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FEEDFORWARD AND FEEDBACK METHODS FOR ACTIVE CONTROL

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Abstract

A consistent framework is presented for the calculation of the optimal performance of feedforward and feedback control systems. In both cases the optimisation problem is transformed into a quadratic form using an internal model of one part of the physical system under control. This approach to feedforward control is fairly well known, but the controller architecture required for active control using feedback has only been recently recognised. It is known as Internal Model Control (IMC) in the control literature and is widely used in H_∞ control. The optimum performance of a feedback control system can thus be readily calculated using techniques already developed for feedforward control. The robustness of such a feedback controller can then be separately assessed using a generalisation of the complementary sensitivity function found in H_∞ control theory. The robustness of a feedback controller can be improved by using various forms of effort weighting, which are already used for adaptive feedforward controllers. By way of example, the performance is calculated of both feedforward and feedback controllers in actively attenuating road noise in cars.

1. INTRODUCTION

In this paper two methods of implementing active control systems are compared: using feedforward control and using feedback control. Feedforward control relies on the existence of some prior knowledge of the disturbance to be controlled, which is contained in the reference signal. Examples of applications in which such reference signals are available are the control of periodic disturbances, in which case the reference signal may be obtained from a tachometer, or the control of random noise in ducts, in which case the reference signal may be obtained from a sensor upstream of the secondary source [1]. In the latter case the prior estimate of the primary disturbance obtained from the reference sensor may be corrupted by acoustic feedback from the secondary source, which complicates both the theoretical estimate of the optimum performance and the adaptation of the controller in practice. The effects of such acoustic feedback can be removed by incorporating a model of the feedback path within the controller to cancel the effects of the physical feedback path. This technique is similar to that of acoustic "echo cancellation", widely used in telecommunications. For a purely feedforward control problem the performance of an optimal controller can be calculated, before implementation, from a knowledge of the statistics of the primary disturbance, and the response from the secondary sources to the error sensors. In order to regularise the least squares solution it is often advisable to incorporate an effort term as well as an error term in the cost function being minimised. Several ways of achieving this are presented below, together with their effect on the formulation of the adaptation algorithms used in practice.

When the primary disturbance is caused by a large number of uncorrelated sources, it becomes impractical to use feedforward control methods because of the very large number of reference signals which would have to be used. Under such circumstances feedback control may be envisaged, in which the residual error signals, measured by the error sensors, are used to drive the secondary sources, usually via an electronic controller. There is a large body of literature on feedback control theory [2,3] but it is not clear how much of this could be usefully applied to the active control of sound and vibration. In this paper we discuss one formulation of the feedback control problem, called Internal Model Control (IMC), which uses an electrical model of the plant to transform the feedback control problem into a feedforward one. This allows the theoretical performance limitations of a feedback control system to be determined using the mathematical tools developed for feedforward control.

The emphasis of this paper is on the consequences of the IMC formulation for feedback control, and particularly on the balance between good performance and robust stability. Some example calculations are also presented of the performance limits of a feedforward and a feedback control system when used to control road noise in cars.

2. FEEDFORWARD CONTROL

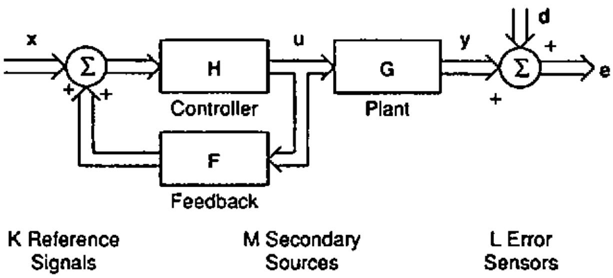


Figure 1. Block diagram of a multichannel feedforward control system with feedback from secondary sources to reference signals.

The optimal performance of a multichannel feedforward control system with a finite-dimensional causal controller will be reviewed in this section. Figure 1 shows the block diagram of such a control system, in which H denotes the M × K matrix of responses of the electrical *controller*, G the L × M matrix of electrical responses from each secondary source input to each error sensor output, which a control engineer would refer to as the *plant*, and F the K × M matrix of responses of the *feedback paths* from secondary sources to detection sensors. The effect of the feedback paths is to complicate the design of the controller, and in extreme cases, to cause instability. If, however, the controller H is implemented as shown in Figure 2, in which an accurate electronic model of the feedback path, $\hat{F}$, is implemented within the controller, the effects of the feedback path can be largely removed. The set of signals feeding the feedforward part of the controller, W are now estimates $\hat{x}$ of the original reference signals x, and the two become equal if the feedback model is perfectly accurate, i.e., $\hat{F} = F$.

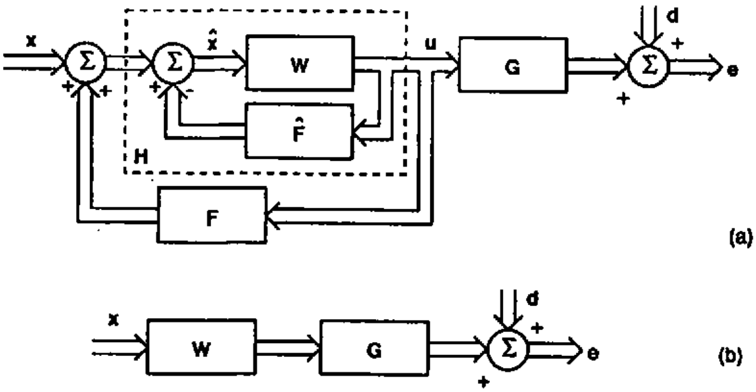


Figure 2. Block diagram of multichannel feedforward control system with feedback cancellation controller, in which $\hat{F}$ is a model of the feedback path F implemented, within the controller, H, (a), and equivalent block diagram if $\hat{F} = F$, (b).

This condition can be assumed to hold for the purposes of calculating the optimum performance, which now becomes a quadratic optimisation problem if the feedforward part of the controller, W, is implemented as an array of FIR filters with coefficients w_{mki} [4]. The plant responses can also be modelled as an array of FIR filters with coefficients g_{lmj}, so that the sampled output of the l-th error sensor can be written

$$e_l(n) = d_l(n) + \sum_{m=1}^{M} \sum_{j=0}^{J-1} g_{lmj}\, u_m(n-j) \qquad (2.1)$$

where $d_l(n)$ is the primary disturbance at the l-th error sensor. The input to the m-th secondary source can also be written as

$$u_m(n) = \sum_{k=1}^{K} \sum_{i=0}^{I-1} w_{mki}\, x_k(n-i) \tag{2.2}$$

where $x_k(n)$ is the sampled version of the k-th reference signal. Using equation (2.2), equation (2.1) can be written as

$$e_l(n) = d_l(n) + \sum_{k=1}^{K} \sum_{i=0}^{I-1} \sum_{m=1}^{M} w_{mki}\, r_{lmk}(n-i) \tag{2.3}$$

where the filtered reference signals are defined as

$$r_{lmk}(n-i) = \sum_{j=0}^{J-1} g_{lmj}\, x_k(n-j). \tag{2.4}$$

Equation (2.3) illustrates the linear relationship between the error signal and the feedforward controller coefficients, and may be re-written in matrix form as [4]

$$\mathbf{e}(n) = \mathbf{d}(n) + \mathbf{R}(n)\mathbf{w} \tag{2.5}$$

The optimum values of the coefficients in the vector **w** depends on the cost function we wish to minimise. The simplest cost function, J_1, is the expectation of the sum of the squared error

$$J_1 = E\left[\sum_{l=1}^{L} e_l^2(n)\right] = E[\mathbf{e}^T(n)\,\mathbf{e}(n)] \tag{2.6}$$

which, using equation (2.5), can be expressed in the general matrix quadratic form [1] as

$$J = \mathbf{w}^T\mathbf{A}\mathbf{w} + 2\mathbf{w}^T\mathbf{b} + c \tag{2.7}$$

where, in this case,

$$\mathbf{A} = E[\mathbf{R}^T(n)\,\mathbf{R}(n)], \quad \mathbf{b} = [\mathbf{R}^T(n)\,\mathbf{d}(n)] \text{ and } c = E[\mathbf{d}^T(n)\,\mathbf{d}(n)].$$

The set of feedforward controller coefficients which minimises equation (2.7) is

$$\mathbf{w}_{(opt)} = -\mathbf{A}^{-1}\mathbf{b} \tag{2.8}$$

which results in the residual cost function

$$J_{min} = c - \mathbf{b}\mathbf{A}^{-1}\mathbf{b} \tag{2.9}$$

The gradient of the cost function with respect to the vector of filter coefficients can also be written as

$$\frac{\partial J}{\partial \mathbf{w}} = 2(\mathbf{A}\mathbf{w} + \mathbf{b}) \tag{2.10}$$

which is equal to $2E[\mathbf{R}^T(n)\,\mathbf{e}(n)]$ in this case. Updating each of the controller coefficients with the negative value of the instantaneous value of this gradient leads to the Multiple Error LMS algorithm [4,5]

$$\mathbf{w}(n+1) = \mathbf{w}(n) - \alpha\,\hat{\mathbf{R}}^T(n)\,\mathbf{e}(n) \tag{2.11}$$

where α is a convergence coefficient, and $\hat{\mathbf{R}}(n)$ is the matrix of reference signals filtered by estimates of the plant's response measured in practice. The algorithm will converge if the real parts of the eigenvalues of the matrix $E\,[\hat{\mathbf{R}}^T(n)\,\mathbf{R}(n)]$ are greater than zero. If the practical estimates of the plant's response are inaccurate, however, the algorithm may become unstable [6].

Because the optimal solution depends on the inverse of the Hessian matrix, $\mathbf{A}$, in the Hermitian quadratic form, the numerical stability of the solution depends on the conditioning of this matrix. The conditioning may be improved by modifying the cost function to include a term proportional to the sum of the squared values of the filter coefficients,

$$J_2 = E[\mathbf{e}^T(n)\,\mathbf{e}(n)] + \beta\,\mathbf{w}^T\mathbf{w} \tag{2.12}$$

which, when written in Hermitian quadratic form has a Hessian matrix $\mathbf{A} = E[\mathbf{R}^T(n)\,\mathbf{R}(n) + \beta\mathbf{I}]$. The coefficient weighting parameter, β, acts in the same way as Tikhonov regularisation, and improves the conditioning of the $\mathbf{A}$ matrix [7]. The addition of the sum of squared filter coefficients in the cost function being minimised has the effect of modifying the appropriate LMS algorithm to be of the form

$$\mathbf{w}(n+1) = \gamma\,\mathbf{w}(n) - \alpha\,\hat{\mathbf{R}}^T(n)\,\mathbf{e}(n) \tag{2.13}$$

where $\gamma = 1 - \alpha\beta < 1$ is a leakage term. This algorithm converges if the real parts of all the eigenvalues of the matrix $E[\hat{\mathbf{R}}^T(n)\,\mathbf{R}(n) + \beta\mathbf{I}]$ are greater than zero. The effect of the leak is to increase the real parts of all these eigenvalues by an amount β, which can stabilize the algorithm even if errors in the estimates of the plant's response become large [6]. Other potential methods of improving the conditioning of the optimisation are briefly presented in the Appendix.

3. FEEDBACK CONTROL

In applications where there are a very large number of partly incoherent processes generating the disturbance, for which reference signals are unavailable, it becomes impossible to implement feedforward control. An example is that of boundary layer excitation of the sound field in an aircraft cockpit [12], but other examples are boundary layer noise in aircraft passenger cabins, and perhaps wind noise and road noise in cars. Under these circumstances active control can still be implemented, but the signals driving the secondary sources now have to be obtained directly from the error sensors, rather than from independent sensors. The block diagram of such a feedback controller is shown in Figure 3. Note that a phase inversion has been assumed in the summing junction of Figure 3, since we are seeking to implement a negative feedback system.

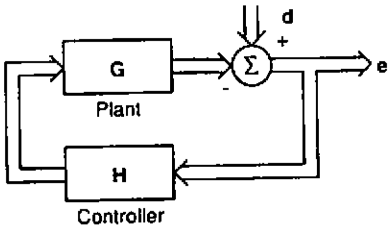


Figure 3. Block diagram of a multichannel feedback controller.

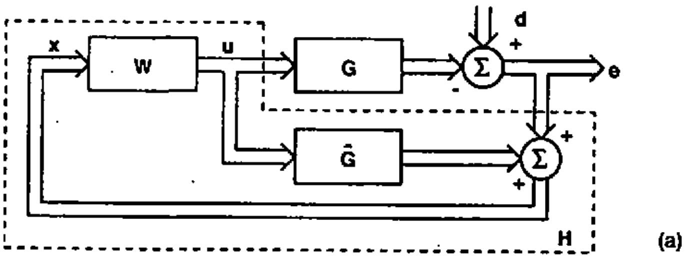
Assuming the signals are in the frequency domain, the vector of errors in Figure 3 can be expressed in terms of vector of disturbances as

$$e(j\omega) = [I + G(j\omega)H(j\omega)]^{-1}\, d(j\omega). \tag{3.1}$$

There are two important aspects to the behaviour of feedback controllers: their performance (the reduction obtained in the disturbance) and their robustness (the ability to remain stable under changing conditions). In very general terms, good performance requires a high "loop gain" ($G(j\omega)H(j\omega)$), whereas robustness requires a lower loop gain. Conventional feedback control theory addresses this fundamental compromise in a number of ways. For an analogue controller, various forms of compensator are typically used, and the robustness is assessed in terms of the resultant gain and phase margins [1]. "Modern" control theory using state space models generally does not directly address this compromise, but it is often noted that the incorporation of effort weighting in the cost function being minimised in optimal or LQG control tends to make the control system more robust [13]. Recent developments in H_∞ control theory have brought the balance between performance and robustness to the centre of the design problem [14]. In this section we will describe an architecture of controller which is widely used in H_∞ control, and which is very similar in philosophy to the feedback cancellation architecture described above in connection with feedforward controllers.

Consider the controller illustrated in Figure 4(a), in which an internal model, $\hat{G}$, of the plant, G, is implemented, which is driven by the same signal as the plant, u. The controller response is thus

$$H(j\omega) = [I + W(j\omega)\hat{G}(j\omega)]^{-1}W(j\omega) \tag{3.2}$$



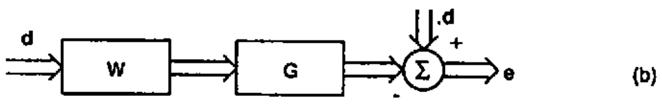


Figure 4. Block diagram of a multichannel feedback controller implemented using Internal Model Control (IMC), (a), and equivalent block diagram if $\hat{G} = G$, (b).

There is again assumed to be a phase inversion as the output of the plant is added to the disturbance to form the error, but now the output from the plant model is added to this observed error to form the signal x. If the plant model is an accurate once, so that $\hat{G} = G$, then the contribution from u in the error will be removed and the signal we are left with will be the disturbance, so that x = d. Under these circumstances the block diagram for the feedback controller (Figure 4(a)) can be re-drawn as in Figure 4(b), which shows that this controller architecture transforms the feedback control problem into an entirely feedforward one, as can be seen by comparing Figure 4(b) with Figure 2(b). The optimal performance of the feedback controller may thus be calculated using the formulation outlined in Section 2. The difference here is that whereas for a truly feedforward system, the performance depended on the cross-correlation between the disturbance and the *reference signal* filtered by the plant response, for feedback control the performance depends on the cross-correlation between the disturbance and the *disturbance* filtered by the plant response.

The controller architecture illustrated in Figure 4(a) was suggested by Newton *et al* in 1957 [15] and is described in detail in the excellent textbook by Morari and Zafiriou [16], who derive a number of its properties including the fact that it is internally stable. It is equivalent to Youla parameterisation [16] for plants which are stable, and in the form shown in Figure 4(a) is known as Internal Model Control, IMC. With this arrangement, the feedforward part of the controller can be arbitrarily specified and the feedback controller will always remain stable; in fact, the IMC architecture parameterises all stable controllers [16], and thus allows the unconstrained design of W for optimum performance without the requirement that the design be checked for stability in the first instance. This approach can be used [17] to provide a simple derivation of the minimum variance control strategy [18], and suggest a more general formulation of this problem.

The obvious danger of IMC is that it may be horribly sensitive to the accuracy of the plant model. This is equivalent to saying that the feedback controller is not "robust", since a small error in the model could be regarded as equivalent to a small change in the physical plant. Robust stability can be evaluated analytically for multichannel controllers using techniques developed for H_∞ control, provided a suitable model can be found for the plant uncertainty. One such model is unstructured multiplicative output uncertainty, in which the plant response is always assumed to be described by the relationship

$$G(j\omega) = [I + \Delta_G(j\omega)]G_0(j\omega) \tag{3.3}$$

where $G_0(j\omega)$ is the nominal plant response, and $\Delta_G(j\omega))$ is the fractional uncertainty, whose norm is bounded by a real but potentially frequency dependent number $B(\omega)$, so that

$$\| \Delta_G(j\omega) \|_\infty < B(\omega) \tag{3.4}$$

For single channel systems this model of uncertainty appears to be a reasonable model for the variations found in practical acoustic plants [17]. Its applicability to multichannel active control systems has yet to be established, but it does lead to a relatively simple expression which has to be satisfied for robust stability [16], which can be written as

$$\| T_0(j\omega)B(\omega) \|_\infty < 1 \tag{3.5}$$

where $T_0(j\omega)$ is the multichannel equivalent of the complementary sensitivity function for the nominal plant,

$$T_0(j\omega) = G_0(j\omega)H(j\omega)[I + G(j\omega)H(j\omega)]^{-1} \tag{3.6}$$

If the norm of the complementary sensitivity function is large, the bound on the fractional plant error must then be small, according to equation (3.5), to maintain robust stability. If a practical estimate of $B(\omega)$ can be obtained, equation (3.5) thus provides a very convenient method of *monitoring* the robust stability of the feedback control system, although it might be burdensome to calculate $T_0(j\omega)$ using equation (3.6). When the controller is implemented using Internal Model Control, however, equation (3.2), the complementary sensitivity function reduces to

$$T_0(j\omega) = G_0(j\omega)W(j\omega) \qquad (3.7)$$

and the robust stability condition is more readily calculated.

The full H_∞ control formulation allows the design of feedback controllers which have an optimum trade-off between performance and robustness. Unfortunately, the mathematical and numerical machinery required to obtain this solution is rather involved, and the solution depends on a number of rather structured assumptions which have to be made about the plant and the disturbance. Another approach to the design of a feedback controller would be to use the Internal Model Control architecture to design a controller which minimised the sum of squared errors with an effort term (an H_2 problem), and increase the weighting on the control term until the robust stability condition is satisfied (an H_∞ condition). In fact any of the cost functions described in the Appendix could be minimised with a similar effect, and it is not clear which of these would give the very best compromise between performance and robustness for a feedback controller in practice. The use of controller coefficient weighting to make the feedback controller robust is described in the following section.

4. ACTIVE CONTROL OF ROAD NOISE IN CARS

In this section we will illustrate some of the properties of feedforward and feedback control, using the active control of random road noise in a car as an example. The theory outlined above provides a consistent formulation for the calculation of the optimum performance of feedforward and feedback control systems from measured data. The 'plant' in this case is from the input to a single secondary loudspeaker (u) to the output of a single error sensor (e), so that the scalar versions of the general, multiple input multiple output, equations derived above are used. The most important component of the response of the plant in determining the performance of the control system, is its overall delay. This delay is partly due to the physical propagation time from loudspeaker to microphone, but in a digital implementation of the controller could also account for the processing delays and the delays through the anti-aliasing and reconstruction filters.

The feedforward control of road noise using reference signals from accelerometers mounted on the body near the suspension points has been described by Sutton and Elliott [19]. The A-weighted spectrum of the measured pressure inside a small car is shown as the solid curve in Figure 5. The residual spectrum predicted after using a feedforward control system with reference signals derived from six accelerometers, is also shown in Figure 5. These predicted results are similar to those achieved in a practical realisation of such a feedforward control system reported by Saunders *et al* [20]. The plant response was assumed to be a pure delay of 1 ms or 5 ms in the calculations used to produce Figure 5, and it can be seen that in this case the attenuations achieved with feedforward control are relatively insensitive to delays in this range.

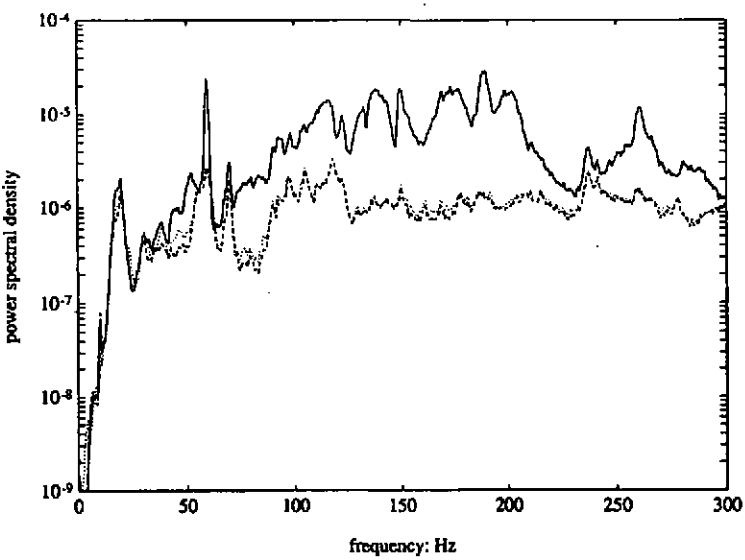


Figure 5. A-weighted power spectral density of the pressure measured in a small car (solid line) and predictions of the residual spectrum with a feedforward control system operating on a plant with a delay of 1 ms (dashed line) or 5 ms (dotted line).

Figure 6 shows the predicted residual pressure spectrum in the car after the implementation of a feedback control system, in which no reference signal is used. The results are again shown for assumed plant delays of 1 ms and 5 ms, and in this case the performance is significantly degraded by the larger delay. The action of the feedback controller can be understood, from Figure 4(b), in which the plant response (G) is a pure delay in this case. The feedforward part (W) of the feedback controller must act as a *predictor* for the disturbance signal d in order to minimise the error. The residual error is then the unpredictable part of the disturbance signal, once the predictable components, over the timescale of the plant delay, have been removed. For the 1 ms delay in Figure 6, the spectrum of the residual error is almost flat, indicating that the residual error is almost white noise, which is uncorrelated from sample to sample. With the 5 ms delay, the residual spectrum has a more 'coloured' spectrum, but the sharp peaks in the original disturbance, which correspond to more predictable components in the pressure signal, have been largely eliminated.

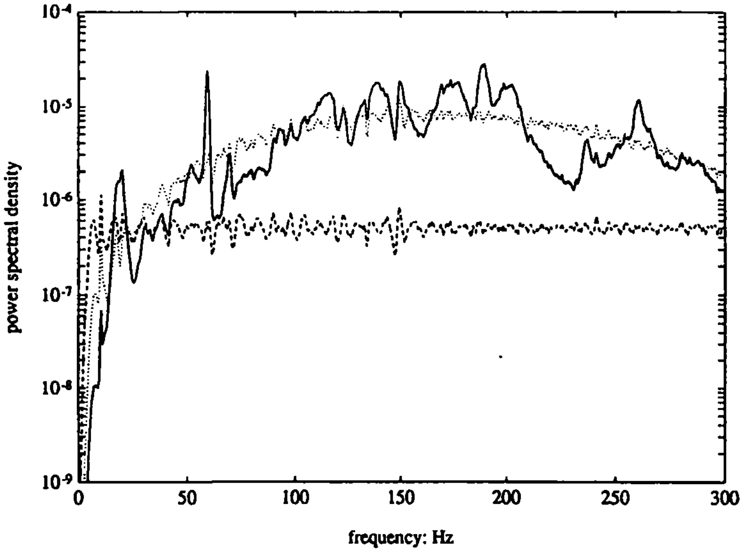


Figure 6. A-weighted power spectral density of the pressure measured in a small car (solid line) and predictions of the residual spectrum with feedback control system operating on a plant with a delay of 1 ms (dashed line) or 5 ms (dotted line).

The variation in the attenuation of the mean square value of the A-weighted pressure with plant delay is plotted in Figure 7, for both feedforward and feedback control systems. It is clear that the performance of the feedforward system is tolerant of plant delays up to about 5 ms, as also shown in Figure 5. The performance of the feedback control system is much more dependent on plant delays in this range, with the overall attenuation falling to less than 1 dB for a 5 ms delay. Notice, however, that for very short delays, the feedback controller can achieve a higher attenuation than the feedforward controller, although the microphone would have to be so close to the loudspeaker to achieve these small delays, that the acoustic effect of control may not be very widespread. Figure 7 provides a vivid illustration of the performance limits of the two types of active control system discussed in this paper. It should be emphasised, however, that the variation of attenuation with delay, as shown in Figure 7, is very dependent on the statistical properties of the disturbance. If the disturbance were a pure tone, for example, both feedforward and feedback control system could give infinite attenuations at a single microphone, regardless of the plant delay. If, on the other hand, the disturbance was white noise, the feedback system would be unable to achieve any attenuation if there were any delay in the plant.

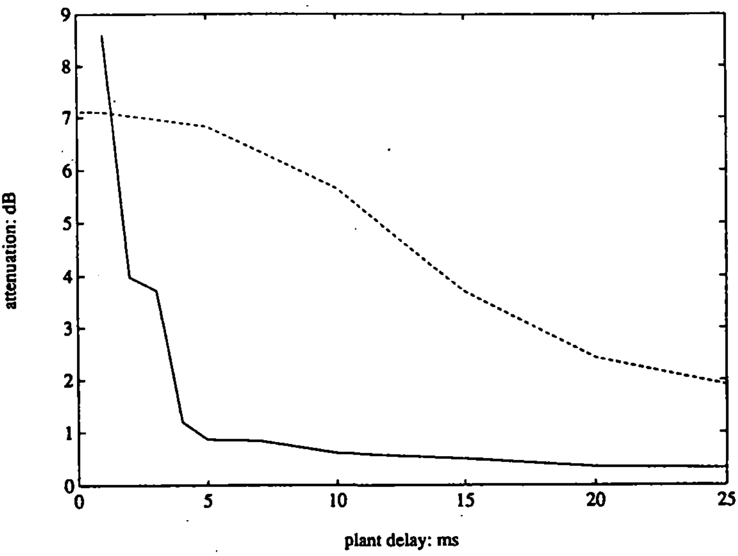


Figure 7. The variation of the attenuation in the mean square value of the A-weighted pressure with delay in the plant for a feedforward control sysem (dashed line) and a feedback control system (solid line).

Having determined the best possible performance of a feedback control system, we now turn to the robustness of the feedback controller to errors or changes in the plant response. This is most graphically illustrated by plotting the frequency response of the open loop system in a Nyquist diagram. The open loop system is HG (see Figure 3) in which H is the complete feedback controller enclosed by dotted lines in Figure 4(a). If the closed loop system is to be stable, the Nyquist plot must not enclose the (-1,0) point on this diagram. The further the Nyquist plot is away from the (-1,0) point at all frequencies, the more robust the feedback controller is to changes in the plant response. If, in the example above, only the mean square error is minimised in the calculations for the optimal feedback controller, the Nyquist plot comes perilously close to the (-1,0) point, as illustrated in Figure 8(a), indicating that this controller is not robust. Only part of the complete Nyquist plot is shown in Figure 8, for very low frequencies and for very high frequencies, because the open loop response over the remainder of the frequency range does not approach the (-1,0) point and obscures the picture. To improve the robustness any one of the methods described in Section 2 could potentially be used. Figure 8(b) shows the Nyquist plot when the simplest of these methods is employed; that of minimising the mean square error and β times the sum of squared filter coefficients (equation (2.12)). It is clear from the Nyquist plot in Figure 8(b), for the value of $\beta = 5 \times 10^{-5}$, that the feedback control system is more robust. A more quantitative measure of the robustness, described in Section 3, is the complementary sensitivity function (T_0, equation (3.6)). For a single input single output system, the maximum value of this function over frequency is equal to the reciprocal of the upper bound on the frequency independent multiplicative uncertainty which can be robustly tolerated in the plant (equation (3.5)). The modulus of the complementary sensitivity function, $T_0(j\omega)$, is plotted in Figure 9 for the feedback controller optimised with $\beta = 0$ (error alone minimised) and $\beta = 5 \times 10^{-5}$ (error and sum of squared filter coefficients minimised). The maximum value of $|T_0(j\omega)|$ is about 17 at low frequencies for $\beta = 0$, corresponding to a maximum fractional uncertainty in the plant or the plant model of about 6% before instability, whereas for $\beta = 5 \times 10^{-5}$, the maximum value of $|T_0(j\omega)|$ is about 3 (at about 400 Hz) corresponding to a maximum fractional plant uncertainty of about 33%. The feedback controller in the latter case is considerably more robust, and will tolerate errors in the modulus of the plant or plant model response of about 2.5 dB and errors in phase of about 20°, whereas with $\beta = 0$, the feedback controller is only robust to errors of 0.5 dB in amplitude and 3.5° in phase.

Although increasing the parameter β clearly improves the robustness of the controller, the performance, in terms of attenuations in mean square values of the error signal, is degraded as β is increased. As β is increased, there is thus a trade-off between performance (attenuation of error signal) and robustness (largest plant uncertainty, given by the reciprocal of the maximum of the complementary sensitivity function, $1/T_{max}$), which is illustrated in Figure 10. For $\beta = 5 \times 10^{-4}$, the attenuation in overall error is about 6 dB, compared with an attenuation of about 9 dB if β were zero. It is doubtful, however, whether a feedback system designed with $\beta = 0$ would be useful in practice because rather small changes in the response of the plant, due for example to movement of people in the car, could make the system unstable.

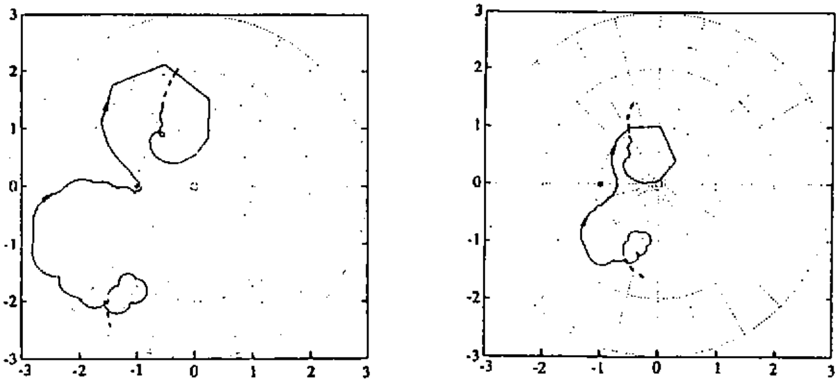


Figure 8. Nyquist plots of the open loop frequency response for the feedback controllers designed to minimise (a) mean squared error only, and (b) mean square error and $\beta \times$ sum of squared controller coefficients ($\beta = 5 \times 10^{-5}$).

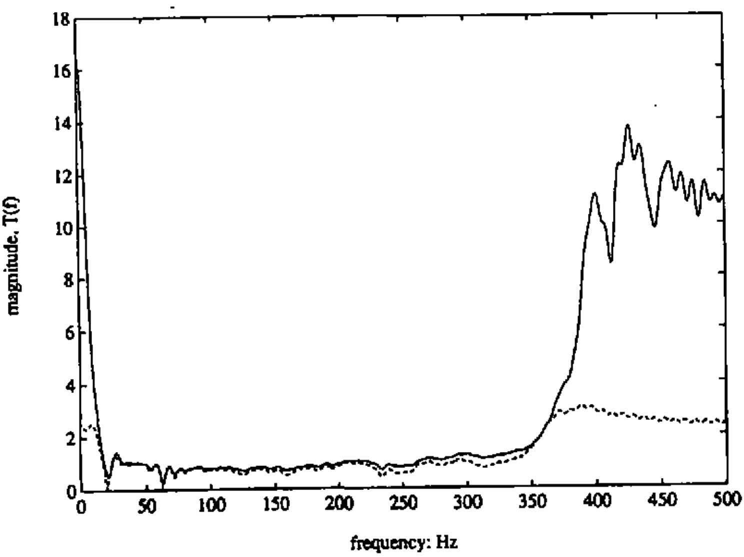


Figure 9. The complementary sensitivity function ($T_0(j\omega)$) for the feedback control systems with the controller designed to minimise mean square error (solid line) and mean squared error and $\beta \times$ sum of squared controller coefficients ($\beta = 5 \times 10^{-5}$).

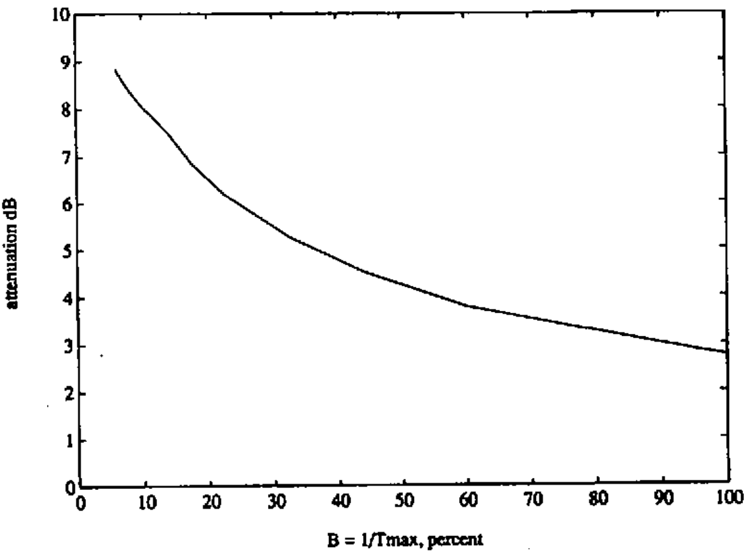


Figure 10. The variation in the attenuation in the mean square error signal, against the maximum fractional plant uncertainty ($1/T_{max}$), as the parameter β is varied for the feedback controller.

5. CONCLUSIONS

A consistent formulation has been presented for calculating the performance of feedforward and feedback control systems. The optimisation problem is transformed into a quadratic one, in both cases, by assuming an internal model of a part of the physical system under control, which can be used to cancel its effects. The "feedback" path in feedforward systems, and the "plant" response in feedback systems, is treated in this way.

To overcome ill-conditioning in the calculation of the optimal solution to this quadratic optimisation problem, a number of methods of regularising the problem are described. These include the incorporation of terms proportional to the sums of squared filter coefficients, or the control effort, into the cost function being minimised. In *adaptive feedforward* control these modifications to the cost function are known to make the control algorithm more robust to errors in the internal plant model used by the algorithm [7]. It is demonstrated in this paper that the simplest of these modifications to the cost function also has the effect of making the stability of a fixed *feedback* controller more robust to such errors.

Example calculations have been performed on internal pressure and accelerometer signals recorded in a car travelling over a rough road. The performance of both feedforward and feedback controllers was calculated, particularly with regard to their variation with delays in the plant. For this disturbance, the potential performance of the feedback control system was better than that of the feedforward system if the plant delay was smaller than about 1.5 ms.

APPENDIX: OTHER MODIFIED COST FUNCTIONS WHICH IMPROVE ROBUSTNESS

The physical effect of the leakage term in equation (2.12) is to prevent the digital controller coefficients converging to large values, when similar reductions in the error could be achieved with smaller values. Using the Wiener-Khinchin relationships, the cost function given by equation (2.12) can be expressed in the frequency domain as

$$J_2 = \int_0^{2\pi} E[e^H(j\omega)e(j\omega) + \beta \; w^H(j\omega)w(j\omega)]d\omega \qquad (A.1)$$

where $e(j\omega)$ is the vector of Fourier transforms of the error signals, and $w(j\omega)$ is the vector of frequency responses of the control filters.

The first term in equation (A.1) is thus the sum of the power spectral densities of the error signals, integrated over the frequency, and the second term is proportional to the sum of the moduli of the frequency responses of the filters in the controller again integrated over frequency. The effect of the leak can be seen to be to discriminate against large values of the frequency response of any of the control filters. It is sometimes desirable to discriminate against large controller responses in certain frequency regions only. This can be achieved using a generalisation of an approach previously suggested for single channel controllers, which relies on deliberately mixing artificially produced measurement noise into the reference signals. Widrow and Stearns [8] show that if the corrupting noise in an electrical cancellation problem is white, the adaptive filter converges to exactly the same solution as it would if there was a leak in the LMS algorithm. For a single channel active control system operating in a duct, it is important to ensure that the controller does not eventually converge to a solution with a large low frequency response, since this will overload the loudspeaker acting as the secondary source. This low frequency overloading will be avoided if measurement noise with a predominantly low frequency spectrum is present in the reference signal. Such measurement noise is generally present naturally in a duct with flow, because of turbulence. Billet [9] has demonstrated that the same effect could be achieved by artificially mixing low pass filtered measurement noise with the reference signal. It is undesirable if this artificial measurement noise is injected into the physical system under control, and so an equivalent effect on the update algorithm of a multichannel system could be achieved with the arrangement shown in Figure 11, in which n denotes a vector of low pass filtered noise signals. If $\hat{G} = G$, the frequency domain cost function minimised by this arrangement is

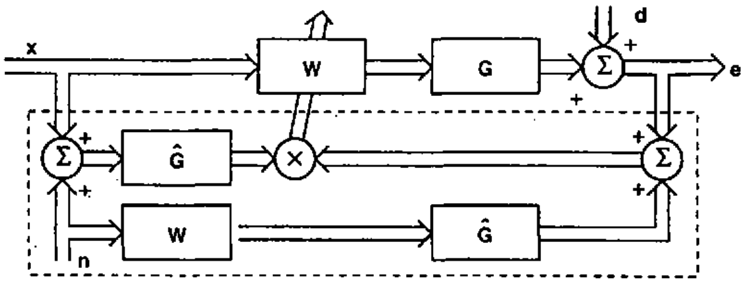


Figure 11. Modification to the LMS adaptation algorithm to minimise a cost function which includes a term proportional to the frequency-weighted control filter responses.

$$J_3 = \int_0^{2\pi} E[e^H(j\omega)e(j\omega) + n^H(j\omega)W^H(j\omega)G^H(j\omega)G(j\omega)W(j\omega)n(j\omega)]d\omega \quad (A.2)$$

where $n(j\omega)$ is the $K \times 1$ vector of Fourier transforms of the noise signals, $W(j\omega)$ is the $M \times K$ matrix of controller frequency responses, and $G(j\omega)$ the $L \times M$ matrix of plant frequency responses.

A cost function with more physical significance is obtained if a term proportional to the sum of squared input signals to the secondary sources, the control effort, is included:

$$J_4 = E[e^T(n)e(n) + \rho\, u^T(n)u(n)]. \quad (A.3)$$

The vector of input signals to the secondary sources can itself by written as

$$u(n) = X(n)w \quad (A.4)$$

where the matrix X(n) contains past and present values of the reference signals. Equation (A.4) enables the cost function J_4 to be again written in Hermitian quadratic form, this time with the Hessian matrix

$$\mathbf{A} = E[\mathbf{R}^T(n)\,\mathbf{R}(n) + \rho\,\mathbf{X}^T(n)\,\mathbf{X}(n)]. \tag{A.5}$$

The LMS algorithm which minimises J_4 can be written

$$\mathbf{w}(n+1) = (\mathbf{I} - \alpha\rho\,\mathbf{X}^T(n)\mathbf{X}(n))\mathbf{w}(n) - \alpha\,\hat{\mathbf{R}}^T(n)\,\mathbf{e}(n) \tag{A.6}$$

The algorithm converges if the real parts of all the eigenvalues of the matrix $E[\hat{\mathbf{R}}^T(n)\,\mathbf{R}(n) + \rho\,\mathbf{X}^T(n)\,\mathbf{X}(n)]$ are positive [10]. Since $\mathbf{X}^T(n)\mathbf{X}(n)$ must be positive definite if the filters are persistently excited, this term will tend to increase the real parts of the eigenvalues and thus stabilise the algorithm even in the presence of errors in the estimates of the plant response. If the properties of the reference signals are such that $\mathbf{X}^T(n)\mathbf{X}(n) = \mathbf{I}$, then clearly equation (A.3) is equal to equation (2.12) and the cost functions are the same. Generally, however, the behaviour of the adaptive algorithms which minimise these two cost functions are not identical [11].

An even more general cost function could be defined to be the weighted sum of the *filtered* error signals

$$f_l(n) = \sum_{p=1}^{P} a_{lp} e_l(n-p) \tag{A.7}$$

and *filtered* secondary source input signals

$$v_m(n) = \sum_{q=1}^{Q} b_{mp}\, u_m(n-q). \tag{A.8}$$

The coefficients of the two filters generating these signals could be chosen to emphasise particular frequency components. Because these signals depend on past samples of the error and secondary input signals, the definition of the vectors of these quantities must incorporate these quantities, for example as

$$\mathbf{f}(n) = \mathbf{A}\,\mathbf{e}(n,P) \tag{A.9}$$

where $\mathbf{e}(n,P) = [\mathbf{e}^T(n-1, \mathbf{e}^T(n-2) \ldots \mathbf{e}^T(n-P)]^T$ is an LP × 1 vector of past errors and **A** is a sparse L × LP matrix containing the coefficients a_{lp}. We can also define

$$\mathbf{v}(n) = \mathbf{B}\,\mathbf{u}(n,Q) \quad \text{(A.10)}$$

where $\mathbf{u}(n,Q) = [\mathbf{u}^T(n-1), \mathbf{u}^T(n-2) \ldots \mathbf{u}^T(n-Q)]^T$ is $MQ \times 1$ and $\mathbf{B}$ is a sparse $M \times MQ$ matrix containing the coefficients b_{mq}.

The most general cost function can now be written as

$$J_5 = E[\mathbf{f}^T(n)\,\boldsymbol{\theta}\,\mathbf{f}(n) + \mathbf{v}^T(n)\,\boldsymbol{\Phi}\,\mathbf{v}(n)] \quad \text{(A.11)}$$

in which $\boldsymbol{\theta}$ and $\boldsymbol{\Phi}$ are positive definite matrix which do not have to be diagonal. The augmented vector of error signals can be written as

$$\mathbf{e}(n,P) = \mathbf{d}(n,P) + \mathbf{R}(n,P)\mathbf{w} \quad \text{(A.12)}$$

where the definitions of $\mathbf{d}(n,P)$ and $\mathbf{R}(n,P)$ follow from that for $\mathbf{e}(n,P)$ above. Similarly, the augmented vector of plant input signals can be written

$$\mathbf{u}(n,Q) = \mathbf{X}(n,Q)\mathbf{w} \quad \text{(A.13)}$$

so that equation (A.11) can once again be written as a Hermitian quadratic function of the vector of controller coefficients $\mathbf{w}$, this time with the Hessian matrix equal to

$$\mathbf{A} = E[\mathbf{R}^T(n,P)\mathbf{A}^T\boldsymbol{\theta}\,\mathbf{A}\,\mathbf{R}(n,P) + \mathbf{X}^T(n,Q)\mathbf{B}^T\boldsymbol{\Phi}\,\mathbf{B}\,\mathbf{X}^T(n,Q)] \quad \text{(A.14)}$$

The LMS algorithm which minimises J_4 can be written as

$$\mathbf{w}(n+1) = [1 - \alpha\,\mathbf{X}^T(n,Q)\mathbf{B}^T\boldsymbol{\Phi}\,\mathbf{B}\,\mathbf{X}^T(n,Q)]\mathbf{w}(n) - \alpha\,\mathbf{R}^T(n,P)\mathbf{A}^T\boldsymbol{\theta}\;\mathbf{f}(n) \quad \text{(A.15)}$$

The various cost functions described here and in Section 2 are summarised in Table 1. All of the adaptive algorithms which minimise a cost function with a term proportional to the mean square controller response, or the mean square control effort, tend to be more robust to errors in the estimated response of the plant.

Table 1 - List of cost functions

$$J_1 = \sum_{l=1}^{L} \int_0^{2\pi} E[\,|E_l(j\omega)|^2]\, d\omega$$ error only

$$J_2 = \sum_{l=1}^{L} \int_0^{2\pi} E[\,|E_l(j\omega)|^2]\, d\omega + \beta \sum_{m=1}^{M} \sum_{k=1}^{K} \int_0^{2\pi} |W_{mk}(j\omega)|^2\, d\omega$$ error and controller

$$J_3 = \sum_{l=1}^{L} \int_0^{2\pi} E[\,|E_l(j\omega)|^2]\, d\omega + \sum_{l=1}^{L} \int_0^{2\pi} E[\,|O_l(j\omega)|^2]\, d\omega$$ error and weighted controller

where $O_l(j\omega) = \sum_{m=1}^{M} \sum_{k=1}^{K} G_{lm}(j\omega)\, W_{mk}(j\omega)\, N_k(j\omega)$

$$J_4 = \sum_{l=1}^{L} \int_0^{2\pi} E[\,|E_l(j\omega)|^2]\, d\omega + \rho \sum_{m=1}^{M} \int_0^{2\pi} E[\,|U_m(j\omega)|^2]\, d\omega$$ error and effort

$$J_5 = \sum_{l=1}^{L} \int_0^{2\pi} E[\,|F_l(j\omega)|^2]\, d\omega + \rho \sum_{m=1}^{M} \int_0^{2\pi} E[\,|V_m(j\omega)|^2]\, d\omega$$ filtered error and filtered effort

where $F_l(j\omega) = A_l(j\omega)\, E_l(j\omega)$ and $V_m(j\omega) = B_m(j\omega)\, U_m(j\omega)$

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