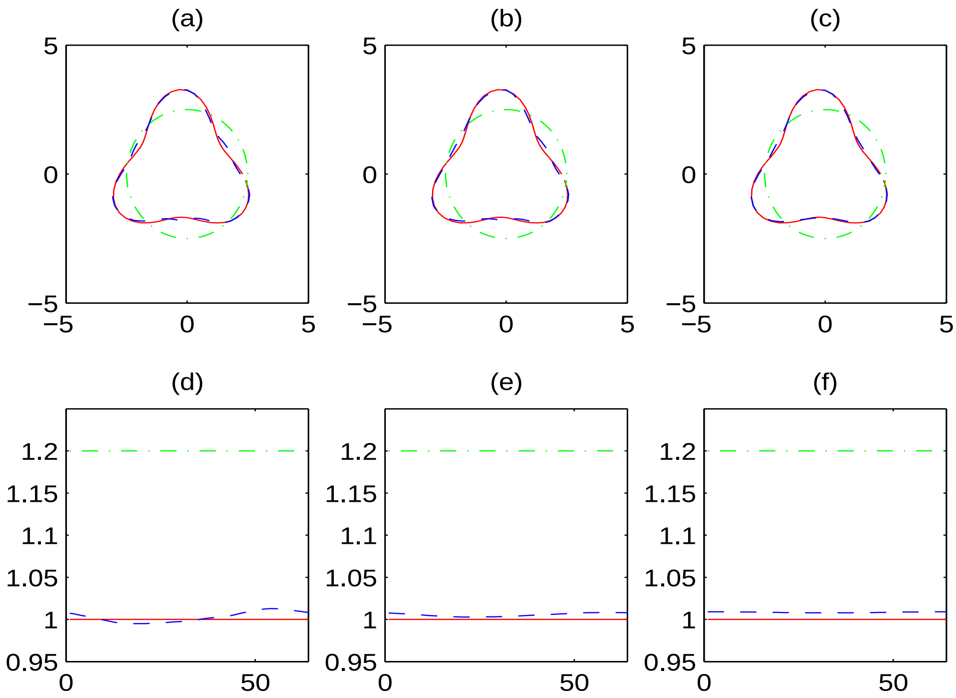

Figure 2: First row: reconstructions of the shape; Second row: reconstructions of the impedance function; First column: the regularization parameter $\lambda = 1$; Second column : $\lambda = 10$; Third column: $\lambda = 100$.

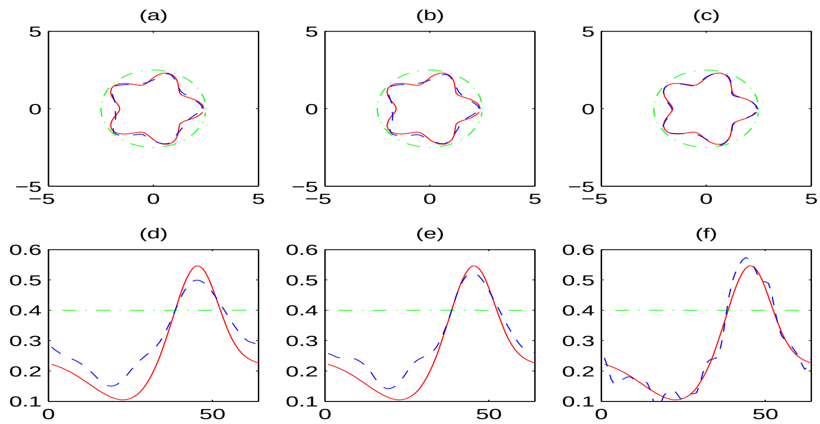

Figure 3: First row: reconstructions of the shape; Second row: reconstructions of the impedance function; First column: the regularization parameter $\lambda = 1$; Second column : $\lambda = 0.5$; Third column: $\lambda = 0$.

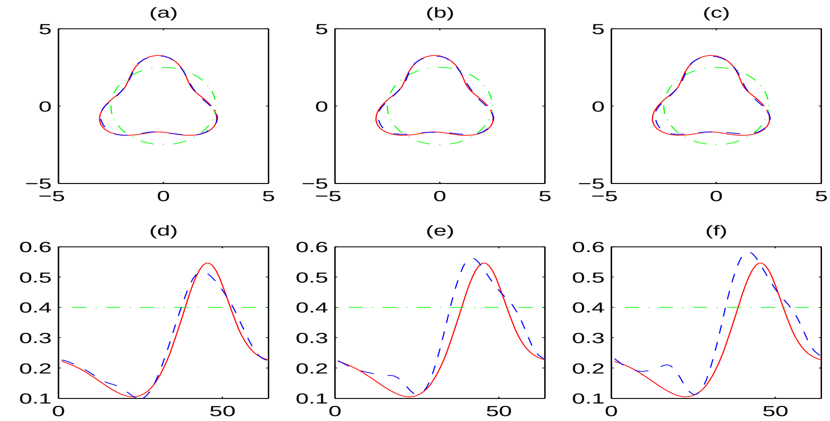

Figure 4: First row: reconstructions of the shape; Second row: reconstructions of the impedance function; First column: noise percentage $\delta = 5\%$, one iteration of $\sigma(x)$ per ten iterations of $\phi(x)$; Second column : $\delta = 10\%$, one iteration of $\sigma(x)$ per two iterations of $\phi(x)$; Third column: $\delta = 20\%$, one iteration of $\sigma(x)$ per two iterations of $\phi(x)$.

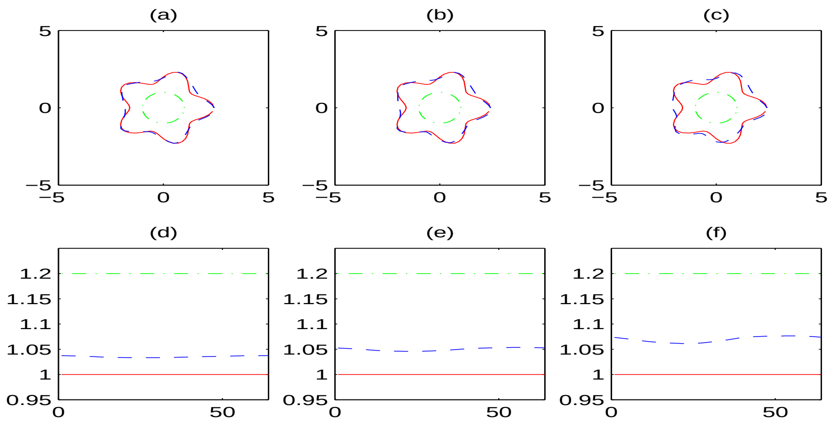

Figure 5: First row: reconstructions of the shape; Second row: reconstructions of the impedance function; First column: noise percentage $\delta = 5\%$, one iteration of $\sigma(x)$ per iteration of $\phi(x)$; Second column : $\delta = 10\%$, one iteration of $\sigma(x)$ per iteration of $\phi(x)$; Third column: $\delta = 20\%$, one iteration of $\sigma(x)$ per iteration of $\phi(x)$.

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