Proceedings of the Institute of Acoustics

ENERGY FORMULATION FOR ONE DIMENSIONAL PROBLEMS

A. LE BOT (1), L. JEZEQUEL (2)

(1) Direction des Etudes et Recherches EDF département AMV 1, avenue du Général de Gaulle 92141 Clamart CEDEX FRANCE
(2) Ecole Centrale de Lyon 36, avenue Guy de Collongue 69131 Ecully CEDEX FRANCE

INTRODUCTION

The theory of elasticity contains the tools for a general analysis of all dynamic structures. In each case (string, rod, beam, membrane, plate etc...), it yields differential equations gouverning the displacements of the structure. Generally, boundary conditions are applied in terms of displacement or force. An alternative method is Statistical Energy Analysis (S.E.A.). Instead of looking for displacement, S.E.A. gives a global energy description without any information on its space distribution.

During the last decade, methods based on the analysis of the energy flow have been developped ([1] to [4]). They lead to equations specifically adapted to high frequencies resolution because of the particular assumptions envolved and the space smoothing process of the energy distribution. We shall call them S.E.F. (Smooth Energy Formulation). However, it remains some difficulties to describe coupled conditions just with total energy and active energy flow. In [5], it is proposed to associate to those quantities, the lagrangian energy and the reactive energy flow. Then one obtains, for longitudinal and transverse vibrations, a formulation as accurate as the displacement one.

In the following study, we shall present, in a deductive form, the pattern which leads to the general energy formulation (G.E.F.). Several examples are dealt with. In each case, we shall show the assumptions that allow one to derive the S.E.F. relationships between total energy and active power.

GENERAL DEFINITIONS

We study the steady state forced oscillations of linear systems harmonically excited with frequency $\omega/2\pi$. Each part of the system has an instantaneous kinetic $E_c(t)$ and potential energy $E_p(t)$ and receive an instantaneous power $P(t)$. The sum and the difference of these energies give respectively the total energy $W(t) = E_c(t) + E_p(t)$ and lagrangian $L(t) = E_c(t) - E_p(t)$. In the harmonic case, the time is removed by taking time-averaged quantities,

$$T = \frac{2\pi}{\omega}, \qquad E_c = \frac{1}{T}\int_0^T E_c(t)dt, \qquad E_p = \frac{1}{T}\int_0^T E_p(t)dt, \qquad W = E_c + E_p, \quad L = E_c - E_p,$$

and complex power $\Pi = P + jQ$ with active P and reactive part Q. For continuous systems, the energies are replaced by energy densities and power by energy flow (it's a vector in the general case). We shall use, however, the same notations W, L and $\vec{\Pi} = \vec{P} + j\vec{Q}$.

ENERGY FORMULATION FOR ONE DIMENSIONAL PROBLEMS

POWER BALANCE IN THE HARMONIC CASE

For a continuous dissipating system, the power balance with instantaneous quantities is

$$-\vec{\nabla}.\vec{P}(M,t) - p_{diss}(M,t) = \frac{\partial W}{\partial t}(M,t),$$

where $\vec{P}$ is the energy flow at point M, p_{diss} the power density being dissipated, and $\frac{\partial W}{\partial t}$ the time rate of change of energy density. In the harmonic case, this relationship contains complex values. It is shown in [6] that if the potential energy is elastic, then the power balance yields the two following forms:

$$\vec{\nabla}.\vec{P} + p_{diss} = 0 \qquad \vec{\nabla}.\vec{Q} + 2\omega L = 0 \tag{1}$$

The interpretation of the first equation is straightforward: in steady state conditions, the active energy flow received by the system is equal to the dissipated power density (the time-averaged rate of change of the energy density vanishes). The second equation shows that the reactive energy flow received by the system is proportional to the lagrangian. So, it has a nonzero divergence even though the system is conservative.The excitations are taken into account by the boundaries conditions discussed below. We shall now determine the dissipated power density p_{diss} and the power $\vec{\Pi}$ flowing through a point in terms of energies W and L. Hence, by substituting these values into (1), we shall derive the equations governing the behaviors of energies W and L and their respective boundary conditions.

First of all, let us consider a viscous damping model. The damping strength is assumed to be proportional to the velocity. It follows that the power dissipated is proportional to the averaged kinetic energy:

$$p_{diss} = \eta(W + L). \tag{2}$$

Let us then consider the case of hysteretic damping. This one is only developped in steady state conditions. One substitutes in Hooke's law a complex Young's modulus. It follows that the power being dissipated is proportional to both, the averaged potential energy and the frequency. So

$$p_{diss} = \eta\omega(W - L). \tag{3}$$

We have two damping models expressed in terms of energies by the identities (2) and (3).

To calculate the energy flow $\vec{\Pi}$ only in terms of energies W and L, we must consider each problem seperatly in such a way that usually no general formula can be proposed. In what follows, the study of several illustration examples is presented.

DISCRET SYSTEM

Let us consider the system shown in figure 1:

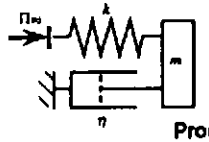


fig.1 Discret system

ENERGY FORMULATION FOR ONE DIMENSIONAL PROBLEMS

We assume that the injected power is known but not the displacement. We have to find the total energy W and lagrangian L with the method explained above. The global form of the power balance (1) is: $P_{inj} - P_{diss} = 0 \qquad Q_{inj} - 2\omega L = 0$ (4)

Moreover the damping model choosed for this system is of viscous type: $P_{diss} = \eta(W + L)$.
These relationships can be set in a matrix form as follows:

$$\begin{pmatrix} \eta & \eta \\ 0 & 2\omega \end{pmatrix} \begin{pmatrix} W \\ L \end{pmatrix} = \begin{pmatrix} P_{inj} \\ Q_{inj} \end{pmatrix}$$

The energy problem can therefore be easily solved. In this case it is obvious that applying the Newton's second law to obtain this simple result would be longer. This simple example shows us that this method is straightforward to obtain energies by knowing injected power. But we have to assume that both active and reactive terms are known.
Following the same procedure one can deal with more elaborate systems constructed with the following elements: mass, spring, hysteretic spring (with complex modulus) and dash-pot (see [6]).

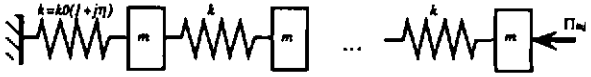


fig.2 Discretization process of an homogeneous rod

The larger the number n of elements is, the more similar the system seems to be compared with an homogeneous rod submitted to longitudinal vibrations: this is a typical discretization process.

LONGITUDINAL VIBRATIONS IN RODS

Let us consider longitudinal vibrations in a rod as shown in figure 3.

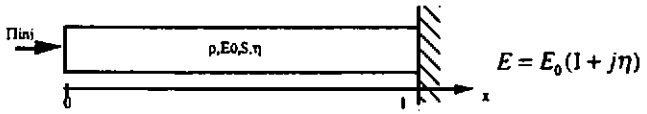


fig. 3 Free-clamped rod

The power balance is: $\dfrac{dP}{dx} + p_{diss} = 0 \qquad \dfrac{dQ}{dx} + 2\omega L = 0.$

We choose here hysteretic damping model, so: $p_{diss} = \eta\omega(W - L)$.
It remains to evaluate $P(x)$ and $Q(x)$ in terms of $W(x)$ and $L(x)$. Let $u(x)$ be the complex displacement of a point of abscissa x, energy densities and energy flow are given by:

$$E_c = \frac{1}{4}\rho S\omega^2 u u^* \qquad E_p = \frac{1}{4}E_0 S\frac{du}{dx}\frac{du^*}{dx} \quad \text{and} \quad \Pi = -\frac{j\omega}{2}ES\frac{du}{dx}u^*$$

ENERGY FORMULATION FOR ONE DIMENSIONAL PROBLEMS

By differentiating E_c and E_p with respect to x and using equation $\frac{d^2u}{dx^2}+k^2u=0$ each time a second derivative appears, we find:

$$\text{with } k_0 \overset{def}{=} \sqrt{\frac{\rho\omega^2}{E_0}} \qquad \begin{pmatrix} P(x) \\ Q(x) \end{pmatrix} = -\frac{\omega}{k_0^2}\begin{pmatrix} 1/\eta & 0 \\ 1/2 & -1/2 \end{pmatrix}\frac{d}{dx}\begin{pmatrix} W(x) \\ L(x) \end{pmatrix}. \tag{5}$$

Hence by substituting this equation into the power balance, we obtain:

$$\frac{d^2}{dx^2}\begin{pmatrix} W \\ L \end{pmatrix} + k_0^2\begin{pmatrix} -\eta^2 & \eta^2 \\ \eta^2 & 4 \end{pmatrix}\begin{pmatrix} W \\ L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6}$$

The equation (5) gives the boundary conditions associated to equation (6). For the example described in figure 3, we write:

$$\begin{pmatrix} P(0) \\ Q(0) \end{pmatrix} = \begin{pmatrix} P_{inj} \\ Q_{inj} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} P(l) \\ Q(l) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Following Wohlever in [4], we now deal with space-averaged quantities:

$$\text{with} \quad \lambda = \frac{\pi}{k_0} \qquad \langle W\rangle(x) = \frac{1}{\lambda}\int_x^{x+\lambda} W(v)dv.$$

As the space-averaged lagrangian vanishes $\forall x \ \langle L\rangle(x)=0$, then (5) and (6) become:

$$\langle P\rangle(x) = -\frac{\omega}{\eta k_0^2}\frac{d\langle W\rangle}{dx}(x) \qquad \frac{d^2\langle W\rangle}{dx^2}(x) - k_0^2\eta^2\langle W\rangle(x) = 0 \tag{7), (8}$$

Relationship (7) gives the boundary conditions associated to equation (8).

Figure 6 (energy densities) presents three types of curve: the first one is computed from the classical displacement formulation, the second one is computed from the method previously presented (G.E.F.) and the third one from the smooth formulation (S.E.F.). There is no difference between the first two. In fact, the general energy formulation assumes nothing more than the displacement formulation.

TRANSVERSAL VIBRATIONS IN BEAMS

Let us consider small transversal vibrations in an Euler-Bernouilli's beam as shown in figure 4.

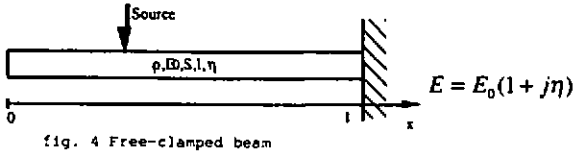


fig. 4 Free-clamped beam

ENERGY FORMULATION FOR ONE DIMENSIONAL PROBLEMS

The power balance has the same form as for rods. Moreover, hysteretic damping model still applies, so (3) remains valid. The main difference lies in the calculation of the energy flow. Here, two sorts of energie flow are present: the first $\Pi_t = P_t + jQ_t$ is carried by shear force and the second $\Pi_f = P_f + jQ_f$ by the moment. Let $u(x)$ be the transverse displacement of a point of abscissa x; energy densities and energy flows are given by:

$$E_c = \frac{1}{4}\rho S\omega^2 uu^*,\quad E_p = \frac{1}{4}E_0 I\frac{d^2u}{dx^2}\frac{d^2u^*}{dx^2},\quad \Pi_t = \frac{j\omega}{2}EI\frac{d^3u}{dx^3}u^*,\quad \Pi_f = -\frac{j\omega}{2}EI\frac{d^2u}{dx^2}\frac{du^*}{dx}.$$

This time, we have to write third and seventh derivatives of energy densities by using equation $\frac{d^4u}{dx^4} - k^4u = 0$ each time a fourth derivative appears; using $k_0 = \sqrt{\left(\frac{\rho S}{E_0 I}\omega^2\right)^{1/4}}$, we get:

$$\begin{pmatrix} P_f \\ Q_f \end{pmatrix} = \frac{-\omega}{16\eta^3}\left\{\frac{(1+\eta^2)^2}{k_0^8}\begin{pmatrix} 8+\eta^2 & -\eta^2 \\ 6\eta & 0 \end{pmatrix}\frac{d^7}{dx^7}\begin{pmatrix} W \\ L \end{pmatrix} + \frac{(1+\eta^2)}{k_0^4}\begin{pmatrix} -128-64\eta^2 & 36\eta^2 \\ -96\eta-34\eta^3 & 34\eta^3 \end{pmatrix}\frac{d^3}{dx^3}\begin{pmatrix} W \\ L \end{pmatrix}\right\}, \tag{9}$$

$$\begin{pmatrix} P_t \\ Q_t \end{pmatrix} = \frac{-\omega}{16\eta^3}\left\{\frac{(1+\eta^2)^2}{k_0^8}\begin{pmatrix} 8+3\eta^2 & -3\eta^2 \\ 2\eta & 0 \end{pmatrix}\frac{d^7}{dx^7}\begin{pmatrix} W \\ L \end{pmatrix} + \frac{(1+\eta^2)}{k_0^4}\begin{pmatrix} -128-96\eta^2 & 28\eta^2 \\ -32\eta-14\eta^3 & 14\eta^3 \end{pmatrix}\frac{d^3}{dx^3}\begin{pmatrix} W \\ L \end{pmatrix}\right\}, \tag{10}$$

By adding these two relationships, we naturally obtain the total energy flow. At last, by substituting this last expression into the power balance, it follows:

$$\frac{d^8}{dx^8}\begin{pmatrix} W \\ L \end{pmatrix} - k_0^4\begin{pmatrix} 16 & -6\eta^2 \\ 6\eta^2 & 0 \end{pmatrix}\frac{d^4}{dx^4}\begin{pmatrix} W \\ L \end{pmatrix} + k_0^8\begin{pmatrix} 0 & 4\eta^2 \\ -4\eta^2 & 16 \end{pmatrix}\begin{pmatrix} W \\ L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{11}$$

It is advised, for better numerical stability, not to use (9) and (10) as boundary conditions, but to call the relationships on energy derivatives with smaller order. For example, at clamped end:

$$(W+L)_{/x=0} = \frac{d(W+L)}{dx}_{/x=0} = \frac{d^2(W+L)}{dx^2}_{/x=0} = \frac{d^3(W+L)}{dx^3}_{/x=0} = \frac{d^3(W-L)}{dx^3}_{/x=0} = 0$$

For simplification purposes, beside the space-averaged energy quantities, we assume also the farfield conditions. Then the lagrangian vanishes and the third derivative of the total energy becomes proportionnal to the first one, $\frac{d^3\langle W\rangle}{dx^3}(x) = \frac{\eta^2 k_0^2}{4}\frac{d\langle W\rangle}{dx}(x)$. Thus, we obtain the smooth energy formulation:

$$\langle P\rangle(x) = -\frac{4\omega}{\eta k_0^2}\frac{d\langle W\rangle}{dx}(x) \qquad \frac{d^2\langle W\rangle}{dx^2}(x) - \frac{\eta^2 k_0^2}{4}\langle W\rangle(x) = 0 \tag{12), (13}$$

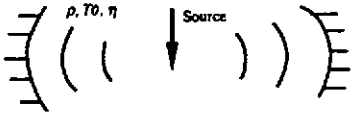
These two equations look like (7) and (8).As (8) and (13) contain $\eta^2 k_0^2$ instead of k_0^2, when using the finite element method, the number of necessary degrees of freedom is much smaller than with a displacement formulation. They are suited for solving in high frequencies domain (see [4]).

ENERGY FORMULATION FOR ONE DIMENSIONAL PROBLEMS

Figures 7 (energy densities) present the curves computed from the classical displacement formulation and the energy formulations. There is again no difference between the classical method and the G.E.F.

VIBRATIONS IN CIRCULAR MEMBRANE

Let us consider the vibrations of a circular membrane as shown in figure10:



$$T = T_0(1 + j\eta)$$

Fig. 5 Circular membrane

All the quantities envolved, are only r dependant. Thus, the power balance can be written:

$$\frac{dP}{dr} + \frac{1}{r}P + p_{diss} = 0 \qquad \frac{dQ}{dr} + \frac{1}{r}Q + 2\omega L = 0$$

We always use hysteretic damping model. Let $u(r)$ be the complex displacement, then energy quantities are expressed by:

$$E_c = \frac{1}{4}\rho\omega^2 uu^* , \qquad E_p = \frac{1}{4}T_0 \frac{du}{dr}\frac{du^*}{dr} \quad \text{and} \quad \Pi = -\frac{j\omega}{2}T\frac{du}{dr}u^* .$$

As previously done, we calculate the derivatives of kinetic and potential energies. But here, the displacement equation is $\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + k^2u = 0$. Let $k_0 \overset{def}{=} \sqrt{\frac{\rho}{T_0}}\omega^2$,

it follows:
$$\begin{pmatrix} P(r) \\ Q(r) \end{pmatrix} = -\frac{\omega}{k_0^2}\left\{ \begin{pmatrix} 1/\eta & 0 \\ 1/2 & -1/2 \end{pmatrix} \frac{d}{dr}\begin{pmatrix} W \\ L \end{pmatrix} + \frac{1}{r}\begin{pmatrix} 1/\eta & -1/\eta \\ 1 & -1 \end{pmatrix}\begin{pmatrix} W \\ L \end{pmatrix} \right\}. \tag{14}$$

Finally, by substituting into the power balance:

$$\frac{d^2}{dr^2}\begin{pmatrix} W \\ L \end{pmatrix} + \frac{1}{r}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\frac{d}{dr}\begin{pmatrix} W \\ L \end{pmatrix} + k_0^2\begin{pmatrix} -\eta^2 & \eta^2 \\ \eta^2 & 4 \end{pmatrix}\begin{pmatrix} W \\ L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{15}$$

As we saw with the rod, relationship (14) gives the boundary conditions associated to equation (15). For the example described in figure 5:

$$\begin{pmatrix} P(r_{min}) \\ Q(r_{min}) \end{pmatrix} = \begin{pmatrix} P_{inj} \\ Q_{inj} \end{pmatrix} \qquad \begin{pmatrix} P(r_{max}) \\ Q(r_{max}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

ENERGY FORMULATION FOR ONE DIMENSIONAL PROBLEMS

The following two relationships give the smooth solution for energy densities and energy flow.

$$\langle P\rangle(r) = -\frac{\omega}{\eta k_0^2}\left\{\frac{d\langle W\rangle}{dr}(r) + \frac{2}{r}\langle W\rangle(r)\right\} \qquad \frac{d^2\langle W\rangle}{dr^2}(r) + \frac{3}{r}\frac{d\langle W\rangle}{dr}(r) - k_0^2\eta^2\langle W\rangle(r) = 0 \quad (16), (17)$$

These two equations represent the smooth energy formulation. A difficulty appears: the second member of (16) is not proportional to the gradient of the space averaged total energy because of the coefficient 2. In other worlds, the first two terms of equation (17) is not a particular form of the laplacien. So, we cannot consider this equation as the heat equation for steady state conditions.
Figure 8 shows the comparison between the classical displacement formulation and the energy ones.

CONCLUSION

The formulations presented above constitute alternative forms for computing the total energy and lagrangian. Taking into account both quantities, they are as accurate as the displacement ones. Moreover coupling conditions can be exactly formulated (see [5] and [6]). But they lead to equations as difficult to solve as the displacement ones. So there is no advantage to take it for high frequency problem. However, the smooth formulations can be deduced. They lead to equations which can be solved by finite element method at high frequencies as the number of degrees of freedom is small. That is the practical interest to use these specific equations rather than the displacement ones. In the case of the circular membrane, it appears that equation (17) cannot be expressed with a laplacien. So, a general form for the equations of the smooth formulation which is able to lead to equations (7), (8), (12), (13), (16) and (17) should be investigated. Besides the differences between energy information and displacement information have to be investigated.

REFERENCES

[1] Belov V.D., Rybak S.A., Tartakovskii B.D., Propagation of vibrational energy in absorbing structures, Soviet Physics Acoustics, vol.23 n°2 1977.
[2] Nefske D.J., Sung S.H., Power flow finite element analysis of dynamic systems: basic theory and application to beams, N.C.A. publication vol.3 1987.
[3] Bernhard R.J., Wohlever J.C., Mechanical energy flow models of rods and beams,Journal of Sound and Vibration, April 1992.
[4] Luzzato E., Approximations and solutions of the vibration energy density equation in beams, Inter-Noise 1991.
[5] Lasé Y., Jezequel L., Analysis of a dynamic system based on a new energetic formulation, Senlis CETIM 1990.
[6] Le Bot A., Formulation énergétique de quelques problèmes vibratoires, rapport EDF DER n°HP-62.93017

ENERGY FORMULATION FOR ONE DIMENSIONAL PROBLEMS

COMPARISON BETWEEN CLASSICAL FORMULATION, G.E.F. AND S.E.F. RESULTS

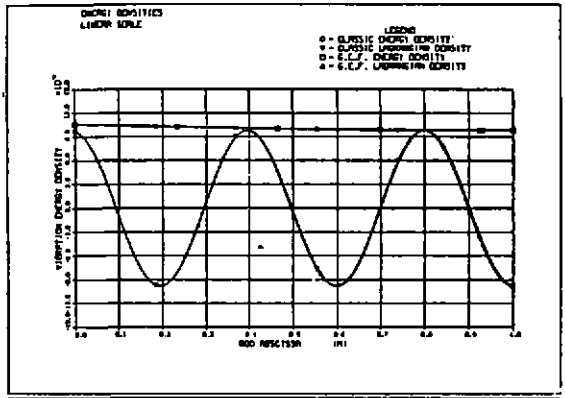


fig.6 Energy densities for a uniform free-clamped rod excited at the free end (left)

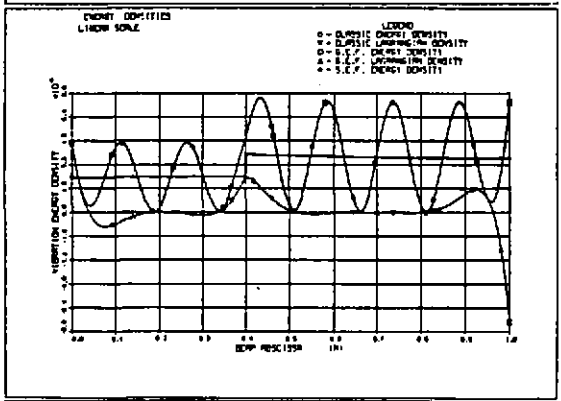


fig.7 Energy densities for a free-clamped beam excited at x=0.4.

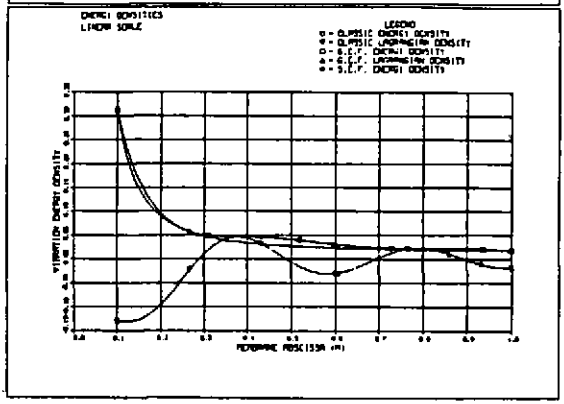


fig.8 Energy densities for a circular membrane excited in its center.