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WAVE MOTION IN A CURVED AND STIFFENED BEAM

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1. INTRODUCTION

Previous research into the propagation of waves in structures has concentrated on understanding the effect of discontinuities upon wave motion in simple straight beam elements. However, in many practical structures wave propagation occurs in more complex curved and stiffened elements, therefore there is a need to extend the analysis to this class of structure. In straight beam flexural and longitudinal (extensional) wave motions are uncoupled. For a curved beam, however, there is interaction between the longitudinal and bending deformations leading to coupled extensional-flexural propagation. In an elementary theory, Love[1] assumed that the centre-line remains unextended during flexural motion, whilst flexural behaviour is ignored when considering extensional motion. In the same reference [1] Love presented equations for thin shells which include the effects of extension of the mid-surface during bending motion. Flügge[2] has also derived equations for thin shells which include extension of the mid-surface during bending motion but has attempted a more careful discard of higher order terms. With some manipulation both these sets of equations can be reduced to expressions applicable to a curved beam[3]. Equations derived specifically for a curved beam are presented by Philipson[4] who included extension of the central line in the flexural wave motion, and also rotary inertia effects. In a development analogous to that to Timoshenko for straight beams, Morley[5] introduced a correction for radial shear when considering the vibration of curved beams. Graff later presented frequency versus wave number data for wave motion in a curved beam, for simple bending[6], and when including higher order effects[7].

In this paper the results of experimental studies of wave motion in a mild steel beam are reported, where it is assumed that the centre-line of the beam forms a plane curve of constant radius of curvature. In section two the harmonic response of a 'semi-infinite' curved beam with a single flexural stiffener is developed by considering the propagating and evanescent waves which travel in both directions along the beam. This method has previously been used to analyse the harmonic response of straight beams on periodic[8] and non-periodic supports[9]. In section three, the experimental apparatus is described and the measurement method outlined. Finally, in section four, the theoretical predictions are compared with laboratory measurements made on a curved and stiffened mild steel beam.

2. THEORY

2.1 **Wave motion in curved beams** The Flügge based analysis involves small displacements of thin beams, hence the assumptions known as 'Love's first approximation' in classical shell theory, can be made[1]. These assumptions impose the following linear relationships between the displacements of a material point U,W in the beam and components of displacement at the undeformed centre-line

$$U(r, s, t) = u(R, s, t) + z\,\phi(s, t) \tag{1}$$
$$W(r, s, t) = w(R, s, t) \tag{2}$$

where u and w are the components of displacement at the centre-line in the circumferential and radial directions, respectively, and ϕ is the rotation of the normal to the centre-line during deformation. A complete list of notation is given in the Appendix. Using these assumptions the equations of motion for a curved beam have previously been derived[3]: Solution of the equations of motions show that three types of positive-going waves can exist and dispersion curves for these waves are shown in Fig 1. The frequency range is expressed in terms of the non-dimensional frequency $\Omega = \omega R/c_o$, where c_o is the phase velocity of extensional waves in a straight bar, whilst the wave number range is expressed in terms of the non-dimensional wave number, kR. From the dispersion curves and associated extensional to flexural wave amplitude ratios, two different frequency regimes can be identified, *above* and *below* the ring frequency, $\Omega = 1$. Above the ring frequency three wave types can exist: flexural travelling wave, flexural near field wave and extensional travelling wave. Below the ring frequency the flexural travelling and near field waves still exist, however, the extensional wave is now evanescent.

2.2 **Point response of an 'infinite' curved beam** By analogy to the infinite system point response functions for straight beams presented in (9) consider now an infinite *curved* beam subjected to an harmonic force $F_o e^{i\omega t}$ acting in the radial direction at the origin s=0. The response due to the force to the right of s=0 can be expressed as the sum of the propagating and evanescent waves shown in the dispersion curves in Fig 1. Thus the flexural displacement is given by

$$w_+(s) = A_1 e^{-k_1 si} + A_2 e^{-k_2 si} + A_3 e^{-k_3 si} \tag{3}$$

where A_1, A_2, and A_3, represent the unknown wave amplitudes and k_1, k_2 and k_3 the wave numbers of the three wave types, ie. flexural travelling, flexural nearfield and extensional, respectively. For clarity of notation the harmonic term, $e^{i\omega t}$, has been omitted. The value of the unknown coefficients, A_i, can be found by considering the relevant equilibrium and compatibility conditions at the excitation point. In this case the following assumptions have been made:

i the extensional motion is zero, ie. $u_+(0) = 0$

ii the slope is zero, ie $\varphi_+(0) = \dfrac{u_+(0)}{R} - \dfrac{\partial w_+(0)}{\partial s} = 0$ (4)

iii the sheer force is half the excitation force, ie.

$$S_+(0) = -EI\,\frac{\partial}{\partial s}\left(\frac{w_+(0)}{R^2} + \frac{\partial^2 w_+(0)}{\partial s^2}\right) = \frac{F_o}{2}$$

The corresponding extensional displacement is given by

$$u_+(s) = B_1 e^{-k_1 si} + B_2 e^{-k_2 si} + B_3 e^{-k_3 si} \tag{5}$$

where the extensional to flexural wave amplitude ratio, (B_i/A_i), is found from the equations of motion. The response to the left of the force can be found in a similar manner.

2.3 **Response of a semi-infinite curved beam with a single flexural stiffener** Consider the semi-infinite curved beam of Fig 2 which is excited by a harmonic force, F_0, acting in the radial direction at $s=s_0$. The response at a position s_r along the beam is due to:

i the forced waves generated by the external force
ii the free waves reflected from the boundary at the free end, and
iii the waves generated by the unknown transverse reaction force at the simple support location

Thus the flexural displacement is given by

$$w(s_r) = L_1 e^{-k_1 s_r i} + L_2 e^{-k_2 s_r i} + L_3 e^{-k_3 s_r i}$$

$$+ A_1 e^{-k_1 (s_r - s_0) i} + A_2 e^{-k_2 (s_r - s_0) i} + A_3 e^{-k_3 (s_r - s_0) i} \tag{6}$$

$$+ D\left[A_{1L} e^{k_1 (s_r - s_1) i} + A_{2L} e^{k_2 (s_r - s_1) i} + A_{3L} e^{k_3 (s_r - s_1) i}\right]$$

where L_1, L_2 and L_3 are the unknown coeficients of the waves reflected from the free end and D the unknown coefficient due to the reaction force on the simple support. The coefficients A_1, A_2 and A_3 are the amplitudes of the infinite curved beam system of section 2.2 where a unit force, $F_0 = 1$, has been assumed and the subscript L indicates left going waves.

The value of the unknown coefficients can be found by satisfying the boundary conditions at the support location and at the free end of the beam. At the support location the following boundary conditions are assumed:

i the transverse displacement is zero, ie $w_+(s_1) = 0$

ii the bending moment is zero, ie $M_+(s_1) = -EI\left(\frac{w_+(s_1)}{R^2} + \frac{\partial^2 w_+(s_1)}{\partial s^2}\right) = 0$ (7)

At the free end of the beam the following boundary conditions are assumed:

i the axial force is zero, ie.

$$N_+(0) = SE\left(\frac{w_+(0)}{R} + \frac{\partial u_+(0)}{\partial s}\right) + \frac{EI}{R}\left(\frac{w_+(0)}{R^2} + \frac{\partial^2 w_+(0)}{\partial s^2}\right) = 0$$

ii the bending moment is zero, ie. $M_+(0) = 0$ (8)

iii the sheer force is zero, ie. $S_+(0) = 0$

The flexural displacement at any point along the beam can now be found by substituting the values of coefficients L_1, L_2 and L_3, and D into Eqn (6), taking acount of the left going or right going nature of the waves.

3. EXPERIMENTAL APPARATUS AND METHOD

The test structure consisted of a curved 5m long mild steel beam, 50 x 6.5mm section, with a constant radius of curvature of 6.366m. A 'semi-infinite' beam was obtained by inserting one end into an anechoic termination consisting of a 1m long box filled with sand to dissipate the energy of the wave motion. To attenuate flexural waves through a range of frequencies foam wedges were inserted, and to attenuate longitudinal waves thin aluminium strips of various lengths (5 - 10mm) were attached at right angles to that portion of the beam which lay in the sandbox. At a distance of 1.45m from the free end of the beam flexural motion was restrained by pinning the beam to a grounded support. A schematic representation of the experimental apparatus is shown in Fig. 3.

The free end of the beam was struck with an instrumental hammer which measured the excitiion force, whilst the response was measured using accelerometers mounted in the circumferential and radial directions. All data were recorded simultaneously in a multi-channel FFT analyser. This was connected to apersonal computer which performed the necessary post processing to calculate point and cross mobility functions.

4. RESULTS

Comparisons of the measured point mobility with theoretical predictions of the point mobility of an *unstiffened* 'semi-infinite' curved beam having the same dimensions and material properties as the test structure have previously been reported[10]. These showed that the measured data contained resonant peaks corresponding to the finite length of the experimental beam, however, the data followed the trend predicted for a 'semi-infinite' curved beam. In this paper, the effect of introducing a flexural stiffener will be reported. Fig. 4 shows the modulus of the point mobility of the beam when the free end is excited in the radial direction. The frequency range is presented on a non-dimensional logarithmic scale from $\Omega = 0.1$ to $\Omega = 10.0$, this corresponds to a dimensional frequency range of 13 Hz to 1300 Hz. Comparison of the measured resonance peaks with the corresponding theoretically predicted natural frequencies shows that the measured data is lower in frequency and in peak amplitude than the corresponding theoretically predicted values. This difference is attributable to the damping in the anechoic termination, whereas the theoretical model assumes on undamped beam. The predicted natural frequencies are approximately those of a straight Euler-Bernoulli beam of length 1.45m with pinned-free boundary conditions. This indicates that the response due to radial excitation is dominated by the effect of the flexural stiffener.

The modulus of the cross mobility (circumferential velocity per unit radial force) is shown in Fig. 5, and the point and cross mobility of the structure when excited in the circumferential direction are shown in Figs. 6 and 7, respectively. The point mobility of the curved and stiffened beam, Fig 6, exhibits similar behaviour to that of the unstiffened beam discussed previously in (10), where it was noted that two frequency regimes can be identified : *above* and *below* the ring frequency, $\Omega = 1$. The cross mobility shown in Fig. 7 exhibits the same characteristics as the cross mobility due to radial excitation shown in Fig. 5. In fact, because of reciprocity the radial response due to circumferential excitation should be identical to the circumferential response due to radial excitation. The close similarity between the data in Figs. 5 and 7 confirms the validity of both the measured and theoretical results.

5. DISCUSSION AND CONCLUSIONS

The paper has reported measurements of the point and cross mobility of a curved and stiffened 'semi-infinite' beam. A corresponding theoretical model was developed by considering the propagating and evanescent waves which travel in both directions along the beam. Analysis of both the measured and predicted data leads to the following conclusions:

i for radial excitation the point response is dominated by the stiffening of the beam
ii for circumferential excitation the point response is dominated by the curvature of the beam.

ACKNOWLEDGEMENT

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APPENDIX : NOTATION

A_i i^{th} coefficient of infinite system point response function in radial direction

B_i i^{th} coefficient of infinite system point response function in circumferential direction

E Young's modulus

F_o magnitude of externally applied force

I second moment of area of beam cross-section

L_i i^{th} coefficient of wave reflected from free end

M bending moment on beam cross-section

N circumferential force on beam cross-section

R radius of curvature

S sheer force on beam cross-section

U displacement of material point in circumferential direction

W displacement of material point in radial direction

c_o wave speed of extentional waves in straight bar

k wave number

r coordinate in radial direction

s coordinate in circumferential direction

t time

u displacement at centre-line in circumferential direction

w displacement at centre-line in radial direction

z coordinate of outward pointing normal to centre-line

Ω non-dimensional frequency

ϕ change of slope of normal to centre-line during deformation

ω radian frequency

Subscripts

+ positive direction

L left going

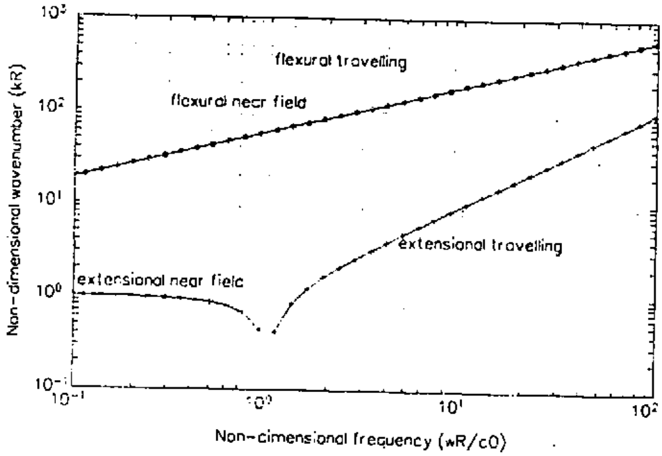

Figure 1: Wave number v. frequency relationship for curved beam ('o' flexural travelling wave and near field wave, '+' extensional travelling wave, 'x' extensional near field wave)

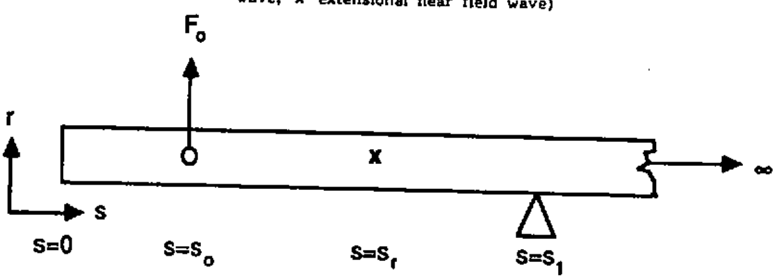

Figure 2 - Semi-infinite curved beam with a single flexural stiffener at $s=s_1$ excited by harmonic force in radial direction at $s=s_0$

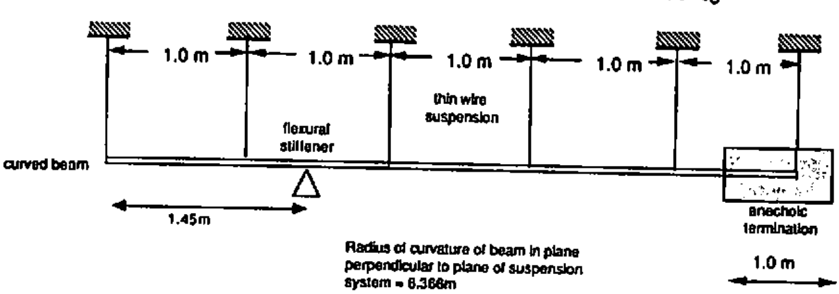

Figure 3 Schematic representation of the experimental apparatus, side view

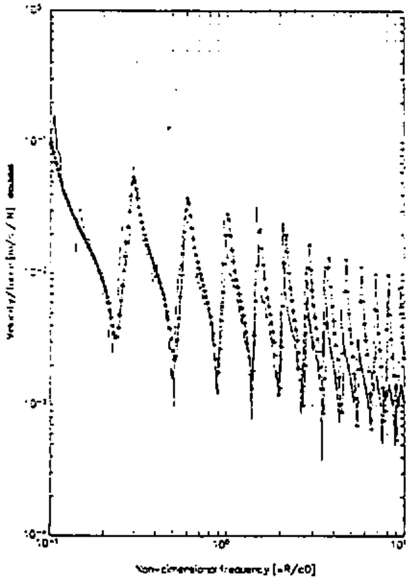

Figure 4 - Comparison of measured and predicted (*) values of point mobility of curved and stiffened beam : Radial Excitation

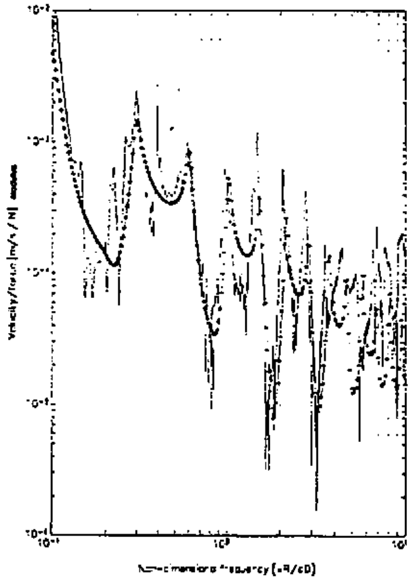

Figure 5 - Comparison of measured and predicted (*) values of cross mobility of curved and stiffened beam : Radial Excitation

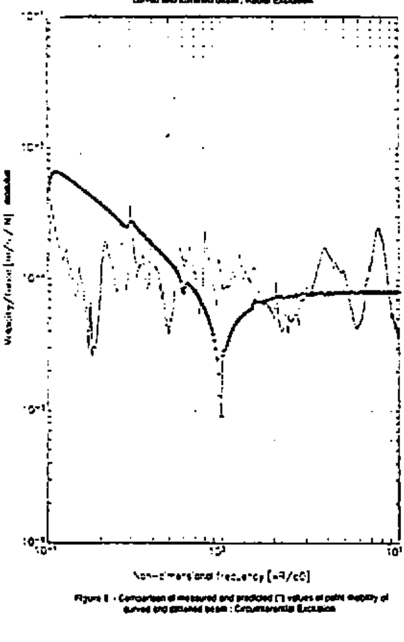

Figure 6 - Comparison of measured and predicted (*) values of point mobility of curved and stiffened beam : Circumferential Excitation

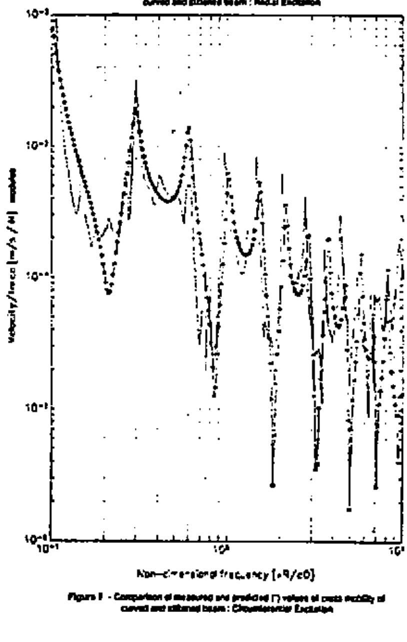

Figure 7 - Comparison of measured and predicted (*) values of cross mobility of curved and stiffened beam : Circumferential Excitation