NUMERICAL METHODS FOR HIGH FREQUENCY ACOUSTIC SCATTERING PROBLEMS

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1 INTRODUCTION

Acoustic wave scattering problems arise in application areas as diverse as the modelling of sonar, acoustic noise barriers, and ultrasound. In many practical applications the characteristic length scale L of the domain is large compared to the wavelength λ. In such cases the small dimensionless wavelength λ / L leads to oscillatory solutions. In order to resolve these solutions, standard numerical schemes with piecewise polynomial approximation spaces are faced with a requirement for a fixed number of degrees of freedom per wavelength, and in order to maintain accuracy this leads to at least a linear growth in the number of degrees of freedom required as λ / L decreases. This debilitating restriction has led to much recent research effort focused on the development of schemes whose computational cost does not increase significantly as λ / L decreases[1].

We consider here the situation in which a time harmonic incident plane wave $u^i(x) = \exp(ikx.d)$, with the unit vector d denoting the direction and with $k = 2\pi / \lambda$ denoting the wavenumber, is scattered by a bounded two dimensional sound soft obstacle Ω, to produce a radiating scattered wave u^s. The total wave $u = u^i + u^s$ then satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0, \text{ in } D = R^2 / \Omega,$$

together with the boundary condition $u = 0$ on the boundary Γ. For a problem such as this where the computational domain is infinite, and under the further assumption that the media is homogeneous, a very popular solver is the boundary element method. Using the fundamental solution

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k \,|\, x - y \,|),$$

for $x, y \in R^2$, $x \neq y$, where $H_0^{(1)}$ denotes the Hankel function of the first kind of order zero, it follows from Green's Theorem that the scattering problem can be reformulated as the boundary integral equation

$$\frac{\partial u}{\partial n}(x) + \int_\Gamma \left(\frac{\partial \Phi(x, y)}{\partial n(x)} + i\eta \Phi(x, y) \right) \frac{\partial u}{\partial n}(y) ds(y) = f(x), \quad x \in \Gamma,$$

where $\partial / \partial n$ denotes normal derivative, $\eta > 0$ is a coupling parameter (ensuring that the integral equation has a unique solution), f is determined by the incident wave, and $\partial u / \partial n$ is to be found.

Standard boundary element methods approximate the whole oscillatory function $\partial u / \partial n$ by piecewise polynomials, with the cost required to maintain accuracy increasing at least linearly with respect to k as described above. Here we instead consider hybrid approximation spaces, incorporating knowledge of the high frequency asymptotic behaviour into the approximation space in order to decrease the dependence on k of the number of degrees of freedom required to maintain accuracy as k increases. We focus our attention on convex scattering obstacles, allowing us to derive the appropriate approximation spaces, as detailed below.

2 HYBRID APPROXIMATION SPACES

Instead of approximating $v := \partial u / \partial n$ directly by piecewise polynomials, the key to achieving efficient approximations at high frequencies is to include the phase of the high frequency asymptotic solution in the approximation space. The general form of the approximation is

$$v(x,k) \approx \sum_{m=1}^{N} k \exp(ik\gamma_m(x)) V_m(x,k), \quad x \in \Gamma,$$

where the phase functions γ_m are chosen *a-priori* and only the unknowns $V_m(x,k)$ are approximated (with respect to x) by piecewise polynomials. Asymptotic analysis can be used to determine the γ_m in such a way that the $V_m(x,k)$ are very much less oscillatory than the original $\partial u / \partial n$, and hence much more amenable to approximation by piecewise polynomials.

2.1 Smooth convex obstacles

For smooth convex obstacles the approach described above has received much attention in recent years, but it is only in the recent paper by Dominguez, Graham and Smyshlyaev[3] that it has been backed up by a rigorous numerical analysis. In this case one takes $N = 1$ and $\gamma_1(x) = x.d$ in the geometrical optics ansatz above, giving

$$v(x,k) \approx \exp(ikx.d) V(x,k), \quad x \in \Gamma,$$

and then as $k \to \infty$ one finds that the equation holds with $V(x,k) \approx 2$ on the illuminated side and $V(x,k) \approx 0$ on the shadow side of the obstacle. Care is required near to the shadow boundary, but full details are given in the literature as to how this difficulty can be resolved[3]. Using a p-version Galerkin boundary element method within this framework to approximate $V(x,k)$, a rigorous error analysis[3] demonstrates that uniformly accurate approximations are obtained as $k \to \infty$ provided the number of degrees of freedom grows like O($k^{1/9}$).

2.2 Convex polygons

The approach described above for smooth convex obstacles is not appropriate when corners are present due to the diffracted waves which occur in this case. For the case of convex polygons, the appropriate form of the geometrical optics ansatz described above is

$$v(x,k) \approx P.O. + k \sum_{m=1}^{N} [\exp(ikx.d_m) V_m^+(x,k) + \exp(-ikx.d_m) V_m^-(x,k)], \quad x \in \Gamma,$$

where $P.O.$ is the physical optics approximation, N now denotes the number of sides of the polygon, the unit vector d_m is parallel to the m^{th} side, and the functions $V_m^{\pm}$ are non-zero only on the m^{th} side (physically the terms in the summation represent the diffracted field). It can then be shown via a rigorous numerical analysis[2] that if the functions $V_m^{\pm}$ are approximated by piecewise polynomials on carefully chosen graded meshes, then uniformly accurate approximations are obtained as $k \to \infty$ provided the number of degrees of freedom grows like O($\log^{3/2} k$).

2.3 Convex curvilinear polygons

For the case of convex curvilinear polygons, a very similar geometrical optics ansatz to that used for convex polygons can be used, specifically,

$$v(x,k) \approx k \exp(ikx.d) V(x,k) + k \sum_{m=1}^{N} [\exp(ikx.d_m) V_m^+(x,k) + \exp(-ikx.d_m) V_m^-(x,k)], \quad x \in \Gamma.$$

The only change here is that the physical optics approximation is not known explicitly in this case, and hence the function $V(x,k)$ must also now be approximated, as well as the functions $V_m^\pm$. Approximating $V(x,k)$ by piecewise polynomials supported on a uniform mesh on the illuminated side only, and approximating the functions $V_m^\pm$ by piecewise polynomials on identical graded meshes to those used in the case of a straight sided convex polygon, numerical results have been produced[4] showing that it appears to be possible to obtain uniformly accurate approximations as $k \to \infty$ provided the number of degrees of freedom grows like O($\log^{3/2} k$), the same rate as for the case of a straight sided polygon.

3 CONCLUSIONS

In this short paper we have outlined the basic ideas behind some recent approaches to the problem of approximating the solution of high frequency acoustic scattering problems in two dimensions by bounded sound soft convex obstacles. By incorporating appropriate oscillatory functions into the approximation space, it is possible to demonstrate via a rigorous numerical analysis that the number of degrees of freedom need grow at a substantially sublinear rate with respect to an increase in the frequency of the incident field or the size of the obstacle, for the case of smooth convex obstacles[3] and for the case of convex polygons[2]. For the case of convex curvilinear polygons, recent numerical results[4] support the conjecture that a similar level of performance can be achieved. For full details of each of these schemes we refer to the relevant references detailed below, and for a full description of the recent developments in this field we refer to the recent survey paper by Chandler-Wilde and Graham[1].

4 REFERENCES

1. S. N. Chandler-Wilde and I. G. Graham, ‘Boundary integral methods in high frequency scattering’, to appear in ‘Highly Oscillatory Problems’, B. Engquist, T. Fokas, E. Hairer, A. Iserles, editors, Cambridge University Press, 2008.
2. S. N. Chandler-Wilde and S. Langdon, ‘A Galerkin boundary element method for high frequency scattering by convex polygons’, SIAM J. Numer. Anal., 45, 610-640, 2007.
3. V. Dominguez, I. G. Graham and V. P. Smyshlyaev, ‘A hybrid numerical-asymptotic boundary integral method for high-frequency acoustic scattering’, Numer. Math., 106, 471-510, 2007.
4. S. Langdon, M. Mokgolele and S. N. Chandler-Wilde, ‘High frequency scattering by convex curvilinear polygons’, submitted for publication, 2007.