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A STUDY OF UNCERTAINTY IN APPLICATIONS OF STATISTICAL ENERGY ANALYSIS TO ONE-DIMENSIONAL AND TWO-DIMENSIONAL STRUCTURAL SYSTEMS.

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1. INTRODUCTION

Statistical Energy Analysis (SEA) provides an estimation of broadband frequency vibration of large and complex mechanical systems. SEA estimates have been found to be (in general) more reliable and acceptable than those obtained from other approaches, especially at high frequencies, when the deterministic analysis of individual modes become less accurate. But the crucial questions are: 'What are the criteria to quantify the reliability?', and, 'What uncertainty is associated with the prediction of response when using this approach?'. These questions arise because of the fact that SEA is based on the concept of subsystems with random parameters in which the time-average response spectrum is calculated as an *average* across an ensemble of similar systems. Such a spectrum is then integrated to give the total energy flow (or level), combining information from all frequencies in an analysis band. In practice, one is usually concerned to estimate the vibrational behaviour of individual physical systems (e.g., a rocket launcher payload) or to infer SEA parameters from tests on a single physical realisation. Since no single physical system can correspond precisely to the assumed ideal theoretical model (ensemble average), practical differences are likely to cause the actual behaviour to deviate greatly from SEA estimate, or lead to a typical inferred parameters. The degree of uncertainty will depend on many factors, such as the range of variation of the dimensions of the coupled subsystems, the fabrication tolerances, the variations in the properties of the materials, the number of modes that are employed in the averaging process, uncertainties in the loss factors, and in the modal densities (or the degree of modal overlap). During the past 30 years most of the work carried out on SEA has been directed toward the prediction of the mean response of coupled systems. Very little attention [1] has been given by the researchers to the problems of uncertainty of response estimates; namely, the variance, the associated probability distributions, and confidence limits. The principal objective of the research programme was to study the uncertainty of the predictions made by the application of Statistical Energy Analysis in terms of perturbation of subsystem geometry. In the examples described in this paper this is achieved by analyzing two cases of multi-mode coupled systems using exact calculations and establishing the influence of the perturbations of the structural parameters on the sensitivity of the predictions of the quantities of interest in SEA. The cases analyzed are coupled beams and line-coupled plates systems.

2. TWO COUPLED BEAMS SYSTEM

The coupled system analyzed consists of two long, steel cantilever beams coupled at their free ends by a non-conservative coupling (damped translational spring). Beam 1 (driven) has a mean length of 3.68 m, a width of 0.025 m and a thickness of 0.003 m. Beam 2 (receiver) has a length of 3.58 m, a width of 0.025 m and a thickness of 0.002 m. Transverse vibration of Beam 1 is generated by a point harmonic force. Vibrational energy is transmitted from beam 1 to beam 2 by shear force and velocity at the coupling point. A solution for the bending displacement of beam i ($i=1,2$) is sought to have the form

$$w_i(x_i,t) = (A_1\, e^{-jk_ix_i} + A_2\, e^{+jk_ix_i} + A_3\, e^{-k_ix_i} + A_4\, e^{+k_ix_i})\, e^{j\omega t} \;, \qquad (1)$$

where A_1 and A_2 represent the amplitudes of the propagating waves; A_3 and A_4 represent the nearfield amplitudes; k_1 and k_2 are the flexural wave numbers of the coupled beams, and ω is the frequency of vibration. The form of equation (1) satisfy the geometric and the natural boundary conditions at the excitation position, the coupling point, and the clamped ends of the coupled beams.Beam damping is taken into account by assuming a complex Young's modulus. This leads to a complex representation for the wave numbers and the wave speeds.

The time-average power injected into the coupled system is given by

$$\bar{P}_{in} = \frac{1}{2}\,\mathrm{Re}\{\tilde{F}\ \tilde{V}_d^*\} \;, \qquad (2)$$

where $\tilde{F}$ and $\tilde{V}_d$ are the applied force and the velocity at the driving point. Re{ } and (*) denote 'real part' and 'complex conjugate' respectively.

The vibrational energy transmitted from the driven beam to the receiver is given by

$$\bar{P}_{12} = \frac{1}{2}\,\mathrm{Re}\{\tilde{S}_{2c}.\tilde{V}_{2c}^*\} \;, \qquad (3)$$

where $\tilde{S}_{2c}$ and $\tilde{V}_{2c}$ are the shear force and the velocity at the coupled end of the receiver beam, respectively. The shear force can be written as

$$\tilde{S}_{2c} = \tilde{K}_c\ \{\tilde{w}_{1c} - \tilde{w}_{2c}\} \;. \qquad (4)$$

where $\tilde{K}_c$ is the complex stiffness of the coupling element.

By using the approximation that only the resonant modes are contributing to the energy content in the frequency band of interest, and by employing the result of simple oscillator energetics, the time-average total energies of the beams ($\bar{E}_1$ and $\bar{E}_2$) are calculated as twice the time-average kinetic energies. The latter is written for a beam of length l and mass per unit length m' as

$$\bar{T}_i = \frac{1}{2}\, m_i'\, \omega^2 \int_0^l \mathrm{Re}\{w_i(x_i)\,.\,w_i^*(x_i)\}\ dx_i \;. \qquad (5)$$

2.1 Perturbation Analysis

In order to investigate the sensitivity of the quantities of interest to small variations in one of the structural parameters, the distribution sampling application of the Monte Carlo method [2] is employed in the present work. The length ratio (l_1/l_2) of the coupled beams is chosen as the input random variable for the purpose of perturbation analysis of the coupled beams system under study. A sample of 32 elements of (l_1/l_2) is drawn randomly from a normally distributed random population. This sample size is justified as being adequate to perform statistical calculations according to the argument of Hodges and Woodhouse [3,4] that the frequency averaging has a somewhat similar effect to including many more configurations of the coupled system. The generated sample has values of mean and standard deviation which provide a coefficient of variation of 10% of the mean length ratio of the coupled beams. For each given value of (l_1/l_2), the time averaged quantities of input power, power flow, and total beams energies of the coupled system are calculated and the results are then integrated over the frequency range of interest (1-50 Hz) to simulate the response to a band of white noise. The computations are carried out for ten, randomly selected, points of excitation and the results are then averaged over the selected points. The reason for exciting the system at different points is to approximate the assumption of 'uncorrelated excitations' made in SEA. The computed quantities form ensembles having a size of 32 elements. Each element in these ensembles represents an observation (output random variable) from one realization of the coupled beams system. Note that the produced ensembles $\bar{P}_{in}$, $\bar{P}_{12}$, $\bar{E}_1$, and $\bar{E}_2$ are for a given value of internal loss factors of the coupled beams, i.e., for a given value of average modal overlap factor (M_{av}). The latter parameter is calculated as $M_{av} = \sqrt{M_1 M_2}$, where M_1 and M_2 are the average modal overlap factors of beam 1 and beam 2 respectively. This is an *ad hoc* relationship which is justified by the fact that M_1 and M_2 were chosen to be similar. The process of calculations described above is then repeated for different M_{av} values to study the effect of the latter parameter on the sensitivity of the predictions. The different values of the average modal overlap factor of the coupled beams are obtained by assuming different values of dissipation loss factors η_1 and η_2. An attempt to obtain a range of M_{av} by assuming different modal densities for the coupled beams (i.e., by assuming different lengths or different cross section areas) was not found to be practicable because of computational difficulties at high modal overlap.

The effects of small variations (perturbations) in the geometry of the coupled system on the sensitivity of the prediction are investigated by calculating the mean and the normalized variance from the computed ensembles of the quantities of interest. The quantities selected for these investigations are the power flow from the driven beam to the receiver beam, and the coupling loss factor of the coupled system, because they represent the most important quantities in SEA formulation. The coupling loss factor is calculated on the basis of the SEA hypothesis [5] as

$$\eta_{12} = \frac{\bar{P}_{12}}{\omega N_1 \left\{ (\bar{E}_1/N_1) - (\bar{E}_2/N_2) \right\}} , \qquad (6)$$

where N_1 and N_2 are the expected number of modal frequencies in beams 1 and 2 respectively in the selected frequency band, and $(\bar{E}_1/N_1)$-$(\bar{E}_2/N_2)$ is the average modal energy difference. Another estimate for the coupling loss factor is obtained from a corresponding infinite coupled beams system by using the travelling wave method [5]. The latter estimate is used for the comparison with those obtained from the finite system. The sensitivity of the predictions is also investigated by studying the cumulative probability distributions of the coupling loss factor and the power flow at different modal overlap conditions. The 95% confidence intervals for the *mean* values of these quantities are estimated on the assumption of a normal distribution, justified by the central limit theorem.

2.2 Results and Discussion

Figure (1) displays the mean values of the coupling loss factor (η_{12}) of the coupled, multi-mode, beams system against the average modal overlap factor (M_{av}) of the coupled system. It also displays the estimate of coupling loss factor of the corresponding infinite system ($\eta_{12\infty}$). Figure (2) displays the mean values of the normalized power flow ($\bar{P}_{12}/\bar{P}_{in}$). Figure (1) shows that the mean of η_{12} has the tendency to increase as M_{av} increases, asymptotically approaching the estimate of coupling loss factor of infinite system as the average modal overlap takes high values ($M_{av} \geq 1$). This result is consistent with those of Davis and Wahab [6] and Davis and Khandoker [7] for cases of coupling of two simply supported beams. Figure (2) shows that the mean value of the normalized power flow is lower at high modal overlap than at low modal overlap. This result may be attributed to the high dissipation in the energy of the driven beam, which is associated with the increase in the internal loss factors of the coupled beams over the range used to increase the modal overlap factors of the beams. Figures (1) and (2) also display the 95 per cent confidence intervals for the calculated means of η_{12} and $(\bar{P}_{12}/\bar{P}_{in})$. The intervals define the degree of confidence that the calculated *means* lie between two certain limits. The probability associated with the confidence interval is called the confidence coefficient. The widths of the intervals demonstrate the effect of M_{av} on the sensitivity of the predictions. The intervals are wider at low modal overlap than at high modal overlap. This suggests that at high modal overlap the probability of obtaining estimates for the above quantities which are close to the means is high. This result is confirmed by calculating the normalized variance (variance/(mean)^2) values for η_{12} and $(\bar{P}_{12}/\bar{P}_{in})$ at different modal overlap conditions. The results are plotted in figures (3) and (4) respectively. These figures show clearly that the normalized variance decreases as M_{av} increases. The normalized variance takes relatively small values at high modal overlap. A possible reason for this is that the frequency response of the coupled beams is dominated by the well separated resonant peaks at low modal overlap which are highly sensitive to small changes in structural details. The changes have no considerable effect on the smoothed frequency response at high modal overlap. The comparison between figure(3) and figure (4) shows that the normalized variance of coupling loss factor is always higher than that of the normalized power flow. This is because the coupling loss factor is affected by the sensitivities of both power flow and the energy difference of the coupled beams to system parameter variations. Figures(9) and (10) display the cumulative probability distributions for a given sample of coupling loss factor and the normalized power flow at different average modal overlap factor values respectively. From these figures it can be emphasized that as M_{av} increases, the probability that any realization of η_{12} or $(\bar{P}_{12}/\bar{P}_{in})$ which is close to the mean value increases. This means that we are certain that the calculated means of the above quantities are more reliable and acceptable at high modal overlap than those at low modal overlap. The way of presentation of the cumulative probability distribution provides a simple, and quick method of testing for normality. It is clearly displayed that as M_{av} increases and takes values equal to unity and greater, the cumulative distribution curves will have the shape of a straight line. This result suggests that the observations came from a normal distribution [8], which is similar to the assumed distribution of the input perturbed parameter l_1/l_2. The results described above are supported qualitatively by experimental investigations (not reported herein).

3. TWO COUPLED PLATES SYSTEM

The coupled system under investigation consists of two rectangular steel flat plates; each has two opposite sides simply-supported. The plates are coupled together along one edge via an elastic, non-conservative coupling element which allows for translational as well as rotational motions of the the coupled edges. The far edges of the plates are assumed to be clamped. Plate 1 has the mean dimensions of 2.75m x 1.5m x 3mm for the length, the width and the thickness respectively. Plate 2 has the mean dimensions of 2.65m x 1.5m x 3mm. The first plate (driven) is excited by a point harmonic force of a unit amplitude. A similar approach to that of the previous model is point harmonic force of a unit amplitude. A similar approach to that of the previous model is followed in this case. An empirical relationship between the variance of coupling loss factor, the number of modes involved and the average modal overlap factor of the coupled system is developed.

3.2 Results and Discussions

Results for the mean of coupling loss factor and the normalized power flow of the two-plate model are shown in figures (5) and (6) respectively. Figure (5) also displays the diffuse field estimate of coupling loss factor of the corresponding infinite system. Similar conclusions to that of two-beam model are drawn. Results for the normalized variances of the coupling loss factor is shown in figure (7). Figure (8) shows the effect of the actual number of modes on the sensitivity of the mean value of coupling loss factor. This figure was obtained by carrying out the analysis in different frequency bands. It shows that, more than five resonant modes are needed (when $M_{av}>1$) in order to obtain good SEA results. Cumulative probability distributions are shown in figures (11) and (12) for the coupling loss factor and the normalized power flow respectively. The results are qualitatively supported by experimental results (not reported herein).

4. CONCLUSION

From the results presented above a general conclusion may be drawn that the travelling wave calculation overestimates the coupling loss factor of the coupled one-dimensional and the two-dimensional systems when the average modal overlap factor is small (less than 1.0). The sensitivity of predictions of the coupling loss factor and the normalized power flow to geometric parameter variation is highly affected by the degree of modal overlap of the coupled system. High variances are observed for the above quantities at low values of modal overlap factor. It is concluded that the coupling loss factor is much more sensitive quantity than the normalized power flow, because of the the presence of the energy difference term in the denominator of the expression for the coupling loss factor. The above conclusions supported by experimental investigation. It is found that the number of resonant modes of coupled plates system needs to be greater than five, at high modal overlap, in order to obtain an acceptable estimate for the coupling loss factor.

5. REFERENCES

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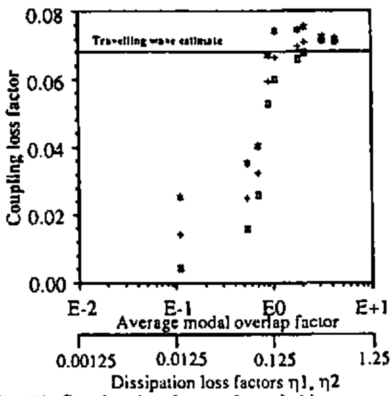

Fig. (1) :Coupling loss factor of coupled beams system
+ Mean value, * # 95% confidence interval

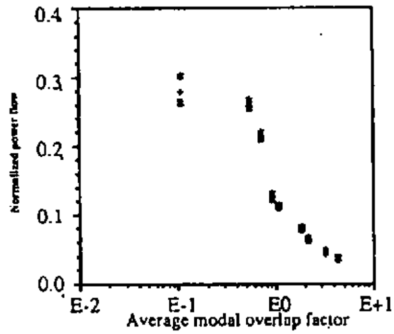

Fig. (2) :Normalized power flow between two coupled beams,
+ mean value, * # 95% confidence interval

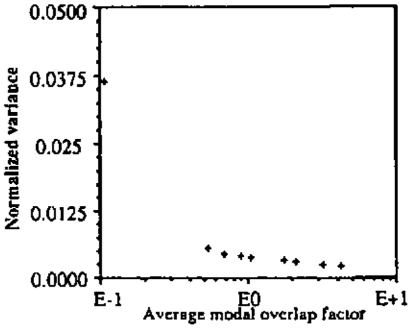

Fig. (4) :Normalized variance of normalized power flow of two coupled beams system

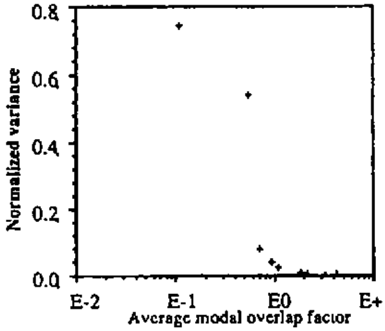

Fig. (3) :Normalized variance of coupling loss factor of two coupled beams system

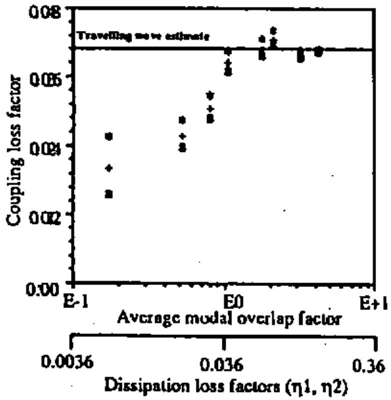

Fig. (5) :Coupling loss factor of two coupled plates system
+ Mean value, * # 95% confidence interval

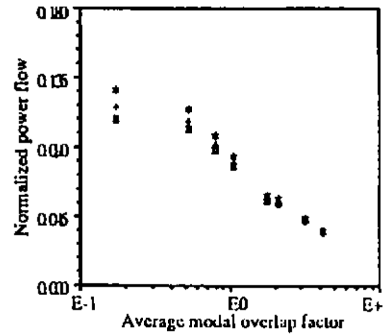

Fig. (6) :Normalized power flow between two coupled plates,
+ mean value, * # 95% confidence interval

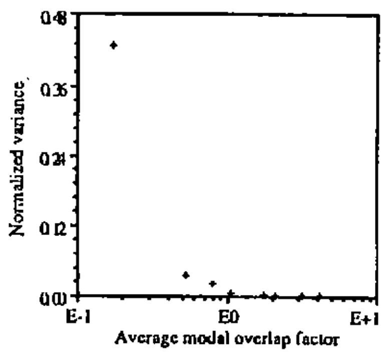

Fig. (7) :Normalized variance of coupling loss factor of two coupled plates system

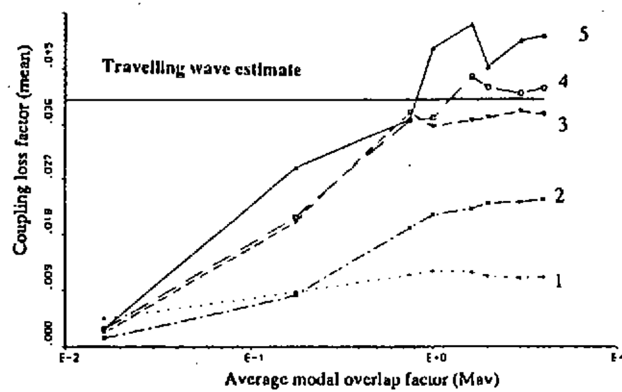

Fig. (8) :Coupling loss factor of two coupled plates system in different frequency bands,
(1) N=3, (2) N=4, (3) N=5, (4) N=8, (5) N=11 (N is the number of modes).

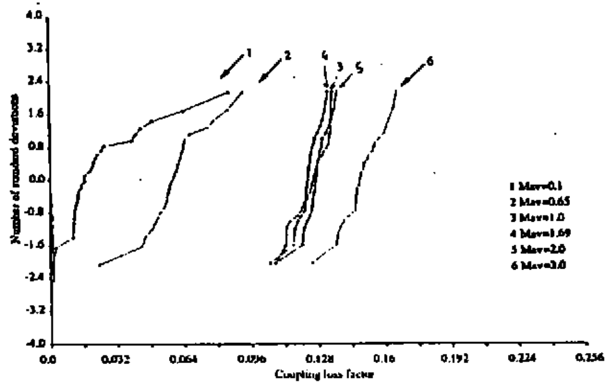

Fig.(9): Cumulative distribution function of coupling loss factor of two coupled beams

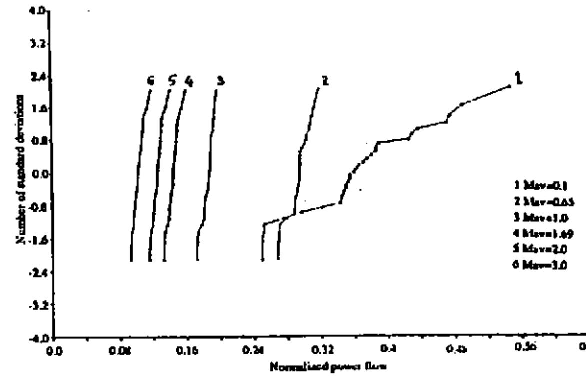

Fig.(10): Cumulative distribution function of normalised power flow between two coupled beams

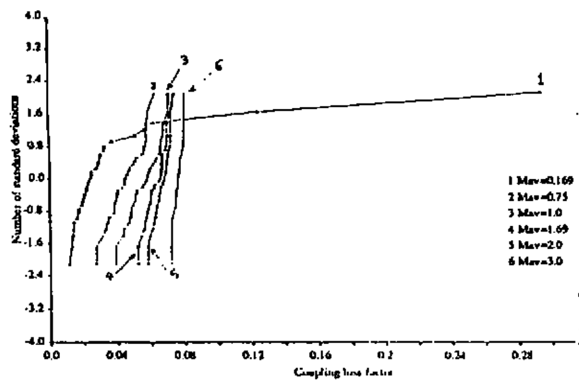

Fig.(11): Cumulative distribution function of coupling loss factor of two coupled plates system

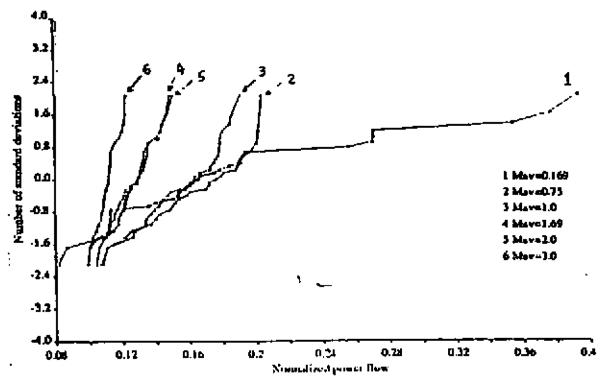

Fig. (12): Cumulative distribution function of normalized power flow between two coupled plates