

# LOCALISED ELASTIC WAVES IN STRUCTURES OF COMPLEX GEOMETRY

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## 1 INTRODUCTION

Localised elastic waves in solids bounded by surfaces of complex geometry are often present in nature, for example as seismic waves resulting from earthquakes and propagating along mountain ridges. They also appear in a variety of engineering applications, where they can be associated with specific vibration modes of different plate-like structures, etc. In the present paper, it is demonstrated that the existence of localised modes of Rayleigh waves on solid surfaces of complex topography or guided quasi-flexural plate waves in different plate-like structures can be understood using the classical definition of open acoustic waveguides utilizing the condition of total internal reflection of Rayleigh or plate waves from the areas surrounding the 'internal' area of wave localisation. It should be noted that the possibility of total internal reflection in structures of complex geometry is often linked to the presence of internal areas on the surfaces characterised by the geometry-dependent local phase velocities of Rayleigh or plate waves that are smaller in the direction of guided wave propagation than their values in the surrounding areas. This is similar to the well-known case of guided wave propagation in atmospheric or underwater Acoustics.

The above-mentioned condition of wave localisation is illustrated in this paper by theoretical calculations of frequency-dependent phase velocities of guided modes for three different cases of localised elastic wave propagation. These are localised Rayleigh waves propagating along solid cylinders of variable curvature, localised flexural waves in slender elastic wedges (also known as wedge elastic waves), and localised quasi-flexural waves in non-circular cylindrical shells. The results for dispersion curves of localised waves are compared with the known solutions, where available.

## 2 TOPOGRAPHIC WAVEGUIDES FOR RAYLEIGH WAVES

### 2.1 General Comments

To understand the physical nature of Rayleigh wave localisation on surfaces of variable curvature it is instructive to apply Geometrical acoustics approximation (GA) to Rayleigh wave propagation on such surfaces. Note that GA is used widely in underwater acoustics or for solving problems of sound propagation in inhomogeneous atmosphere<sup>1</sup>. However, in the acoustics of solids its use is not so frequent, which can be partly explained by the complexity of real inhomogeneous solid structures, such as bodies with curved surfaces, plates of variable thickness, noncircular shells of arbitrary shape, etc. In the same time, the use of GA to describe wave propagation in such inhomogeneous solids is very efficient, and in many cases it can serve as a key to understanding of the physical nature of wave propagation in such solids and structures<sup>2</sup>.

Geometrical acoustics approximation is an asymptotic high-frequency solution (sometimes called ray-tracing solution) to the differential equations with boundary conditions describing wave propagation in any particular medium or structure. In this section, a brief overview of the developments of the geometrical acoustics theory for Rayleigh waves in solids of complex topography is given, based mainly on the original results of the present author. Initially, the propagation of Rayleigh waves along arbitrary curved surfaces is considered. The obtained results

are then applied to the description of localised modes in the so-called smooth topographic waveguides for surface waves.

## 2.2 Propagation of Rayleigh Waves on Curved Surfaces

It is well known that if a Rayleigh wave propagates along a curved surface its velocity changes due to the effect of curvature<sup>2</sup>. Generally, curved surfaces represent anisotropic and inhomogeneous media for propagating Rayleigh waves. As it is obvious from general geometry, only a spherical surface is both isotropic and homogeneous. A circular cylindrical surface is anisotropic, but homogeneous, and a non-circular cylindrical surface is both anisotropic and inhomogeneous (in the direction perpendicular to the cylindrical axis). Thus, if the surface of a solid body is curved, then there are two main features associated with propagation of high-frequency Rayleigh waves: these are the anisotropy of the local wave velocity and the inhomogeneity of the medium due to a variable surface curvature.

The starting point for the geometrical acoustics approximation for Rayleigh waves on curved surfaces of arbitrary form is the high-frequency asymptotic expression for the local Rayleigh wave velocity as a function of two local radii of surface curvature<sup>2-4</sup>:

$$c = c_0 \left( 1 + a_u \frac{1}{k_0 \rho_u} + a_v \frac{1}{k_0 \rho_v} \right). \quad (1)$$

Here  $c_0$  is the Rayleigh wave velocity on a flat surface,  $k_0 = \omega/c_0$  is the corresponding wavenumber,  $\rho_u$  and  $\rho_v$  are the radii of the surface curvature in the direction of wave propagation and in the direction perpendicular to it respectively,  $a_u$  and  $a_v$  are the non-dimensional coefficients that depend on Poisson's ratio of the medium. Note that expression (1) has been established by several researchers, including the present author (see Ref 2 for details). The values of  $a_u$  and  $a_v$  for all values of Poisson's ratio can be found in Ref 2. The important fact to be mentioned here is that  $a_u$  is always positive, whereas  $a_v$  is always negative. The latter feature is paramount for Rayleigh wave localisation, and, as will be discussed in the next section, it is responsible for the existence of smooth topographic waveguides for Rayleigh waves.

If the two radii of curvature  $\rho_u$  and  $\rho_v$  are known as functions of surface coordinates, the usual formalism of geometrical acoustics (in scalar approximation) can be applied to describe either vertical or horizontal component of a propagating Rayleigh wave in the arbitrary point of the curved surface. In order to do so, one should initially establish the trajectories of all possible rays that can be traced from a chosen point of Rayleigh wave excitation. After the ray trajectories have been established, the solution for a wave propagating along any particular trajectory, using a surface coordinate  $s$  measured along the trajectory, can be written in the form<sup>1,2</sup>

$$u = A(s) e^{i \int k(s) ds}, \quad (2)$$

where  $A(s)$  and  $k(s) = \omega/c$  are slowly varying functions of  $s$  describing the amplitude and the local wavenumber of the Rayleigh wave.

To calculate ray trajectories of Rayleigh waves propagating over surfaces of variable curvature it is convenient to use the Hamiltonian approach<sup>2,3</sup>. For example, in the case of a smooth noncircular cylinder shown in Figure 1, the calculated ray trajectories<sup>3</sup> of Rayleigh waves propagating from the point of excitation located in the area of maximum curvature are shown in Figure 2. It can be seen that Rayleigh waves incident at the angles larger than 82 degrees experience total internal reflection and thus become captured within the area of maximum curvature corresponding to the minimum of phase velocity in the x-direction. This constitutes a waveguide effect of such inhomogeneous surfaces.

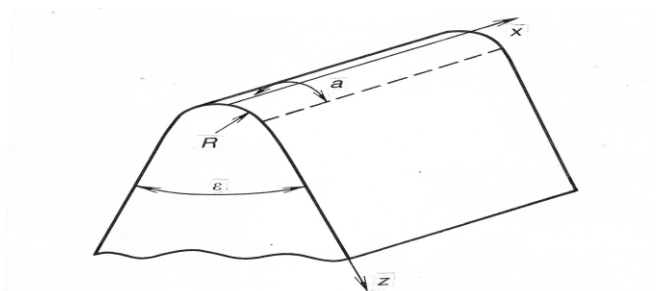


Figure 1. Geometry of a noncircular cylinder.

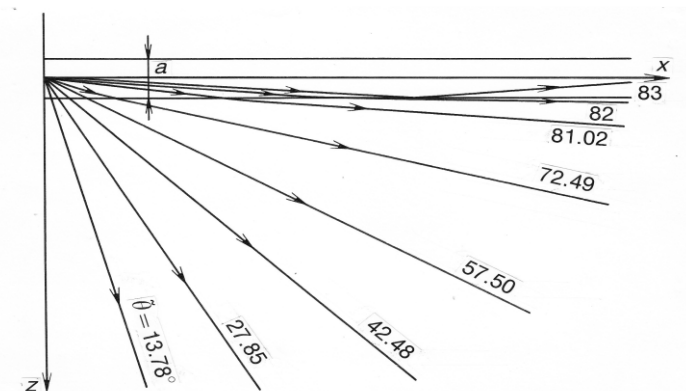


Figure 2. Ray trajectories of Rayleigh waves propagating at arbitrary angles in respect of the axis of a noncircular cylinder<sup>3</sup>.

### 2.3 Smooth Topographic Waveguides for Rayleigh Waves

As was shown above, non-circular cylindrical surfaces can support guided Rayleigh waves. If the waveguiding properties are attributed to the influence of surface geometry, as in the case considered, then the associated waveguides are often called 'topographic waveguides'. The need to take into account guiding properties of surfaces of complex geometry appears in seismology and in different applications of ultrasonic non-destructive testing. For example, in condition monitoring using acoustic emission, it is often necessary to be able to predict the most likely directions of propagation for Rayleigh waves radiated by a developing crack. Since the likely locations of emerging cracks can be anticipated, it is important to be able to predict possible paths of guided waves propagation, where placement of acoustic emission sensors would be most efficient.

A rigorous analysis of topographic waveguides is rather difficult. As a rule, it cannot be done analytically and requires numerical approaches. However, there are several important cases that can be considered using approximate analytical approaches. Among these cases are smooth topographic waveguides (see Figure 3), for which the minimum radius of surface curvature is greater than the Rayleigh wavelength. Such waveguides can be considered in geometrical acoustics approximation on the basis of the asymptotic expression (1) for the local velocity of Rayleigh waves.

The first geometrical acoustics consideration of smooth topographic waveguides and the first physical interpretation of the Rayleigh wave localisation in these and similar topographic structures have been given by the present author<sup>4</sup> (see also the monograph<sup>2</sup>, where a special chapter is devoted to Rayleigh waves on curved surfaces of arbitrary form (Chapter 9), and another chapter (Chapter 10) describes waves in topographic waveguides, including smooth topographic waveguides (see Section 10.5)). The geometrical acoustics theory of smooth topographic waveguides provides a clear and physically explicit explanation of the reason for the presence of waveguide effect in such structures. Namely, it demonstrates that the existence of propagating localised modes of Rayleigh waves in smooth topographic waveguides can be explained by the presence of an 'internal' area on the surface with the curvature-dependent local phase velocity of Rayleigh waves that is smaller in the direction of guided wave propagation than its values for adjacent flat surfaces (with zero curvature). The sequence of this is the possibility of total internal reflection of the curvature-modified Rayleigh waves from the surrounding areas of zero or negative surface curvature<sup>2, 4</sup>.

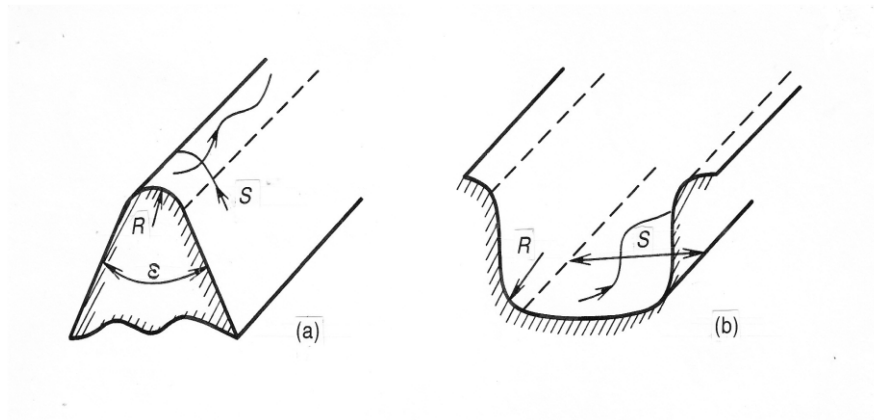


Figure 3. Ridge and groove types of smooth topographic waveguides<sup>4</sup>.

For illustration purposes, let us consider a brief derivation of the dispersion equation for symmetric guided modes of the smooth topographic ridge-type structure shown in Figure 3(a). This structure is formed by a part of a circular cylinder of radius  $R$  and by two flat surfaces positioned at the angle  $\varepsilon$ . Applying formula (1) to a Rayleigh wave propagating along a circular cylindrical surface at an angle  $\Phi$  in respect of the element of cylinder and using Euler's formulas for the radii of curvature  $\rho_u$  and  $\rho_v$  in the case of circular cylinder of radius  $R$ , one can obtain the following expression for Rayleigh wave velocity on the curved part of the structure<sup>2</sup>:

$$c = c_0 \left\{ 1 + \frac{1}{k_0 R} \left[ (a_u - a_v) \sin^2 \Phi + a_v \right] \right\}. \quad (3)$$

The dependence of the velocity  $c$  on the angle  $\Phi$  in (3) describes the anisotropy of Rayleigh wave velocity on circular cylindrical surfaces, even if the material is isotropic. We remind the reader that formula (3) is valid for  $k_0 R \gg 1$ .

The described smooth ridge-type structure with the surface of variable curvature can be considered as a three-layered plane medium in respect of Rayleigh wave propagation, where the internal (curved) area is characterised by the Rayleigh wave velocity  $c$  described by formula (3) and the two side (flat) areas are characterised by Rayleigh wave velocity on the flat surface  $c_0$ . Therefore, in order to analyse guided surface wave propagation along a solid cylinder of variable radius, one can apply the standard dispersion equation for a three-layered scalar medium, in which one should

take into account the anisotropy of Rayleigh wave velocity  $c$  in the internal area. For example, for the  $m$ -th symmetric guided mode this equation has the form<sup>2</sup>:

$$\frac{a}{2} \left[ k_c^2(\Phi) - \gamma^2 \right]^{1/2} = m \frac{\pi}{2} + \tan^{-1} \left( \frac{\gamma^2 - k_0^2}{k_c^2(\Phi) - \gamma^2} \right)^{1/2}, \quad (4)$$

where  $\gamma = k_c \cos \Phi$  is the constant (wavenumber) describing wave propagation in a waveguide,  $a = (\pi - \varepsilon)R$  is thickness of the internal layer (see Figure 3(a)), and  $k_c(\Phi) = \omega/c(\Phi)$  is the wavenumber of a Rayleigh wave on the curved surface, where Rayleigh wave velocity  $c(\Phi)$  is defined by formula (3). Combining the expressions (3) and (4) and resolving them in respect of  $\gamma$ , one can obtain, after some simple rearrangements, that the expression for  $\gamma$  for the lowest order mode takes the form:

$$\gamma = k_0 \left( 1 + \frac{\beta_1}{2k_0 R} - \frac{\beta_1^{1/2}}{\pi - \varepsilon} \frac{1}{(k_0 R)^{3/2}} \right), \quad (5)$$

where  $\beta_1 = -2a_v$ . As expected, the expression (5) describes the waveguide propagation of Rayleigh waves at the velocity that is slightly lower than the velocity of Rayleigh waves on a flat surface  $c_0$ . The expression (5) coincides in form with the first terms of the expansion for  $\gamma$  earlier obtained by a direct method of solving the corresponding boundary problem<sup>5</sup> (see also other references in the monograph<sup>2</sup>). However, it is important to emphasise that the geometrical acoustics approach described in this section is physically explicit and incomparably simpler. The amplitude distributions of guided modes in the lateral direction can be easily constructed using the expression (5) (the details are not shown here for brevity). It is useful to mention though that the wave amplitudes decay exponentially away from the curved area.

In light of the above, it is interesting to note that research papers continue to appear in which the authors attempt to give other physical interpretations of the waveguide effect of surface topography, which are different from the above-mentioned one based on geometrical acoustics. For example, in the paper<sup>6</sup>, the authors consider asymptotically and numerically Rayleigh wave propagation in a topographic waveguide formed by a smooth ridge-type elevation over a flat surface. Although the factual results obtained by the authors seem to be correct, their physical interpretation of the Rayleigh wave localisation phenomenon does not look convincing. In contrast to that, the approach based on geometrical acoustics approximation described in this section provides a clear physical explanation of the guiding properties of surface topography for all structures of this type and gives numerical results for phase velocities of guided surface modes that are in good agreement with numerical calculations and with experiments.

### 3 LOCALISED FLEXURAL WAVES IN SLENDER WEDGES

In addition to Rayleigh waves on curved surfaces considered in the previous section, very important are also Lamb waves propagating in plates of variable thickness. The most important modes of Lamb waves are lowest order antisymmetric and symmetric modes. In noise and vibration engineering, these modes are usually called flexural and compression waves respectively. In what follows, a particular type of plates of variable thickness will be discussed – plates with local thickness variable in the lateral direction and reducing to zero at the plate edge. Especially interesting is the case of linear reduction of a plate local thickness with the distance, i.e. the case of slender elastic wedges of linear profile (Figure 4). To develop a geometrical acoustics theory of flexural or compression wave propagation in slender wedges one can start from the corresponding reduced equations for flexural and compression waves respectively and seek the solutions of these equations in the form similar to equation (2). It can be shown that equation (2), containing local wavenumbers of flexural and compression waves respectively, satisfies both these equations

asymptotically at relatively high frequencies<sup>7, 8</sup>. In practice though this often means that frequencies are well within the range of practically used frequencies.

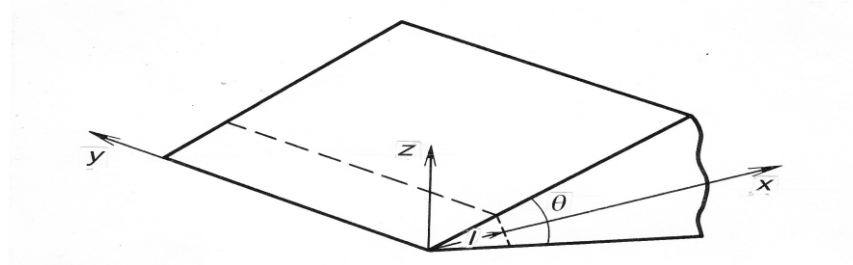


Figure 4. Geometry of a slender elastic wedge.

### 3.1 Geometrical Acoustics Theory of Wedge Elastic Waves

One of the important applications of geometrical acoustics to antisymmetric Lamb modes is the development of the analytical theory of localised elastic waves propagating along the tips of slender elastic wedges. Such localised waves are also known as 'wedge elastic waves' or 'wedge acoustic waves'. Their existence has been first predicted using numerical calculations<sup>9,10</sup>. The geometrical acoustics theory of wedge elastic waves has been first developed by the present author<sup>7,8</sup> (see also the monograph<sup>2</sup>). For the case of wedges of linear profile, this theory is based on the geometrical acoustics approach to flexural wave propagation in a slender wedge considered as a plate with a local variable thickness  $d = x\theta$ , where  $\theta$  is the wedge apex angle and  $x$  is the distance from the wedge tip measured in the middle plane (see Figure 4). These wedges can be considered as continuously varying media for propagation of flexural waves because the velocity of the latter depends on the local wedge thickness proportionally to the square root from  $d$ .

In the framework of the geometrical acoustics approach, the velocities  $c$  of the localised flexural modes propagating in  $y$  direction can be determined as the solutions of the Bohr - Sommerfeld type equation<sup>2, 7, 8</sup>:

$$\int_0^{x_t} [k^2(x) - \beta^2]^{1/2} dx = \pi n, \quad (6)$$

where  $\beta = \omega/c$  is yet unknown wavenumber of a propagating wedge mode,  $k(x)$  is a local wavenumber of a flexural wave in a plate of variable thickness,  $n = 1, 2, 3, \dots$  is the mode number, and  $x_t$  is the so called ray turning point (the point of total internal reflection) that can be determined from the equation  $k^2(x) - \beta^2 = 0$ . For example, in the case of a wedge in vacuum  $k(x) = 12^{1/4} k_p^{1/2} (\theta x)^{-1/2}$ ,  $x_t = 2\sqrt{3} k_p / \theta \beta^2$ , and  $k_p = \omega/c_p$ , where  $\omega$  is circular frequency,  $c_p = 2c_l(1 - c_t^2/c_l^2)^{1/2}$  is the so called plate wave velocity,  $c_l$  and  $c_t$  are propagation velocities of longitudinal and shear acoustic waves in a plate material. Performing the integration in (6), which in this case can be done analytically, and solving the resulting algebraic equation yields the extremely simple analytical expression for velocities of different modes of wedge elastic waves<sup>7,8</sup>:

$$c = \frac{c_p n \theta}{\sqrt{3}}. \quad (7)$$

It can be seen from (7) that all wedge modes are non-dispersive, which was expected for a wedge of linear profile described by a single non-dimensional parameter, angle  $\theta$ . Note that the expression

(7) agrees well with the earlier developed numerically based theories<sup>9, 10</sup> and with the available experimental results. The analytical expressions for amplitude distributions of localised wedge modes are rather cumbersome<sup>7</sup> and are not displayed here. Figure 5 illustrates the first three modes calculated in the geometrical acoustics approximation<sup>7</sup>.

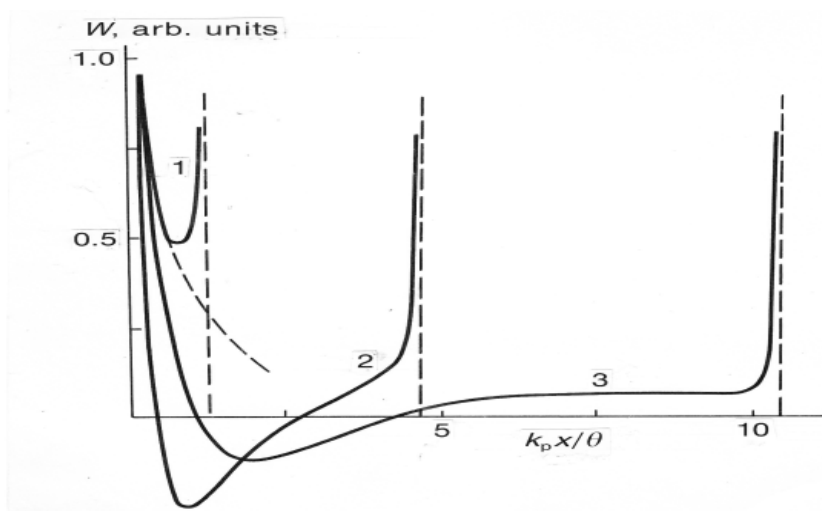


Figure 5. First three modes of wedge acoustic waves calculated using geometrical acoustics approximation<sup>7</sup>.

The structure of the localised wedge modes shown in Figure 5 agrees well with the results of the known numerical calculations, with the exception of the clearly seen singularities in the areas marked by dashed vertical lines. Beyond these lines, which show the positions of caustics (ray congestion zones), the modes do not penetrate into the depth of a wedge. The above singularities manifest the well-known limitation of all geometrical acoustics (optics) theories that become invalid near caustics<sup>1, 2</sup>. The above-mentioned geometrical acoustics theory of wedge acoustic waves can be generalised to consider localised modes of quadratically-shaped elastic wedges<sup>11</sup>, wedges immersed in liquids<sup>12</sup>, cylindrical and conical wedge-like structures (curved wedges)<sup>13</sup>, wedges of general power-law shape<sup>14</sup>, truncated wedges<sup>7</sup>, and wedges made of anisotropic materials<sup>15, 16</sup>.

## 4 LOCALISED WAVES IN NONCIRCULAR SHELLS

Non-circular cylindrical shells are relatively simple configurations (see Figure 6) that are used widely as elements of different complex structures, especially in aeronautical, automotive and oceanic engineering. They are also used frequently for modelling structural vibrations and structure-borne interior noise in all means of transportation<sup>17</sup>. Note that obtaining analytical solutions for vibrations of non-circular cylindrical shells is extremely difficult since it requires solving a system of governing differential equations with variable coefficients that describe the effects of variable curvature.

The geometrical acoustics approach to the description of waveguiding properties of shells and calculation of their resonant vibrations simplifies the problem substantially<sup>18</sup>. As in the case of Rayleigh waves and flexural waves in elastic wedges, the physical reason for waveguide propagation along quasi-flat areas of non-circular cylindrical shells is the difference between quasi-flexural wave velocities in their quasi-flat and curved parts. In particular, for waveguide propagation to become possible it is necessary that the velocity of quasi-flexural waves in the adjacent curved

areas is higher than in the quasi-flat internal area. This is always the case for quasi-flexural wave propagation in the near-axial directions of curved shells<sup>19</sup>.

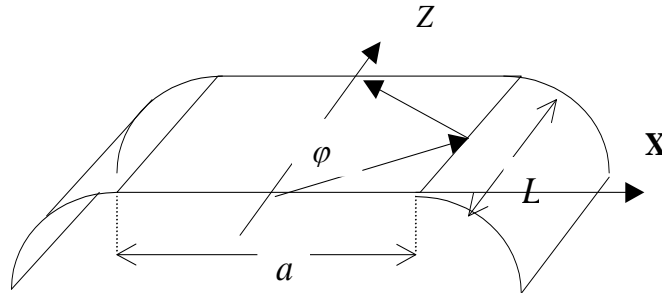


Figure 6. Waveguide propagation in a non-circular cylindrical shell.

#### 4.1 Geometrical Acoustics Theory of Localised Waves in Shells

It is convenient to consider waveguide propagation of flexural waves in a simple non-circular elastic shell comprising an infinitely long flat plate (strip) of thickness  $h$  and width  $a$  bounded by fragments of two cylindrical shells having equal radii  $R$  (Figure 6). Like in the similar case for Rayleigh waves, such a structure can be considered as a three-layered anisotropic medium for flexural waves<sup>18</sup>. The middle layer (a flat isotropic strip) is characterised by a lower phase velocity of flexural waves in comparison with the velocities of quasi-flexural waves in the near-axial direction in the adjacent cylindrical shells. Waveguide propagation in such a three-layered medium can be described in the same way as Rayleigh waves in topographic waveguides considered in Section 2.

The general dispersion equation in this case can be written in the form similar to equation (4), keeping in mind that in this case the internal area of the waveguide represents a flat plate<sup>18</sup>:

$$\frac{a}{2} \left[ k_{pl}^2 - \gamma^2 \right]^{1/2} = m \frac{\pi}{2} + \tan^{-1} \left[ \frac{\gamma^2 - k_{sh}^2(\varphi)}{k_{pl}^2 - \gamma^2} \right]^{1/2}. \quad (8)$$

Here  $\gamma$  is a yet unknown wavenumber of a guided flexural wave propagating in the above-mentioned three-layered system,  $k_{pl} = (\omega^2 \rho_s h / D)^{1/4}$  is the wavenumber of flexural waves in a flat plate, where  $\rho_s$  is the mass density of the plate material and  $D$  is plate flexural rigidity,  $k_{sh}(\varphi)$  is the angular-dependent wavenumber of flexural waves in a circular cylindrical shell, where  $\varphi$  is the wave propagation angle, and  $m = 0, 1, 2, 3, \dots$  is the mode number. Note that  $\gamma = k_{pl} \cos \varphi$ .

Let us consider waveguide propagation at frequencies lower than the ring frequency. In this case the expressions for  $k_{sh}(\varphi)$  are generally too complex to be described analytically. It is well known<sup>19</sup> that quasi-flexural waves in shells, being in fact curvature-modified Lamb modes, are governed by bending and membrane effects, which influence the expressions for quasi-flexural wave velocities that are anisotropic. Their functional appearance depends on the characteristic parameters of the shell, in particular on its ring frequency  $\omega_r$  and on its mean radius of curvature  $R$ .

In what follows we limit ourselves with the case of small wave propagation angles  $\varphi$ , for which  $k_{sh}(\varphi)$  is known to be extremely small. To simplify things even further, we assume that  $k_{sh}(\varphi) = 0$  for all  $\varphi$  in the range of interest. In this case, the dispersion equation (8) can be rewritten as



$$\frac{a}{2} k_{pl} \varphi = m \frac{\pi}{2} + \tan^{-1} \left[ \frac{1 - \varphi^2}{\varphi^2} \right]^{1/2}, \quad (9)$$

which results in the simple approximate solution for  $\gamma$ :

$$\gamma = k_{pl} \left[ 1 - \frac{(m+1)^2 \pi^2}{2a^2 k_{pl}^2} \right]. \quad (10)$$

It follows from this solution that the velocities of guided modes are higher than the velocity of flexural waves in a flat plate forming the internal area, but lower than the velocities of quasi-flexural waves in the adjacent fragments of circular shells. It can be also shown that the waveguide effect in this case is rather strong, and the energy of a guided wave is almost entirely concentrated in the internal flat plate area<sup>18</sup>.

## 4.2 Resonant Frequencies of Vibration of Finite Shells

The expression (10) for the wavenumbers of guided flexural waves in non-circular cylindrical shells can be used for calculations of resonant frequencies and mode shapes of resonant vibrations of finite non-circular shells<sup>18</sup>. To do so one should consider a finite length  $L$  of the above-mentioned noncircular shell (see Figure 6) and assume for simplicity that at  $z = 0$  and at  $z = L$  the system is subject to simply supported boundary conditions. Then the distribution of the resulting elastic field along  $z$ -axis formed by incident and reflected guided waves can be expressed in the form  $\sin(\gamma z)$ . Using the condition  $\sin(\gamma L) = 0$ , one can obtain that  $\gamma L = \pi n$ , where  $n = 1, 2, 3, \dots$ . Using this equality together with formula (10) for  $\gamma$ , one can obtain the following expression for resonant frequencies of a finite shell in the case of frequencies that are lower than the ring frequency:

$$\omega_{mn} = \left( \frac{D}{\rho_s h} \right)^{1/2} \left[ \frac{\pi^2 (m+1)^2}{a^2} + \frac{\pi^2 n^2}{L^2} \right]. \quad (11)$$

One should keep in mind that expression (11) is valid for  $n \gg m$ , which corresponds to low values of the propagation angle  $\varphi$  considered in the above-mentioned approximate solution. The values of resonant frequencies defined by the expression (11) are in good agreement with the numerical calculations and with the experimental measurements<sup>18</sup>. It is remarkable that formula (11) coincides with the well-known expression for resonant frequencies of simply supported flat plates having the dimensions  $L$  and  $a$  (note that in the present paper we assume that  $m = 0, 1, 2, 3, \dots$ , whereas in the plate theory usually  $m = 1, 2, 3, \dots$ ). This coincidence reflects the fact that at frequencies lower than the ring frequency the waveguide effect provided by two circular shells attached to a flat plate at opposite edges is very strong and almost the whole vibration energy is concentrated in the flat plate area.

## 5 CONCLUSIONS

In the present paper, localised elastic waves in three different types of solid structures have been considered theoretically. These are localised Rayleigh waves propagating along solid cylinders of variable surface curvature, localised flexural waves in slender elastic wedges (also known as wedge elastic waves), and localised quasi-flexural waves in non-circular cylindrical shells. For all these cases, derivations of the expressions for dispersion curves of localised waves have been discussed and the results compared with the known solutions, where available.

It has been demonstrated that in all cases under consideration the possibility of wave localisation can be explained by the existence of total internal reflection in structures of complex geometry due to the presence of internal areas of the surfaces characterised by the geometry-modified local phase velocities of Rayleigh or plate waves that are smaller in the direction of guided wave propagation than their values in the surrounding areas. This is similar to the condition of guided wave propagation in open waveguides considered in atmospheric or underwater Acoustics.

## 6 REFERENCES

1. L.M. Brekhovskikh and O.A. Godin, *Acoustics of Layered Media*, Vols. I and II, Springer, Berlin, Heidelberg (1990) and (1992).
2. S.V. Biryukov, Yu.V. Gulyaev, V.V. Krylov and V.P. Plessky, *Surface Acoustic Waves in Inhomogeneous Media*, Springer, Berlin, Heidelberg (1995).
3. V.V. Krylov, Transmission of Rayleigh waves through smooth large-scale surface irregularities, *Soviet Physics - Acoustics*, 34(6), 613-618 (1988).
4. V.V. Krylov, Distinctive characteristics of guided surface-wave propagation in complex topographic structures, *Soviet Physics - Acoustics*, 33(4), 407-411 (1987).
5. L.O. Wilson and J.A. Morrison, Propagation of high frequency elastic surface waves along cylinders of general cross section, *Journal of Mathematical Physics*, 16(9), 1795-1805 (1975).
6. S.D.M. Adams, R.V. Craster and D.P. Williams, Rayleigh waves guided by topography, *Proceedings of the Royal Society A*, 463, 531-550 (2007).
7. V.V. Krylov, Conditions for validity of the geometrical-acoustics approximation in application to waves in an acute-angle solid wedge, *Soviet Physics - Acoustics*, 35(2), 176-180 (1989).
8. V.V. Krylov, Geometrical-acoustics approach to the description of localised vibrational modes of an elastic wedge, *Soviet Physics - Technical Physics*, 35(2), 137-140 (1990).
9. P.E. Lagasse, Analysis of a dispersionfree guide for elastic waves, *Electronics Letters*, 8(15), 372-373 (1972).
10. A.A. Maradudin, R.F. Wallis, D.L. Mills and R.L. Ballard, Vibrational edge modes in finite crystals, *Physical Review B*, 6(4), 1106-1111 (1972).
11. V.V. Krylov, Localised acoustic modes of a quadratic solid wedge, *Moscow University Physics Bulletin*, 45(6), 65-69 (1990).
12. V.V. Krylov, On the velocities of localised vibration modes in immersed solid wedges, *Journal of the Acoustical Society of America*, 103(2), 767-770 (1998).
13. V.V. Krylov, Localised vibration modes propagating along edges of cylindrical and conical wedge-like structures, *Journal of Sound and Vibration*, 227(1), 215-221 (1999).
14. V.V. Krylov and A.L. Shuvalov, Propagation of localised flexural vibrations along plate edges described by a power law, *Proceedings of the Institute of Acoustics*, 22(2), 263-270 (2000).
15. A.L. Shuvalov and V.V. Krylov, Localised vibration modes in free anisotropic wedges, *Journal of the Acoustical Society of America*, 107(1), 657-660 (2000).
16. A.P. Mayer, V.V. Krylov and A.M. Lomonosov, Guided acoustic waves propagating at surfaces, interfaces and edges, *Proceedings of the IEEE Ultrasonics Symposium*, 2046-2052, Orlando, USA (2011).
17. L. Cremer, M. Heckl and E.E. Ungar, *Structure-borne Sound*, Springer, Berlin, Heidelberg (1988).
18. V.V. Krylov and V.B. Georgiev, New approach to investigation of resonant vibrations of noncircular shells based on the theory of coupled waveguides, *Journal of Mechanics of Materials and Structures*, 2(9), 1761-1771 (2007).
19. F. Fahy, *Sound and Structural Vibration*, Academic Press, London (1985).