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SOUND GENERATION IN A DUCT WITH A BULK-REACTING LINER

Alan Cummings

Department of Engineering Design and Manufacture, University of Hull, Hull HU6 7RX

INTRODUCTION

Aerodynamic sound generation in air-moving ductwork is an important aspect of the acoustics of building services. In ducts without internal acoustic linings, a knowledge of the aerodynamic noise source characteristics would enable the sound field in the duct to be found if the duct walls could be assumed rigid (see Davies and Ffowcs Williams [1] and Nelson and Morfey [2], for example). In the case of acoustically lined ducts or package silencers, however, the situation is complicated by the fact that the sound absorbent is usually in the form of continuous layers of a porous material, placed parallel to the airflow, and these do not present a locally reacting impedance surface to the sound field in the duct.

In this case, care must be taken in the use of an eigenfunction expansion for the sound field in the duct. The eigenfuctions are not orthogonal, in the usual sense, on the duct cross-section, and this means that the customary method of finding the sound field radiated by a point source or a source distribution in the air flow passage by matching the eigenfunction series to the source *via* the inhomogeneous wave equation has to be modified. An associated problem lies in the completeness of the eigenfunctions, which is not straightforward to prove in the present case. If the eigenfunction series is not complete, then it cannot, strictly speaking, be used to represent the sound field.

Concern about the completeness of eigenfunctions is, perhaps, a minor issue because all evidence indicates that these eigenfunctions do form a complete set. Nilsson and Brander [3], for example, who examined sound transmission in a circular flow duct with a bulk-reacting liner, found that their results demonstrated completeness, though this was not rigorously proved.

In the present study, the case of sound generation in a duct with a bulk-reacting liner and containing negligible mean fluid flow is analysed; the acoustic absorbent is assumed to behave like an equivalent fluid. Numerical calculations are presented for a two-dimensional geometry; the two-dimensional case contains all the essential elements of the three-dimensional problem and is thus adequate to demonstrate the phenomena involved. In the specific two-dimensional example studied, a dipole source is chosen, largely because this would ordinarily be the predominant source type in a practical situation. Some conclusions of general interest may be drawn from the results.

THEORY

Figure 1 shows the geometry of a uniform lined duct with arbitrary cross-section, having a point source at $(x_0, \mathbf{y}_0)$, $\mathbf{y}$ being a position vector on the cross-section of the duct.

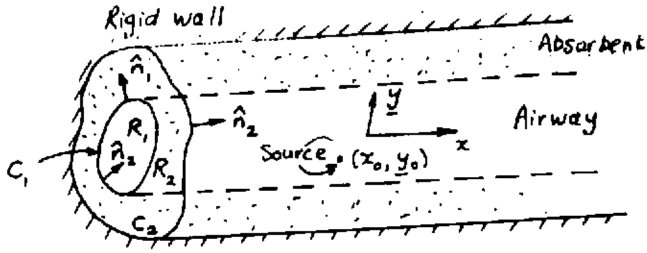

Figure 1. A lined duct

Modal Representation of the Sound Field, and the Eigen-Equation

It is assumed here that there is no mean gas flow in the "airway" R_1, and that the porous sound-absorbing medium in R_2 behaves like an equivalent fluid. Then, for a harmonic time-dependence, the Helmholtz equation in R_1, and its equivalent in R_2, apply:

$$(\nabla^2 + k^2)p = 0 \text{ in } R_1; \qquad (\nabla^2 + k_a{}^2)p = 0 \text{ in } R_2, \tag{1a,b}$$

where p = acoustic pressure, $k = \omega/c$, $k_a = \omega/c_a$, ω = radian frequency, c = sound speed, c_a = (complex) effective sound speed in the porous medium. We take a modal solution to eqs. (1a,b) for (say) positive travelling modes:

$$p(x,y;t) = e^{i\omega t}\sum_{n=1}^{\infty} A_n \Psi_n(y) e^{-ik_{xn}x}, \qquad \Psi_n = (\Psi_n^{(1)}, \Psi_n^{(2)}), \tag{2a,b}$$

where $\Psi_n^{(1,2)}$ are the modal eigenfunctions in R_1 and R_2, A_n is a modal coefficient and k_{xn} is the axial wavenumber. The boundary conditions are:

$$\nabla\Psi_n^{(2)}\cdot\hat{n}_2 = 0 \text{ on } C_2; \quad \Psi_n^{(1)} = \Psi_n^{(2)} \text{ on } C_1; \quad \rho_a\nabla\Psi_n^{(1)}\cdot\hat{n}_1 = \rho\nabla\Psi_n^{(2)}\cdot\hat{n}_1 \text{ on } C_1, \tag{3a,b,c}$$

where $\hat{n}_1$ and $\hat{n}_2$ are the outward unit normals to R_1 and R_2 and ρ, ρ_a are the gas density in R_1 and the (complex) effective gas density in R_2.

Scott [4] was the first to analyse sound propagation in a two-dimensional duct with a bulk liner; the geometry is shown in Figure 2. Scott's analysis

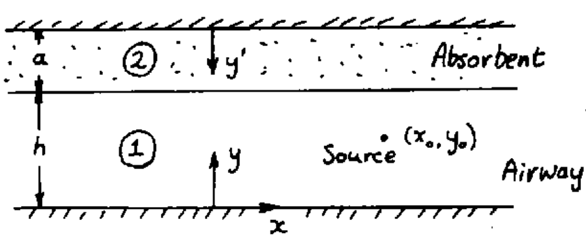

Figure 2. A two-dimensional lined duct

yields the eigenfunctions

$$\Psi^{(1)} = \cos \kappa y; \quad \Psi^{(2)} = \cos \kappa h \cos \kappa_a y' / \cos \kappa_a a, \tag{4a,b}$$

where $y' = a+h-y$, $\kappa = (k^2-k_x^2)^{1/2}$, $\kappa_a = (k_a^2-k_x^2)^{1/2}$; subscripts "n" on Ψ, κ, κ_a and k_x have been omitted for brevity. Scott's eigen-equation is

$$\rho\sqrt{k_a^2-k_x^2}\ \tan a\sqrt{k_a^2-k_x^2} \ + \ \rho_a\sqrt{k^2-k_x^2}\ \tan h\sqrt{k^2-k_x^2} = 0. \tag{5}$$

Eqs. (4a,b) and (5) apply in the present two-dimensional analysis. The eigen-equation was solved in the following way. The porous medium was assumed to consist of a fibrous material such as glass fibre, and to be homogeneous and isotropic. The empirical formulae of Delany and Bazley [5] are thus applicable, and in these the real and imaginary parts of the propagation coefficient and the characteristic impedance are expressed in terms of eight coefficients. If these are all put equal to zero, then the porous material has the properties of the fluid contained in its pores and is non-dissipative. The eigen-equation was first solved for the hard-wall duct case (for which $k_{xn} = \{k^2-[n\pi/(a+h)]^2\}^{1/2}$) and then the eight coefficients were all put equal to a small, fixed proportion of their actual values. The hard-wall wavenumbers were used as initial values in a Newton-Raphson iterative solution to eq. (5), and a new set of wavenumbers generated. The coefficients were increased again and the new wavenumbers used as initial values in the iterative solution to eq. (5). This process was repeated until the coefficients took on their actual values. Then the last set of iterated waveumbers taken to be the required set of k_x values.

Orthogonality Properties and Completeness of Eigenfunctions

If eqs. (2a,b) are substituted into (1a,b), we have

$$(\nabla_t^2 + \kappa^2)\Psi_n^{(1)} = 0, \qquad (\nabla_t^2 + \kappa_a^2)\Psi_n^{(2)} = 0, \tag{6a,b}$$

where ∇_t^2 is the two-dimensional Laplacian operator on the duct cross-section. Eq. (6a) is multiplied by $\Psi_m^{(1)}$ and (6b) by $\Psi_m^{(2)}$, then these equations are integrated over R_1 and R_2 respectively. Green's theorem is employed, and the boundary conditions (3a-c) applied. Appropriate manipulation of the resulting equations yields an orthogonality property in weighted space:

$$\iint_{R_1} \Psi_m^{(1)} \Psi_n^{(1)} dR_1 + (\rho/\rho_a) \iint_{R_2} \Psi_m^{(2)} \Psi_n^{(2)} dR_2 \Big\} \begin{array}{l} = 0, \quad m \neq n \\ = R\Lambda_n, \quad m = n \end{array}, \tag{7}$$

where $R = R_1 + R_2$. Equation (7) is not the more usual orthogonality relationship in which $\Psi_m \Psi_n$ would be integrated over R, but contains the weighting factor ρ/ρ_a in R_2.

To date, completeness of the eigenfunctions has not been proved analytically, but numerical demonstrations have been carried out in the two-dimensional case. Suppose we wish to represent a δ-function pressure distribution by a sum of eigenfunctions:

$$\delta(y - y_1) = \sum_{n=1}^{\infty} A_n \Psi_n(y). \tag{8a}$$

It may readily be shown that

$$A_n = \overset{(1)}{\Psi}_n(y_1)/\Lambda_n(a+h) \tag{8b}$$

if $0 \leq y_1 \leq h$, where Λ_n is found from eq. (7).

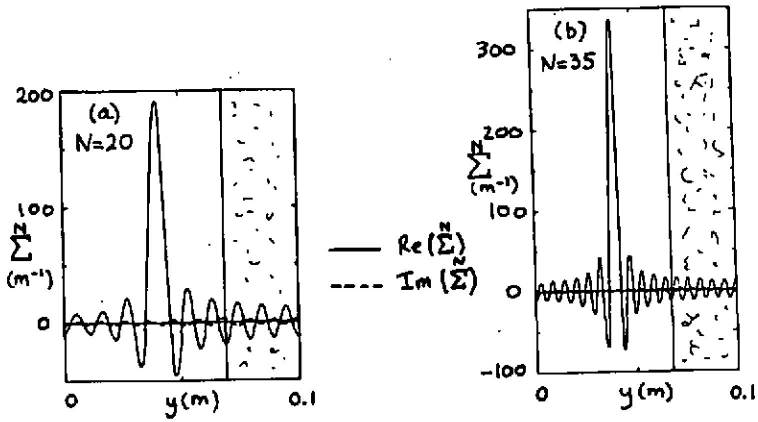

Figure 3. Synthesis of a δ-function by a finite sum of eigenfunctions

Figures 3(a,b) show the right-hand side of eq. (8a), summed to N terms, with N=20 and N=35, for a duct with a+h=0.1 m, y_1=0.04 m, frequency=600 Hz, and for a steady flow resistivity of the absorbent, $\sigma=10^4$ SI rayl/m. Convergence is clearly indicated as N is increased. The area under the main peak approaches unity as $N\to\infty$ and the peak value approaches ∞. If an infinite sum of modes can accurately reproduce a δ-function, then any other function $f(y_1)$ can also be represented by the same eigenfunctions:

$$f(y_1) = \int_0^{a+h}\delta(y-y_1)f(y)dy = [1/(a+h)]\sum_{n=1}^{\infty}[\overset{(1)}{\Psi}_n(y_1)/\Lambda_n]\int_0^{a+h}\Psi_n(y)f(y)dy \tag{9}$$

(if $0 \leq y_1 \leq h$). Then the set of eigenfunctions must be complete. If y_1 lies within the absorbent layer, similar arguments hold, but a weighting factor of ρ/ρ_a appears in equation (9).

Sound Generation in a Lined Duct

Consider sound generation by a monopole source located at (x_o, y_o) in the airway of an infinite lined duct (see Figure 1). The inhomogeneous wave equation in R_1 is

$$(\nabla^2 + k^2)p = -i\omega\rho q\delta(x-x_o)\delta(y-y_o)\delta(z-z_o), \tag{10}$$

where $q=Qe^{i\omega t}$ is the source volume velocity and $\mathbf{y}=\hat{j}y+\hat{k}z$. We may take solutions (2a,b) to the homogeneous equations (1a,b) and match them to the source *via* eq. (10) (see Cummings [6]). This must be done separately in R_1 and R_2. The modal solutions are inserted into the wave equations and then these are integrated over a volume enclosing the source point. A weighting factor of ϱ/ϱ_a is introduced, the weighted orthogonality property (7) is employed and, after appropriate manipulation, it is found that

$$p = (e^{i\omega t}\omega\rho Q/2R)\sum_{n=1}^{\infty}[\Psi_n^{(1)}(y_0)\Psi_n(y)e^{-ik_{xn}|x-x_0|}/k_{xn}\Lambda_n], \tag{11}$$

where Λ_n is found, *a priori*, from eq. (7). We now consider two nearby, oppositely phased, monopoles, separated by distance $l=\hat{i}l_x+\hat{j}l_y+\hat{k}l_z$. This is a dipole, equivalent to a point force of amplitude

$$F_i = i\omega\rho l_i Q \tag{12}$$

if the source spacing is sufficiently small, where $\rho l_i Q$ is the dipole moment and i=x,y or z. The total sound field from this dipole may readily be shown to be

$$p = (e^{i\omega t}/2R)\sum_{n=1}^{\infty}\{\Psi_n^{(1)}(y_0)\Psi_n(y)e^{-ik_{xn}|x-x_0|}\ \times$$
$$[\mathrm{sgn}(x-x_0)F_x\Psi_n^{(1)}(y_0)k_{xn} - iF_y\partial\Psi_n^{(1)}/\partial y|_{y_0} - iF_z\partial\Psi_n^{(1)}/\partial z|_{y_0}]/k_{xn}\Lambda_n\}. \tag{13}$$

Two dipoles may, of course, be combined to form a point quadrupole. The two-dimensional equivalent of eq. (13) is readily found by replacing R by a+h, discarding the third term in brackets, and putting F_i equal to force per unit width of duct.

Some Observations

In many cases, aerodynamic noise is generated at frequencies where only the fundamental acoustic mode can propagate in the equivalent rigid-walled duct. If a dipole source consisting of a flow obstacle (perhaps a strut) is placed in a hard-walled duct, then the fluctuating lift force would not radiate any sound energy since $\partial\Psi_n/\partial y$ and $\partial\Psi_n/\partial z$ are both zero. Only the fluctuating drag would generate sound. If, however, a lining were applied, then these transverse derivatives would generally be non-zero even for the fundamental mode, and the lift force could couple to the sound field and radiate energy.

A rather strange phenomenon was noted, concerning some of the higher order modes in the two-dimensional case. At sufficiently high frequencies, where these modes are strongly attenuated, a "cut-off" effect is observed, wherein the real part of k_x (for a positive propagating mode) goes from a positive value, through zero, and then takes on a small negative value, as the frequency is lowered. It would appear at first sight that the direction of propagation is reversed because of this sign change, though the imaginary part of k_x is still negative, consistent with energy decay in the direction of propagation. Closer examination reveals, however, that there is a small power flow in the *negative* direction in the airway, but a *positive* power flow in the liner, that outweighs the negative energy flow in the airway. Thus the net power flow is, in fact, still positive, in keeping with positive propagation. It is not at present clear whether this phenomenon is of any practical significance.

AN APPLICATION: A CIRCULAR CYLINDER IN A CROSS-FLOW

Suppose that, in a two-dimensional duct, a circular cylinder is located at $(0,y_o)$ in the airway and that a low-speed airflow passes along the duct, with a velocity that has negligible effects on sound propagation, and no significant convective effect on sound generation.

Keefe [7] and Gerrard [8] measured the fluctuating lift and drag forces on cylinders in a cross-flow. We may take Keefe's data as being - possibly - preferable because the forces were measured directly rather than being inferred from perimetral pressure measurements, as they were in Gerrard's work.

We assume an airflow speed of 15 m/s and a cylinder diameter of 14.25 mm which, according to Keefe's data, would give an essentially sinusoidal lift force at 200 Hz and a drag force at 400 Hz. The amplitudes of these forces are F_y=1.09 N/m and F_x=0.122 N/m respectively. Other data are: σ=10000 SI rayl/m, a= 0.03 m, h=0.07 m, y_o=0.05 m. In the calculations, the summation in eq. (13) was truncated at 20 terms.

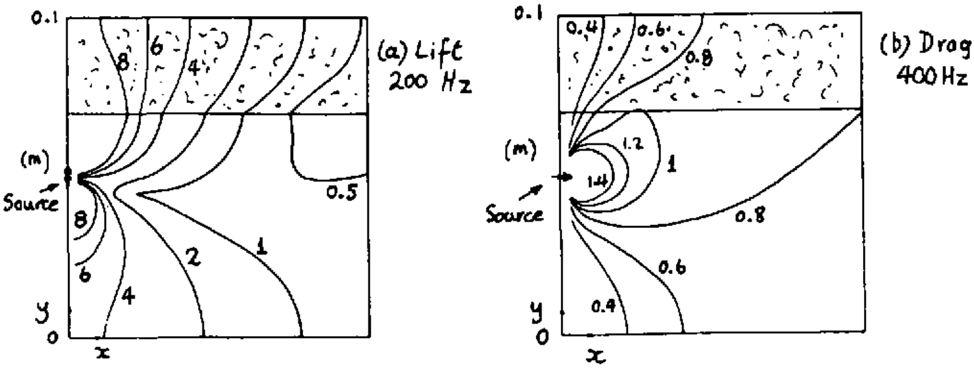

Figure 4. Near field sound pressure contours of the flow dipole

Figures 4(a,b) show near field contours (for $x>0$) of sound pressure amplitude in Pa for the sound fields radiated by the lift and drag forces respectively, if the cylinder is regarded as being acoustically compact. In Figure 4(a) we see that, close to the source, there is the characteristic dipole "figure of eight" pattern aligned across the duct axis, though this is distorted by the presence of the liner and, of course, also by the duct walls. The refractive effects of the liner on the sound field are clear. The drag dipole in Figure 4(b) displays a dipole pattern close to the source, but (as expected) this is aligned parallel to the duct axis. The effects of the liner are again noted.

In Figure 5, the axial variation in sound pressure level along the wall at y=0 is plotted, both for the lift and drag forces. We immediately note that the near field is quite extensive, reaching to about 0.2 m from the source in the case of the sound field from the lift. This is understandable since the lift force excites the n=2 mode much more strongly than the n=1 (fundamental) mode,

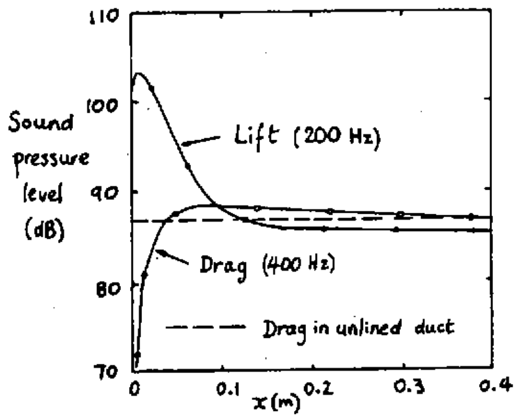

Figure 5. Axial variation in sound level from the flow dipole, at y=0

but the n=2 mode decays much more rapidly. For x>0.2 m, only the n=1 mode carries significant energy. The near field of the drag dipole extends less far, only to about 0.1 m, and this is because the n=1 mode is excited more strongly than the n=2 mode, and decays less quickly. The amplitude of the n=3 mode equals that of the n=1 mode in the source plane, but decays very much more rapidly. For x>0.1 m, the fundamental mode carries virtually all the power.

Although the sound level from the drag dipole is slightly higher than that from the lift near to the source, the former falls off more rapidly than the latter (because of the greater decay rate of the n=1 mode at the higher frequency), and after several metres, the 200 Hz tone from the lift dipole would dominate the sound field. For comparison, the sound level from the same drag force in an unlined duct of width 0.1 m is shown. This is similar in magnitude to that in the unlined duct but does not, of course, decay with distance. The lift force generates no sound energy in the unlined duct.

SOME PRACTICAL IMPLICATIONS

The above example has demonstrated that an acoustic lining in a duct can cause the lift dipole exerted by a flow obstacle on the surrounding fluid to couple to the sound field at low frequencies, whereas it would not do so in an untreated duct. Since the fluctuating lift is ordinarily greater than the drag, this could be a significant effect in practical circumstances.

Consider for example the situation in which a flow obstacle in a duct (perhaps an internal bracing strut, or turning vanes at a bend) generates a level of aerodynamic noise such that an acoustic lining must be applied to attenuate the noise. If the lining is placed along the duct wall in the same plane as the flow obstacle, then the situation could be made *worse*, rather than better, by the lift force being able to radiate sound energy into the duct. The lining should be placed some distance from the source to guarantee noise reduction.

SOUND GENERATION IN A DUCT WITH A BULK-REACTING LINER

DISCUSSION

A theoretical treatment of sound generation in a duct having a bulk-reacting lining, but containing negligible mean flow, has been described. If flow were taken into account, the weighted orthogonality properties of the eigenfunctions would be lost because one of the boundary conditions would become eigenvalue-dependent. The problem would then become less easily manageable.

In the case of air-conditioning ducts, the neglect of mean flow would normally be unimportant.

Further investigation of practical aerodynamic noise sources in lined ducts would be of value. The inclusion of the effects of a perforated facing sheet, used to retain the absorbent, in the model would also be desirable.

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