24th INTERNATIONAL CONGRESS ON SOUND AND VIBRATION
23–27 July 2017, London

SIMULTANEOUS EXCITATIONS-INDUCED WIDEBAND FREQUENCY RESPONSE IN NONLINEAR COUPLED PERIODIC PENDULUMS

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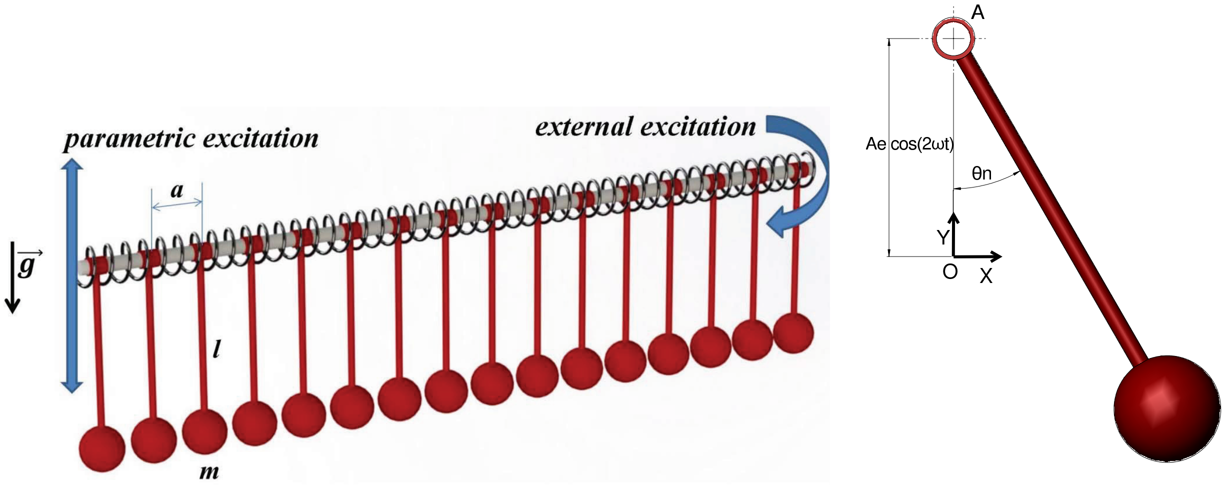
The collective dynamics of a chain of coupled pendulums is investigated under simultaneous external and parametric excitation. The purpose of this study is to track the frequency response of the considered system in term of bifurcation topology with respect to the excitation amplitudes and structural imperfections. The equations of motion are derived and solved using the harmonic balance method coupled with the asymptotic numerical method. Several numerical simulations are performed in the case of six coupled pendulums in order to analyze the complexity of the frequency responses in terms of stability and bifurcation topology. Remarkably, wideband responses are displayed thanks to simultaneous excitations compared to the case of a single excitation.

Keywords: coupled pendulums, simultaneous excitations, imperfection, nonlinear dynamics, bifurcation topology

1. Introduction

Nonlinear periodic oscillators have been used in a wide range of applications extending from chains of coupled pendulums [1] and Josephson junction arrays [2] to gravitational and high-energy physics models [3, 4]. For instance, Ikeda et al. investigated the behavior of intrinsic localized modes (ILMs) for an array of coupled pendulums subjected to horizontal [5] and vertical [6] sinusoidal excitation.

The array of coupled pendulums was recently modeled and studied by Bitar et al. [7] and the effect of modal interactions has been investigated. Moreover, the excitation of several modes lead to multimodal solutions [8] obtained for three coupled pendulums. For a large number of oscillators, Ivancevic $et\ al.$ [9] transformed the equations of motions of a coupled oscillator to one Sine-Gordon equation and reviewed the essential dynamic of a nonlinear excitation in living cellular structures. Khomeriki $et\ al.$ [10] studied the tristability of a chain of pendulums driven periodically in one end and free at the other end. There are two types of excitation: external excitation [11, 12] and parametric excitation [13, 14] (excitation of the base). The first type of excitation was simulated by Braiman $et\ al.$ [11] on a chain of coupled damped pendulums with a free end boundary condition. The authors showed that when the chain is perfectly periodic, the oscillations become chaotic. However, their motion becomes ordered when imperfections are added. The effect of imperfection was also tested numerically [12] on an array of 128 pendulums. The parametric excitation was studied numerically by Alexeeva $et\ al.$ [14] and experimentally by Chen $et\ al.$ [13]. The authors showed that a high positive imperfection level can extend the region of stability of the system while a negative imperfection is more exposed to oscillatory instabilities. Also, array of coupled pendulums under external and



(a) (b)

Figure 1: Array of coupled pendulums under simultaneous parametric and external excitations.

parametric excitation is used to study the intrinsic localized modes[15]. The use of two excitations may improve the existence and stability regions of solitons.

In this paper, we derived the equations of motion describing the nonlinear dynamics of an array of coupled pendulums under simultaneous external and parametric excitations. the Harmonic Balance Method coupled with Asymptotic Numerical Method is used to transform the system of equation to a Fourier series and determine the nonlinear frequency response. By comparing the obtained results of two excitations to those with one excitation, we show that simultaneous excitations can extend the stable region without shifting the frequency response or breaking the symmetry of the array.

2. Design and model

The considered system, depicted in Figure 1, is composed of an horizontal axle A. Along this axle, at equally spaced intervals, there are N_{pen} equal pendulums. Each pendulum consists of a rigid rod, attached perpendicularly to the axle, with a mass m at the end. At rest, all the pendulums point down the vertical. a is the distance between two pendulums, g is the gravity acceleration, θ_n is the angle between the n^{th} pendulum and the downward vertical, k_2 is the linear torque constant and k_4 is the cubic torque constant. By neglecting the mass of the rigid rod, all the pendulums have the same moment of inertia $I = ml_n^2$, where l_n is the length of the n^{th} pendulum. The considered system is excited by two forces at the drive frequency ω. The first one is an external force $Fcos(\omega t)$ applied to one or several pendulums, and a parametric force $4A_e\omega^2 cos(2\omega t)$ due to the base excitation of the system. The kinetic and potential energy of the system can be written as:

$$V = \sum_n \tfrac{1}{2}k_2(\theta_n - \theta_{n+1})^2 + \tfrac{1}{2}k_2(\theta_n - \theta_{n-1})^2 + \tfrac{1}{4}k_4(\theta_n - \theta_{n+1})^4 + \tfrac{1}{4}k_4(\theta_n - \theta_{n-1})^4 \\ -mgl(\sin(\theta_n) + A_e\cos(2\omega t)) \tag{1}$$

$$T = \sum_n \frac{1}{2} m v_n^2 \tag{2}$$

The potential energy V consists of two parts: the strain energy due to the elongation of the spring and the gravitational potential energy. T is the kinetic energy due to the velocity v_n of the moving mass, where

$$v_n = \overrightarrow{\dot{r}_{OA}} + \vec{\omega} \times \overrightarrow{r_{AP}} \tag{3}$$

where

$$\overrightarrow{\dot{r}_{OA}} = A_e\omega_e \sin(\omega t)\hat{j} \qquad \vec{\omega} = \dot{\theta}_n\hat{z} \tag{4}$$

$$\overrightarrow{r_{AP}} = l(\sin(\theta_n)\,\hat{i} + \cos(\theta_n)\hat{j}) \tag{5}$$

we applied Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_n}\right) - \frac{\partial L}{\partial \theta_n} = Q_n \tag{6}$$

with $L = T - V$ and Q_n is the non-conservative generalized external forces applied to the n^{th} pendulum (in this case Q_n is the sum of the friction force and the external excitation). Hence, the n^{th} pendulum's equation of motion is determined as follows:

$$\begin{aligned} &ml_n^2\tfrac{d^2\theta_n}{dt^2} + \alpha l_n\tfrac{d\theta_n}{dt} + k_2\left(2\theta_n - \theta_{n+1} - \theta_{n-1}\right) + k_4\left((\theta_n - \theta_{n-1})^3 + (\theta_n - \theta_{n+1})^3\right) = \\ &-ml_n\left[g + 4A_e\omega_e^2\cos\left(2\omega_e t\right)\right]\sin\left(\theta_n\right) + fcos(\omega_e t); \quad n = 1, 2, .., N \end{aligned} \tag{7}$$

In our case we choose a fixed boundary conditions ($\theta_0 = 0$ and $\theta_{N+1} = 0$). By expanding $sin(\theta_n)$ in Taylor series up to the third order, Equation (7) can be written as:

$$\begin{aligned} &ml_n^2\tfrac{d^2\theta_n}{dt^2} + \alpha l_n\tfrac{d\theta_n}{dt} + k_2\left(2\theta_n - \theta_{n+1} - \theta_{n-1}\right) + k_4\left((\theta_n - \theta_{n-1})^3 + (\theta_n - \theta_{n+1})^3\right) \\ &= -ml_n\left[g + 4A_e\omega_e^2\cos\left(2\omega_e t\right)\right]\left(\theta_n - \tfrac{1}{6}\theta_n^3\right) + fcos(\omega_e t); \quad n = 1, 2, .., N \end{aligned} \tag{8}$$

Equation (8) presents a system of Duffing oscillators coupled respectively with linear and nonlinear coefficients k_2 and k_3 subjected to external and parametric excitations. To solve the obtained system of nonlinear equations we used the Harmonic Balance Method (HBM) and the Asymptotic Numerical Method (ANM), described in the next section.

3. Solving procedure

To study the colective dynamics of the array of coupled pendulums, we solve Equation (8) and we plot the frequency response of the solution amplitude. To do so, we rewrite the system nonlinearities into quadratic terms. Cochelin and Vergez [16] used this technique in order to follow the periodic solutions of dynamical systems when a control parameter is varied. Equation (8) is transformed into a quadratic one and the unknown variables are decomposed into truncated Fourier series using the HBM. We derive the algebraic system and we solve it using the ANM. The latter has been used recently to investigate pull-in instability in circular capacitive micromachined ultrasonic transducers [17].

3.1 Quadratic recast

Let us consider an autonomous system of differential equations:

$$\dot{Y} = f(Y, \lambda, t) \tag{9}$$

where Y is a vector of unknowns, f is a smooth nonlinear vector valued function and λ is a real parameter. The dot denotes the derivative with respect to time t. First, we transform the system (9) into a new system where the nonlinearities are quadratic, which can be written as follows:

$$m\left(\dot{Z}\right) = c\left(\lambda, t\right) + l\left(Z\right) + q(Z, Z) \tag{10}$$

with Z is the vector of unknowns, $c(\lambda, t)$ is a constant vector with respect to Z, $l(Z)$ is the linear vector and $q(Z, Z)$ is the quadratic vector. To transform the system (9) into (10) we need to predefine some variables:

$$\begin{aligned} &u_n = \theta_n \qquad && v_n = \dot{\theta}_n \qquad && w_n = \theta_n^2 \qquad && x_n = \theta_n^3 \\ &a = \cos(2\omega t) && b = \omega && c = \omega^2 = b^2 && d = \omega^2\cos\left(2\omega t\right) = c \times a \end{aligned} \tag{11}$$

The vectors $m(\dot{Z})$, c, $l(.)$ and $q(.,.)$ become :
$m(\dot{Z}) = [u_1, ..., u_N, v_1, ..., v_N, 0, ..., 0]^t$
$c(t) = [\underbrace{0, .., 0}_{N}, \underbrace{\frac{f}{ml_n^2}cos(\omega t), .., \frac{f}{ml_n^2}cos(\omega t)}_{N}, \underbrace{0, .., 0}_{N}, a, 0, 0, 0]^t$
$l\left(Z\right) = [\underbrace{v_1, .., v_N}_{N}, \underbrace{f_1, .., f_N}_{N}, \underbrace{w_1, .., w_N}_{N}, \underbrace{x_1, .., x_N}_{N}, a, b, c, d]$
where $f_i = -\mu v_i - \omega_0^2 u_i - k_L\left(2u_i - u_{i+1} - u_{i-1}\right)$; $i = 1, .., N$
$q\left(Z, Z\right) = [\underbrace{0, .., 0}_{N}, \underbrace{g_1, .., g_N}_{N}, \underbrace{u_1 u_1, ..u_N u_N}_{N}, \underbrace{w_1 u_1, .., w_N u_N}_{N}, 0, 0, bb, ca]^t$
with $g_i = -\gamma u_i w_i - \delta\left(u_i d - \frac{1}{6}x_i d\right) - k_{NL}(u_i - u_{i-1})^3 - k_{NL}(u_i - u_{i+1})^3$; $i = 1, ..N$
we define the variables as:
$\mu = \frac{\alpha}{ml_n}, \omega_0^2 = \frac{g}{l_n}, \gamma = -\frac{\omega_0^2}{6}, \delta = \frac{4A_e}{l_n}, k_L = \frac{k_2}{ml_n^2}, k_{NL} = \frac{k_4}{ml_n^2}$

The number of equations to implement is $N_{eq} = 4N_{pen} + 4$ where N_{pen} is the number of pendulums, therefore the length of the vectors c, l and q is equal to the number of equations N_{eq} and $Z = (u_1, ..., u_N, v_1, ..., v_N, w_1, ..., w_N, x_1, ..., x_N, a, b, c, d)$

3.2 The harmonic balance method (HBM)

In order to apply the HBM on our system, $Y(t)$ is decomposed into a truncated Fourier series with H harmonics:

$$Z\left(t\right) = Z_0 + \sum_{k=1}^{H} Z_{c,k}\cos\left(k\omega t\right) + Z_{s,k}\sin(k\omega t) \tag{12}$$

By collecting all the components of the Fourier series in a vector U, the unknown variables to determine become the components of the vectors U.

$$U = \left[Z_0^t, Z_{c,1}^t, Z_{s,1}^t, Z_{c,2}^t, Z_{s,2}^t, \ldots, , Z_{c,H}^t, Z_{s,H}^t\right]^t \tag{13}$$

Therefore, the size of U is equal to $(2\mathrm{H} + 1) \times \mathrm{N_{eq}}$. Where N_{eq} represents the number of equations of the system (10). Replacing Equation (12) into Equation (10), the system becomes:

$$\omega M\left(U\right) = C + L\left(U\right) + Q(U, U) \tag{14}$$

The new operators $M(U)$, C, $L(U)$ and $Q(U,U)$ depend only on the operators $m(\dot{Z})$, $c(t)$, $l(Z)$ and $q(Z,Z)$ of Equation (10). The final system (14) contains $(2\mathrm{H}+1)\times\mathrm{N_{eq}}$ for $(2\mathrm{H}+1)\times\mathrm{N_{eq}}$ unknowns U plus the angular frequency ω and the continuation parameter λ.

3.3 The asymptotic numerical method (ANM)

Applying HBM, a new system is obtained:

$$R(U) = 0 \tag{15}$$

where $R \in \mathbb{R}^{N_v}$ and $U = [U^t, \lambda, \omega] \in \mathbb{R}^{N_v}$ with $N_v = (2\mathrm{H} + 1) \times \mathrm{Neq} + 2$. The final system (13) is quadratic with respect to U and ω. Thus, the application of the ANM is quite straightforward. We obtain:

$$R\left(U\right) = L_0 + L\left(U\right) + Q(U, U) \tag{16}$$

Where L_0, $L(U)$ and $Q(U,U)$ are respectively constant, linear and quadratic vectors.

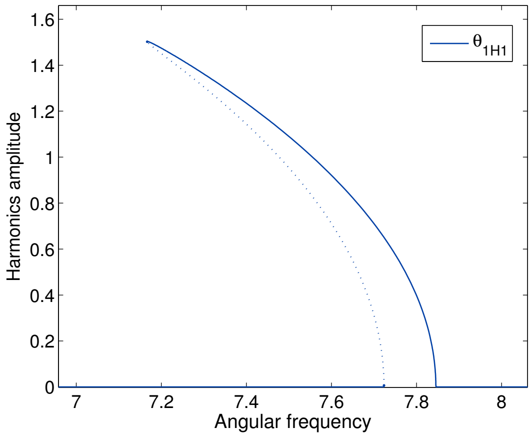


Figure 2: Frequency response of the harmonic 1 ; 3 and 5 of one pendulum.

4. Results and discussion

As a first step, we use the HBM and ANM to plot the frequency-response of one pendulum. Figure 2 represents the amplitude of the harmonics 0,1,3 and 5 of the pendulum, where the system is excited using a parametric force. The system parameters are $m = 0.03\ kg$; $l = 0.25\ m$; $A_e = 0.0029\ m$; $k_2 = 0.02\ Nm$; $k_4 = 0.001\ Nm$ and $\alpha = 0.001\ Nm^{-1}s^{-1}$. The equation of motion associated to one pendulum is determined as follows:

$$\begin{array}{l} ml^2\frac{d^2\theta_1}{dt^2} + \alpha l\frac{d\theta_1}{dt} + k_2\left(2\theta_1 - \theta_0 - \theta_2\right) + k_4((\theta_1-\theta_0)^3 + (\theta_1-\theta_2)^3)) = \\ -ml[g + 4A_e\omega_e^2\cos(2\omega_e t)](\theta_1 - \theta_1^3/6) + f cos(\omega_e t) \end{array} \tag{17}$$

or in canonical form:

$$\begin{array}{l} \frac{d^2\theta_1}{dt^2} + \mu\frac{d\theta_1}{dt} + k_L\left(2\theta_1 - \theta_0 - \theta_2\right) + k_{NL}((\theta_1-\theta_0)^3 + (\theta_1-\theta_2)^3)) = \\ -[\omega_0^2 + \delta\omega_e^2\cos(2\omega_e t)](\theta_1 - \theta_1^3/6) + F_{ext} cos(\omega_e t) \end{array} \tag{18}$$

where $\mu = \alpha/ml$; $k_L = k_2/ml^2$; $k_{NL} = k_4/ml^2$; $\delta = 4A_e/ml^2$; $F_{ext} = f/ml^2$ and $\omega_0 = \sqrt{g/l}$ presents the natural frequency of a free simple pendulum, however in our case the pendulum is coupled to the frame from the two sides ($\theta_0 = \theta_2 = 0$). Therefore, the natural frequency of the pendulum is $\omega_p = \sqrt{\omega_0^2 + 2k_L}$.

In all simulations, the solid and dashed lines represent the stable and unstable steady-state solutions, respectively. In Figure 2, the amplitude of the first harmonic u_{1H1} dominates the other harmonics. Therefore, the solution amplitude of the equation of motion is almost equal to the value of u_{1H1}.

In order to investigate the effect of adding an external force, we plot the frequency-responses of an array of six pendulums under different types of excitations. The equation of motion (8) of the system with 6 degrees of freedom can be written as:

$$M\ddot{\theta} + B\dot{\theta} + K\theta + G\left(\theta, t\right) = F \tag{19}$$

where

$$M = \begin{bmatrix} ml^2 & & 0 \\ & \ddots & \\ 0 & & ml^2 \end{bmatrix}; B = \begin{bmatrix} \alpha l & & 0 \\ & \ddots & \\ 0 & & \alpha l \end{bmatrix}; K = \begin{bmatrix} mgl+2k_2 & -k_2 & \dots & 0 \\ -k_2 & \ddots & & \vdots \\ \vdots & & \ddots & -k_2 \\ 0 & \dots & -k_2 & mgl+2k_2 \end{bmatrix}; F = \begin{bmatrix} f \\ \vdots \\ f \end{bmatrix}$$

Where M represents the mass matrix, B the damping matrix, K the stiffness matrix with dimension $N \times N$ and G contains the nonlinear terms and F is the excitation vector with dimension $N \times 1$.

Table 1: Comparison between different types of excitations

	Parametric	Parametric with imperfection	Simultaneous excitations
Number of branches	3 ✗	6 ✓	6 ✓
Frequency shift with respect to parametric excitation	-	shifted ✗	not shifted ✓
Symmetry of the system	symmetric ✓	not symmetric ✗	symmetric ✓

The eigenfrequencies ω_ν and the eigenvectors θ_ν of the corresponding linear system can be computed by solving the following eigenvalue problem:

$$\left(K - \omega_\nu^2 M\right) \theta_\nu = 0; \quad \nu = 1, .., N \tag{20}$$

The frequency response curves bend to the left and exhibit soft-spring characteristics beyond the critical Duffing amplitude [18, 19, 20, 21, 22] due to the negative sign of the cubic nonlinearity. Figure 3(a) presents a frequency response of the system, perfectly periodic, (19) under parametric excitation ($l_n = l_0$; $n = 1, \cdots, 6$). We can notice that pendulums 1 and 6, 2 and 5, 3 and 4 vibrate at identical amplitude due to the symmetry of the structure. Moreover, we notice that the resonant stable solution is limited to a three frequency range [6.39 6.47], [7.42 7.52] and [8.52 8.67] rad/s while the trivial solution $\theta_n = 0$ is elsewhere. The number of the resonant range is equal to three due to the mode of excitation (we excite all the pendulums with the same force). In order to break the symmetry of the system, an imperfection is introduced into one pendulum. Figure 3(b) and Figure 3(c) display the frequency responses of the system parametrically excited when an imperfection is introduced by increasing by 10% the length of the pendulums located in the 2^{nd} and the 3^{rd} position,respectively. Comparing this configuration to the previous one, we remark that the system loses its symmetry and each pendulums has a specific frequency response. In this case, the number of resonant regions and stable solutions increase.

In Figure 3(d), the system is simultanesously excited with parametric and external forces. We perform numerical simulations for the case of double excitations: the parametric excitation was applied to the whole system and a small external force was applied on the 1st pendulum. Remarkably, new branches are added approving a non-zero solution. Compared to the previous cases, we can see the existence of new stable branches and the structure still symmetric. These results are summarized in Table. A quantitative comparison of the two last configurations requires a profound investigation.

5. Conclusion

The dynamic behavior of an array of nonlinear oscillators subjected to different types of excitations (external, parametric and combination of both) has been modeled. The collective dynamics of a chain of coupled pendulums has been numerically investigated while comparing three different configurations. We showed that the introduction of imperfections in the system increases the resonant regions but it breaks the system symmetry, while remarkably, the same property was obtained in a symmetric fashion when the perfectly periodic system is subjected to simultaneous parametric and external excitations. Future works concern a quantitative comparison between the performances of each configuration and the robustness analysis of the resulting dynamical behavior.

Acknowledgements

This project has been performed in cooperation with the Labex ACTION program (contract ANR-11-LABX-01-01).

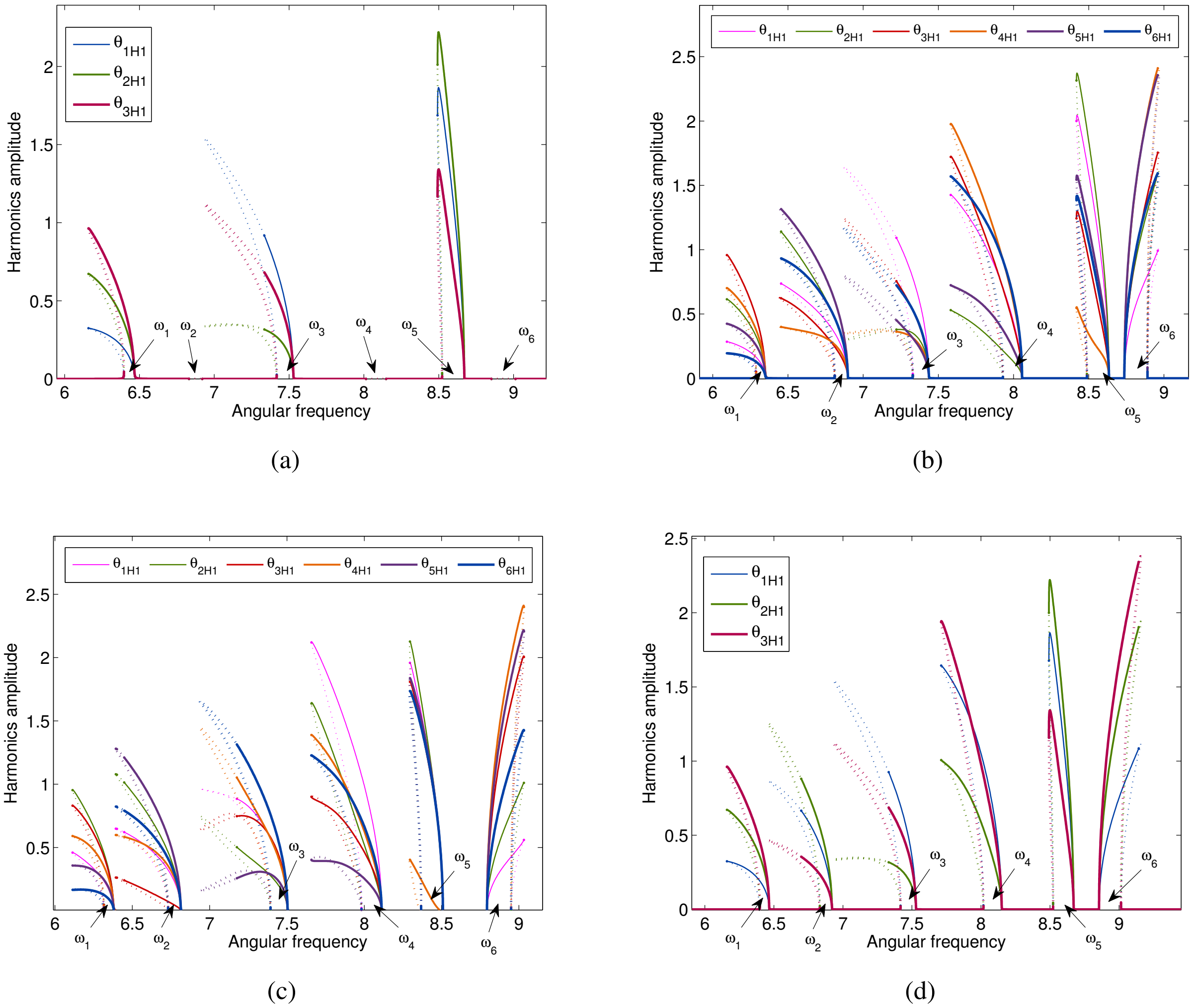


Figure 3: Frequency responses of six coupled pendulums. (a) parametric excitation, (b) parametric excitation with an imperfection in the 2^{nd} pendulum ($l_2 = 1.1l_0$), (c) parametric excitation with an imperfection in the 3^{rd} pendulum ($l_3 = 1.1l_0$), and (d) simultaneous excitations (the external force is applied on the first pendulum).

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