BRITISH ACOUSTICAL SOCIETY

71SC12

Spring Meeting
5-7th April 1971

APPROXIMATE SCALING LAWS OF ACOUSTIC ARRAY INTERACTION

A. Freedman

For a fixed set of parameters, computing the performance of an array of N transducers when taking acoustic coupling into account usually involves the solution of a real matrix equation of order 2N. If N is large, the computation for just one set of parameters can be a considerable task. In array interaction studies many parameters are variable, and the computational effort to cover even a limited range of variables can be formidable indeed. One approach to reducing the magnitude of effort required is to search for scaling laws, even of an approximate nature, which will enable conclusions drawn from a study of one array to be applied to a family of arrays. This paper, which is based on Ref 1, describes the derivation of such approximate scaling laws for planar arrays of piezo-electric elements vibrating either within an infinite, plane, rigid baffle or with no baffle present; the transducer faces are deemed to be rigid and circular, with diameter less than $\lambda/2$ and, for convenience, the medium will be assumed to be water. All transducers of an array are presumed identical. The several scaling laws which will be deduced can be applied in combination, providing the restrictions applicable to each individual law are complied with.

Over a limited frequency band, a piezo-electric transducer may be represented by the well known Van Dyke equivalent circuit of Fig 1, where upper and lower case subscripts denote electrical and mechanical units, respectively. Instead of specifying the equivalent circuit components directly, we specify the following quantities: f_o, mechanical resonant frequency of an isolated transducer in water; a/λ_o, piston radius in wavelengths at f_o; ρ, density of water; c, velocity of sound in water; η_{ma}, mechano-acoustic efficiency of an isolated transducer in water; Q_w, motional Q factor of an isolated transducer in water; k, coupling coefficient; B, electrical to mechanical transform coefficient; tan δ, dielectric loss factor. From the foregoing, the impedances of all the circuit components of Fig 1 may be deduced in terms of quantities which are constant for a specified element and of the following three calculable quantities: (a) the self radiation resistance at f_o, $R_{r(s)o}$; (b) the self radiation reactance at f_o, $X_{r(s)o}$; (c) the normalised frequency, f/f_o.

For the sake of some generality we allow an optional series or parallel tuning inductance, L_E, and an optional series input resistance, R_S, with the restriction that the impedances at f_o are each assumed to bear some fixed ratio to $R_{r(s)o}$.

The self radiation-resistance and reactance at any frequency can, of course, be expressed in terms of their respective values at f_o and of functions of f/f_o, while the mutual radiation impedance is given (Ref 2) to a fair degree of accuracy, for elements up to $\lambda/4$ diameter (or up to greater diameters with decreasing accuracy), by

$$Z_{r(m),g} = R_{r(m),g} + jX_{r(m),g} = R_{r(s),g} \sum_{\substack{h=1 \\ h \neq g}}^{N} \frac{u_h}{u_g} \left(\frac{\sin kd_{gh}}{kd_{gh}} + j \frac{\cos kd_{gh}}{kd_{gh}} \right),$$

where d_{gh} represents the distance between the gth and hth elements and the u's represent the velocities. To the degree of approximation implicit, it can be deduced that $R_{r(m)}$ and $X_{r(m)}$ are each only functions of f/f_o and of $R_{r(s)o}$.

Thus, for a fixed element specification, all impedances of Fig 1 have been expressed in terms of constants, of $R_{r(s)o}$, of $X_{r(s)o}$ and of functions of f/f_o. If $L_{r(s)}$ and L_m are grouped together, even the dependence on $X_{r(s)o}$ disappears.

For stated boundary conditions (such as the presence of an infinite, rigid baffle or the absence of a baffle) and for given value of a/λ_o, the quantities $R_{r(s)o}/\rho cA$ and $X_{r(s)o}/\rho cA$ are fixed. As the piston area, A, is $\pi(a/\lambda_o)^2(c/f_o)^2$, $R_{r(s)o}$ and $X_{r(s)o}$ are each equal to a constant times f_o^{-2}. Thus the impedances of the equivalent circuit components may each be expressed as a function of f/f_o and as proportional to f_o^{-2}. Hence we may deduce the following dependence factors for the parameters E, B and f_o (which we shall call scaling law 1).

Quantity	Power of dependence on f_o	B	E
Piston velocity	2	1	1
Force on piston	0	1	1
Total radiation impedance	-2	0	0
Radiation impedance/ρcA	0	0	0
Radiated power	2	2	2
Radiated power/piston area	4	2	2
Input current	2	2	1
Input impedance	-2	-2	0
Input power	2	2	2

The values of f_o, B and E used for portraying the curves of array performance in this study were f_o = 1 kH, B = 1 Newton/Volt and E = 1 Volt. With the help of scaling law 1 the curves can be interpreted easily for any values of f_o, B and E.

With constant voltage drive and no external series impedance, the force across the input to the mechanical part of the equivalent circuit is unaffected by the values of k and of tan δ; hence the acoustic performance is then unaffected by these two parameters. (Scaling law 2.)

Using the equivalent circuit formulae and the formulae for the self radiation impedance of small, circular pistons, the total mechanical impedance, $Z_{m,r}$, (see Fig 1) for an isolated element can be shown to satisfy

$$Z_{m,r} \approx R_{r(s)o} (1 + j2Q_w \Delta)/\eta_{ma} ,$$

where Δ is the fractional bandwidth, $(f - f_o)/f_o$, and is assumed small. Thus, for two different isolated elements for each of which the specification is identical, except for Q_w, $Z_{m,r}$ will be the same for both elements at equal values of $Q_w \Delta$. Thus curves of a parameter depending only on $Z_{m,r}$ will, when plotted on a generalised frequency scale of $2Q_w \Delta$, be the same for both elements (to within the accuracy implicit in the above equation). When there is a constant voltage source and no series impedance in the electrical feed, this applies to the acoustic quantities such as piston velocity, force on piston and radiated power. (Scaling law 3a.) It does not hold, in general, for electrical quantities such as input current, input impedance and input power, because the expressions for these quantities contain the terms Q_w and Δ separately and not only in product form. It does however hold for these electrical quantities if the input to each of the two elements is parallel-tuned to f_o, assuming they have negligible dielectric loss and the same value of $Q_w k/(1 - k^2)^{\frac{1}{2}}$, for Q_w and Δ then only occur in the product form in the relevant formulae. (Scaling law 3b.)

It is not immediately obvious that the ability to plot parameters independently of Q_w will hold for elements in an array, as the mutual radiation impedance formulae do not show an obvious dependence on $Q_w \Delta$. However, computational checks have been made for some small arrays. Such a comparison is presented for two arrays in which the specification is identical except that in the one case the elements have a Q_w of 6, in the other of 12. Each array consists of 9 abutting pistons (3 rows x 3 columns) in an infinite, rigid baffle. The element specifications, equivalent circuits and curves for velocity magnitude and radiated power are shown in Fig 2; the curves are for constant voltage drive with no steering or tapering. There are three unique element positions in each array, namely corner, mid-side and centre; thus three separate full line curves are shown on each graph; in addition, the variations of the same parameter for an isolated element are depicted by a broken line curve. It will be seen that the severe interaction effects follow very similar curves for the two arrays over a range of values of $2Q_w \Delta$ from about -2 to +2. Scaling law 3a thus appears to apply to arrays, and similar computational evidence indicates that this is also true of scaling law 3b.

It was indicated earlier that, with $L_{r(s)}$ and L_m grouped together, the impedances of the equivalent circuit could all be expressed in terms of constants, of $R_{r(s)o}$ and of functions of f/f_o. Thus, to a certain degree of approximation, if two arrays have the same specification, apart from the element size, a/λ_o, the equivalent circuit components within each array will be scaled to the value of $R_{r(s)o}$ applicable to that array. Thus, the following approximate scaling can be expected: piston velocity, radiated power, input current and input power will be proportional to $1/R_{r(s)o}$, whereas mutual radiation impedance, total radiation resistance and input impedance will be proportional to $R_{r(s)o}$. (Scaling law 4.) Quantities, such as radiation reactance and force on piston, which depend on $L_{r(s)}$ or L_m separately will not necessarily scale. An example of scaling law 4 is shown in Fig 3. On the right hand graph of each pair, the expected highest point of the array curves as scaled from the left hand graph is indicated by a short horizontal line. Quite respectable agreement can be observed between the sets of curves.

We next compare an array in an infinite, rigid baffle and an unbaffled array with otherwise identical specification. Denoting "baffled" and "unbaffled" quantities by the subscripts b and u, respectively, one finds (using results from Refs 3 and 4) that, over the range of small a/λ_o which concerns us, the ratios $R_{r(s),b}/R_{r(s),u}$ and $X_{r(s),b}/X_{r(s),u}$ are, to a rough degree of approximation, constant. Hence, similarly to scaling law 4, the two arrays will scale approximately in the ratio or inverse ratio of the respective values of $R_{r(s)o}$. (Scaling law 5.) An example of scaling law 5 is portrayed in Fig 4, and the agreement there is again quite reasonable.

We note that the equivalent circuits of the two arrays in Fig 4 differ significantly. One can deduce (by a combination of scaling laws 4 and 5) that two similar arrays, one baffled and one unbaffled, yet both with elements having (nearly) the same equivalent circuits, will yield approximately the same performance if a/λ_o for the baffled array is one sixth less than for the unbaffled one.

<u>References</u>

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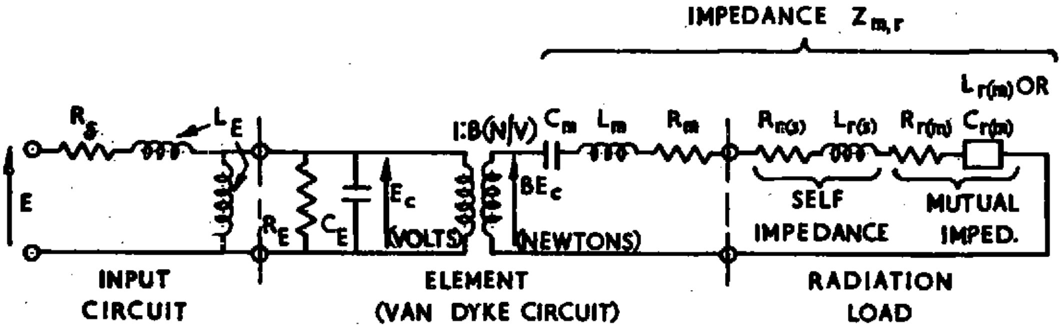
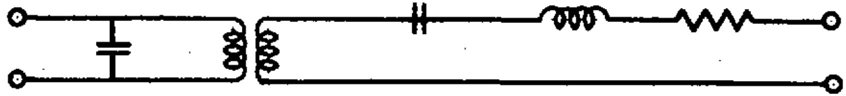


Fig 1 Equivalent circuit of a piezo-electric element, its radiation load and the assumed input components.

$0.1\lambda_o$ Geometry for each array

Element	f_o(kH)	a/λ_o	η_{ma}	Q_w	B(N/V)	k	tan δ
A	1	0.05	0.90	6	1	0.28	0
B	1	0.05	0.90	12	1	0.28	0

	μF	V : N	m/N	kg	N/m/s
A	0.213	1 : 1	18.1×10^{-9}	0.28	146
B	0.107	1 : 1	9.1×10^{-9}	1.67	146



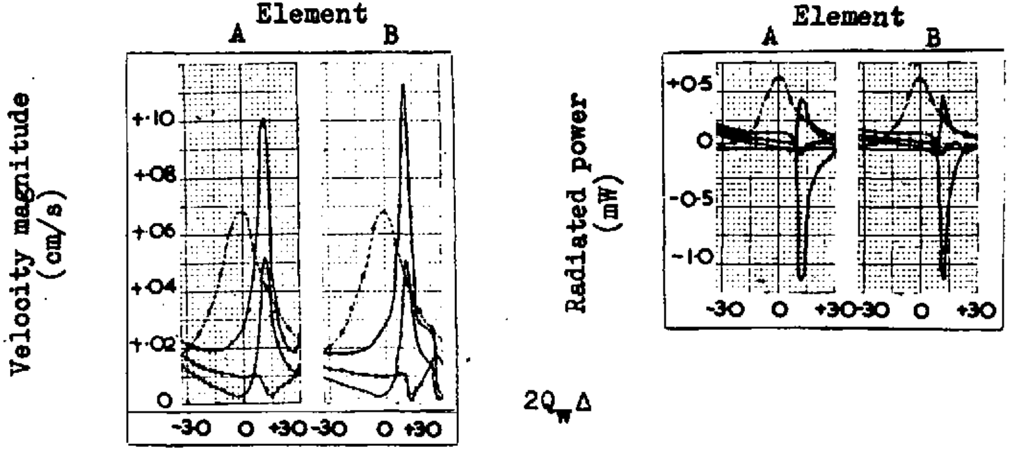
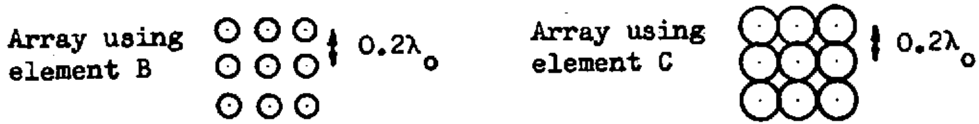


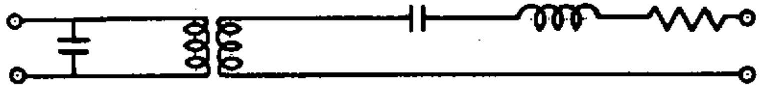
Fig 2 Example of scaling law 3a



Element	f_o(kH)	a/λ_o	η_{ma}	Q_w	B(N/V)	k	tan δ
B	1	0.05	0.90	12	1	0.28	0
C	1	0.10	0.90	12	1	0.28	0

	μF	V : N	m/N	kg	N/m/s
B	0.107	1 : 1	9.1×10^{-9}	1.7	146
C	0.007	1 : 1	0.6×10^{-9}	34.3	2230

Constant voltage.
No steering or tapering.



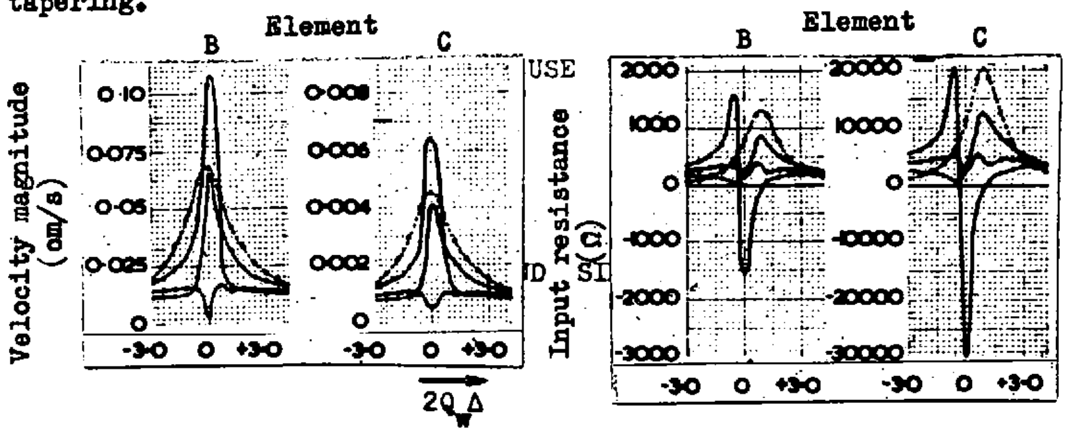


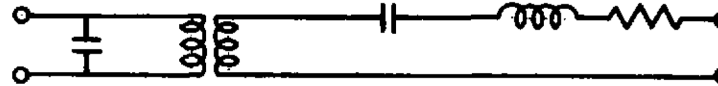
Fig 3 Example of scaling law 4

array in infinite, rigid baffle uses element B. Unbaffled array uses element D.

Element	f_o(kH)	a/λ_o	η_{ma}	Q_w	B(N/V)	k	tan δ
B	1	0.05	0.90	12	1	0.28	0
D	1	0.05	0.90	12	1	0.28	0

	µF	V : N	m/N	kg	N/m/s
B	0.107	1 : 1	9.1×10^{-9}	1.67	146
D	0.212	1 : 1	18.0×10^{-9}	0.55	74

Constant voltage. No steering or tapering.



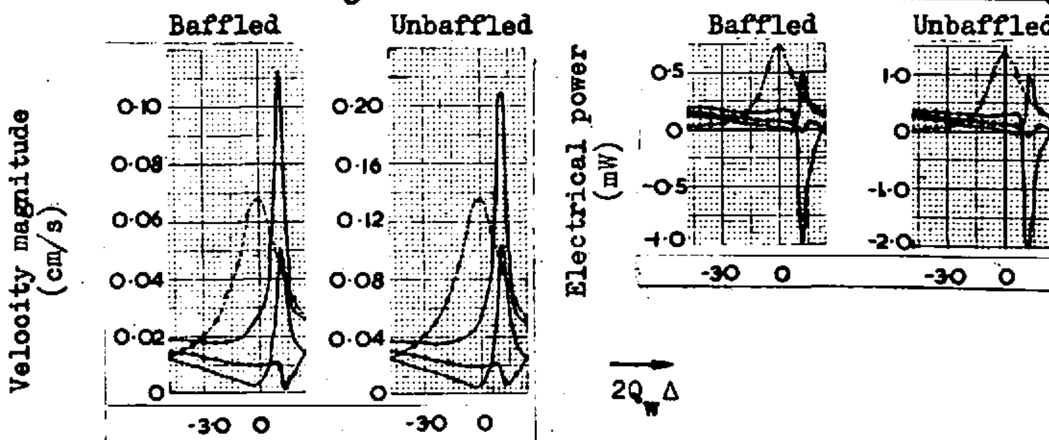


Fig 4 Example of scaling law 5