ANALYTICAL METHODS FOR ONE-DIMENSIONAL IMPACT PROBLEMS INVOLVING LAYERED ELASTIC MEDIA

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A homogeneous semi-infinite flyer impacts a finite layered Goupillaud-type medium attached to a homogeneous half space. After the initial impact in the direction perpendicular to the layers, the stress wave propagates through the medium. The Goupillaud-type finite layered medium allows for the stress to be modeled discretely through a linear dynamical system. Here we analyze the short and long term behavior of stress by means of the coefficient matrix of the system and its eigenvalues.
Keywords: discrete model, wave propagation, eigenvalues.

1. Introduction

Here we study a one-dimensional normal impact problem, where an infinite linear elastic homogeneous flyer collides/welds with a stationary Goupillaud-type elastic layered target, attached to a halfspace, see Fig. 1.

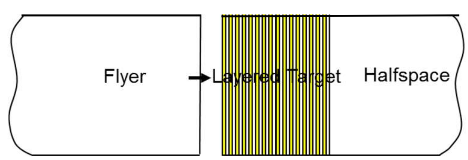

Figure 1: Semi-Infinite Flyer Impacting a Layered Target on Halfspace.

The target has a finite length *L* and is initially at rest. As shown in Fig. 2 and [1-4], after the initial impact, the stress waves meet and split at the same time at each layer interface, since the Goupillaud-type layered medium of the target provides equal wave travel time for each layer. Distinct stress sequences denoted by $\{S_i(k)\}_{k=1}^{\infty}$, $i = 0,1,\dots m,$ develop over time. This results in a discrete model and a linear system of coupled recursive relations with constant coefficients given below:

$$\begin{cases} s_0(k+1) = \frac{1-\alpha_0}{1+\alpha_0}\, s_0(k) + \frac{2\alpha_0}{1+\alpha_0}\, s_1(k) + \frac{v_{imp} z_{imp}}{1+\alpha_0}\, \delta_k\{0\} \\ s_i(k+1) = -\, s_i(k) + \frac{2\alpha_i}{1+\alpha_i}\, s_{i+1}(k) + \frac{2}{1+\alpha_i}\, s_{i-1}(k+1)\,, \quad i = 1,2,\dots m-1 \\ s_m(k+1) = -\, \frac{1-\alpha_0}{1+\alpha_0}\, s_m(k) + \frac{2}{1+\alpha_m}\, s_{m-1}(k+1) \end{cases} \tag{1}$$

Here $k = 0,1,2,3,\dots,$ $\delta_k\{0\}$ represents the discrete version of the Dirac measure, while the impedance ratios of two consecutive layers are represented by $\alpha_i = z_i/z_{i+1}$, $i = 0,1,2,\ldots,m$. The boundary condition at the impact face *x=0* is taken from [5], while the continuity conditions of stress and displacement apply across each layer interface. It was concluded in [3, 4] that over time all the stress sequences approach the same value, (steady-state solution) l_s. This limit stress value does not depend on the properties of the finite layered target as shown below:

$$l_s = \frac{v_0 z_0 z_{m+1}}{z_0 + z_{m+1}}. \tag{2}$$

Here v_0 and z_0 are the velocity and impedance of the flyer, while z_{m+1} is the impedance of the halfspace.

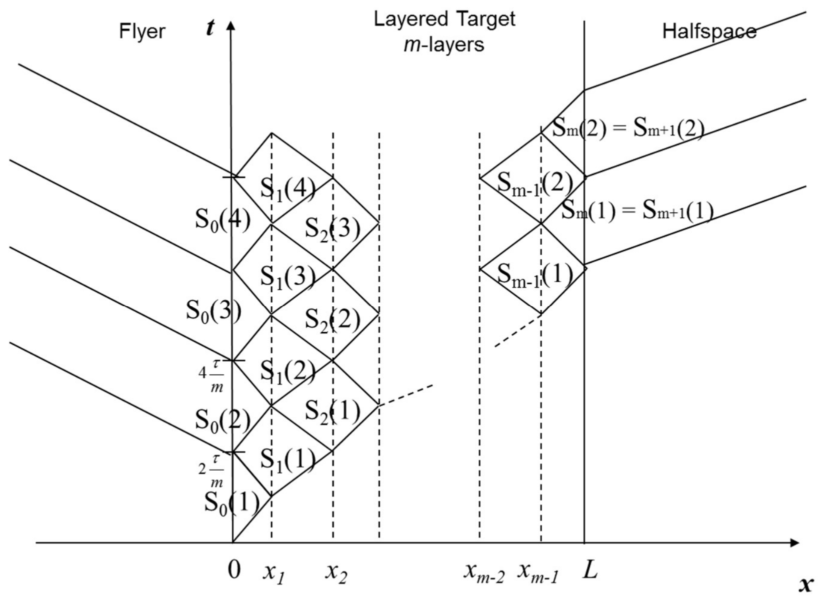

Figure 2: Discrete Model. Lagrangian Diagram.

The main purpose of this work is to gain new insight and independently prove the results from [4], by taking an alternative approach and treating the coupled linear system of the stress terms in Eq. (1), as a linear dynamical system, evaluating and analyzing the coefficient matrix and its eigenvalues.

2. General solution of our discrete dynamical system

The system of the recursive relations for the stress terms developed in [4] and Fig.2, can be written as:

$$\vec{S}(k+1) = A_m \vec{S}(k), \tag{3}$$

Here $\vec{S}(k) = [s_0(k), s_1(k), \ldots, s_i(k) \ldots s_m(k)]^T$, $k = 0,1,2,3,\ldots$, A_m is the $(m+1)\times(m+1)$ coefficient matrix, and m represents the number of layers of the target. The description of A_m for the three-layer case $(m = 3)$ is given below:

$$A_3 = \begin{bmatrix} \sigma_0 & \mu_0\alpha_0 & 0 & 0 \\ \mu_1\sigma_0 & (\mu_0\,\mu_1\alpha_0 - 1) & \mu_1\alpha_1 & 0 \\ \mu_1\,\mu_2\sigma_0 & \mu_2(\mu_0\,\mu_1\alpha_0 - 1) & (\mu_1\,\mu_2\alpha_1 - 1) & \mu_2\alpha_2 \\ \mu_1\,\mu_2\mu_3\sigma_0 & \mu_2\mu_3(\mu_0\,\mu_1\alpha_0 - 1) & \mu_3(\mu_1\,\mu_2\alpha_1 - 1) & \mu_2\,\mu_3\alpha_2 - \sigma_3 \end{bmatrix} \tag{4}$$

Equation (4) can be generalized for any number of layers *m*.

The entries of the coefficients matrix are given in terms of the layer impedance ratios:

$$\mu_i = \frac{2}{1+\alpha_i}, \ \sigma_i = \frac{1-\alpha_i}{1+\alpha_i} \quad \text{for } i = 0,1,2,\dots,m. \tag{5}$$

The general solution for our discrete dynamical system in Eqs. (3) - (5) is:

$$\vec{S}(k) = C_0\lambda_0^k\vec{v}_0 + C_1\lambda_1^k\vec{v}_1 + \cdots + C_m\lambda_m^k\vec{v}_m. \ k = 1,2,3,\dots \tag{6}$$

Here $\vec{v}_i$ is the eigenvector of the single eigenvalue λ_i, while C_i is their corresponding constant, $i = 0,1,2,\dots m$. The eigenvalues of the A_m matrix are determined by solving the characteristic equation $|A_m - \lambda I| = 0$.

2.1 Expectations about the eigenvalues

Based on [4] and Eq. (2), in order for all the stress terms to approach the same non-zero value l_s as k approaches infinity, the following statements must be true.

2.1.1 All the eigenvalues λ_i of the coefficient matrix A_m except one, must satisfy $|\lambda_i| < 1$, for $i = 0,1,2,\dots,m$.

Verifying this statement for a large number of layers becomes computationally intensive. Here is a demonstration for the case of a one layer target ($m = 1$). The characteristic equation is

$$|A_m - \lambda I| = \begin{vmatrix} \sigma_0 - \lambda & \mu_0\sigma_0 \\ \mu_1\sigma_0 & \mu_0\mu_1\alpha_0 - \sigma_1 - \lambda \end{vmatrix} = -\lambda^2 + (-\sigma_0 - \mu_0\mu_1\alpha_0 + \sigma_1)\lambda - \sigma_0\sigma_1 =$$

$$=(\lambda - 1)(\lambda + \sigma_0\sigma_1)=0 \tag{7}$$

Thus, the two eigenvalues are $\lambda_0 = 1$ and $\lambda_1 = -\sigma_0\sigma_1$, where $|\lambda_1| = |-\sigma_0\sigma_1| < 1$ as expected, since $|\sigma_0\sigma_1| = \frac{(1-\alpha_0)(1-\alpha_i)}{(1-\alpha_0)(1-\alpha_1)}<1$.

2.1.2 There must be a single eigenvalue of unity, $\lambda_0 = 1$, with its corresponding $(m+1)\times 1$ eigenvector of ones $[1,1,1,\dots,1]^{\mathrm{T}}$, for all $m \geq 1$.

Indeed, it can be shown by inspection/induction that each row in the coefficient matrix A_m sums to one. After applying this fact, we replace each term of the first column of $A_m - \lambda I$ with the sum of the terms in the respective row. Since this operation does not change the value of the determinant $|A_m - \lambda I|$, after factoring the common factor $(1-\lambda)$, the characteristic equation becomes:

$$|A_m - \lambda I| = \begin{vmatrix} (1-\lambda) & \\ (1-\lambda) & \\ & \dots \\ \vdots & \\ (1-\lambda) & \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & \\ 1 & \\ & \dots \\ \vdots & \\ 1 & \end{vmatrix} = 0. \tag{8}$$

Equation (8) shows that $\lambda = 1$ is an eigenvalue of A_m for any number of layers $m \geq 1$, as previously demonstrated in Eq. (7). Using similar argument, it can be shown by inspection that the eigenvector is a vector of ones.

2.1.3 When A_m is a Markov matrix, all the stress sequences converge to the same finite value.

Based on the discussion in subsubsection 2.2.2, a coefficient matrix with positive entries is a Markov matrix, since the sum of each row of the matrix is already equal to one. A Markov matrix satisfies statements 2.2.1 and 2.2.2. In addition, from Eqs. (2) and (6), one can also derive that $C_0 = l_s$. As a result, we conclude that all the stress sequences converge to the same value l_s.

For instance, the medium consisting of a flyer, a three-layered target and halfspace with impedance ratios $\alpha_0 = \frac{9}{10}$, $\alpha_1 = \frac{16}{19}\,\alpha_2 = \frac{28}{35}\ \alpha_3 = \frac{2}{3}$, from Eqs. (4) - (5), is characterized by the following system coefficient matrix:

$$A_3 = \begin{bmatrix} 1/19 & 18/19 & 0 & 0 \\ 2/35 & 1/35 & 32/35 & 0 \\ 4/63 & 2/63 & 1/63 & 56/63 \\ 24/315 & 12/315 & 6/315 & 273/315 \end{bmatrix} \tag{9}$$

The system matrix A_3 in Eq. (9) is a Markov Matrix and has four eigenvalues, two of which are complex conjugate, as shown below:

$$\lambda_0 = 1$$

$$\lambda_{1,2} \approx 0.07932 \pm 0.218444i \ \text{ with } |\lambda_{1,2}| \approx 0.232399 < 1 \tag{10}$$

$$\lambda_3 \approx -0.194897$$

With the exception of the unity eigenvalue, the rest of the eigenvalues in Eq. (10) have a magnitude less than one, a fact that based on Eq. (6) guarantees the convergence of all the stress sequences to the same finite value.

Further understanding of the convergence of the stress terms when the system matrix is not a Markov Matrix remains to be explored in the future.

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