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SOME ASPECTS OF THE PARAMETRIC AND NONLINEAR VIBRATION OF STRUCTURES

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INTRODUCTION

Parametric and nonlinear effects in structural vibration are often regarded as uncommon curiosities which manifest themselves only under exceptional conditions. While it is true that extremes of excitation or response virtually require that these effects be taken into consideration, they are essentially always 'there' and may, even under what could be regarded as very moderate conditions, exert a considerable influence on the behaviour of the structure.

The study of parametric vibration of structures is concerned with the behaviour of structures in which the parameters — inertia, stiffness and the like — are regarded as explicit functions of time. It represents a level of linear modelling one stage more refined than the usual constant coefficient model. The variation of the parameters in time is usually taken to be small relative to their average values but in a large class of problems of practical importance, elastic mechanisms, the dynamics of which must be considered as within the realm of structural dynamics, this is often not the case.

Retention of nonlinear terms to some degree in the modelling of a structure allows refinement of its predictive capabilities and extends its range of application (to larger amplitudes for instance). Further, the nonlinear model may introduce the means of explaining or predicting behavioural phenomena which were outside the limits of the linear theory. Nonlinear mathematical models of structures will often include parametric effects i.e. variable coefficients, and vice-versa so that various features of the two become interwoven.

This paper describes a few of the features of some work by the author and his colleagues and students on simple one degree of freedom parametrically excited structures and on nonlinear modal interaction phenomena in multi-degree structures.

PARAMETRIC INSTABILITY

The solution of equations with explicit time dependent coefficients may under certain circumstances exhibit enormous growth from relatively small initial conditions. Normally only nonlinearities in the equations can limit the growth. This sort of behaviour is referred to as parametric instability.

For a periodic system, that is a system of linear equations with periodic coefficients, the solutions are characterised by the monodromy matrix. This matrix relates the solution matrix of the equations after one period of the coefficients to the initial condition matrix. Its eigenvalues are complex and if any of them has a magnitude greater than unity then the solution will grow each period, that is it will be unstable. Computing the monodromy matrix and its eigenvalues can thus be used for establishing the instability zones of a periodic system.

Long period inputs

Most work on parametric vibration is concerned with the response to excitation having a period of the same order as that of the structure. So called main parametric resonance occurs when the excitation period is about one half the natural period.

Long period parametric excitation occurs naturally if a structure is subjected to two or more inputs whose frequencies are not simple multiples of one another. The overall period can then be very great so that many cycles of oscillation of the structure occur in it (in theory it can be of indefinite extent as with the period of $(\cos\omega t + \cos(\pi/3)\omega t)$. The study of systems under these long period excitations can also shed light on their behaviour under random parametric excitation.

One type of problem that has been examined is the stability of the null solution of the equation

$$\ddot{y} + 2\nu\omega\dot{y} + \omega^2\{1 + \varepsilon f(t)\}y = 0 \qquad (1)$$

where f(t) has the form

$$f(t) = \sum_{r=-n}^{n} \cos\{\Omega(1 + r\alpha)t + \phi_r\} \qquad \text{(integar r)} \qquad (2)$$

comprising $(2n + 1)$ unit amplitude sinusoids in the neighbourhood of Ω, the frequency spacing α being taken small. The initial phase angles ϕ_r are fixed and can be chosen from a rectangular distribution on $(0,2\pi]$ generated in a digital computer. This f(t) is a form of Rice noise. For fixed ϕ_r, f(t) is a pseudo-random function with period $2\pi/(\alpha\Omega)$ which is long for $\alpha \ll 1$.

Stability maps on the plane of $(\Omega/2\omega)$ against ε have been produced for this problem, generally concentrating on the region around $(\Omega/2\omega) = 1$. It is found that the unstable zones are for small damping a number of closely neighbouring areas like fingers which more or less present a common ε at which instability begins. For closely spaced inputs $\alpha < \nu$ the stability boundary is more complicated but the value of ε at which instability begins is very approximately equal to the equivalent value for Gaussian white noise excitation.

Telegraph signal input

The problem in this case is again described by the equation (1) but with f(t) now in the form of the random telegraph signal which takes only the values ± 1 with certain statistical properties including a specifiable average crossing rate. Related to this problem is that in which f(t) is a pseudo-random signal having statistical properties like those of the telegraph signal but with a time history which is periodic.

This problem was tackled numerically by simulating the telegraph switching using one of the standard algorithms for a random number with a uniform distribution over an interval. The solution can then be developed in the computer (there is no need for integration as the problem has piecewise constant coefficients) up to some 'reasonable' time T. At T the 'pseudo-monodromy' matrix of the solution is obtained and its eigenvalues examined for stability. In doing this we are pretending that we have reached the end of one period of a pseudo-random telegraph signal.

It is found that this process can lead to good estimates of the stability boundary for the truly random telegraph signal.

MODAL INTERACTION

The normal modes of a linearised constant coefficient structure can be used as co-ordinates to describe the motion of the same structure when its modelling is extended to include parametric effects and nonlinearities which may stem from inertial, elastic or other sources. These nonlinearities are approximated by quadratic and cubic terms which are generally small and have little effect on the behaviour except when certain conditions obtain. In such situations response estimations using the linear constant coefficient model are quite useless. The smallness of the nonlinear and parametric terms does not prevent them from having an overwhelming effect on the response in the course of time. These added terms can be looked on as providing a coupling or an energy interchange mechanism between the linear modes. This coupling can be dominant in determining the response of modes which are involved in so-called internal resonance. Internal resonance can exist between two or more modes depending on the degree of nonlinearity admitted into the equations. Thus with quadratic nonlinearities two modes i and j having linear natural frequencies ω_i and ω_j are in internal resonance if $\omega_i = 2\omega_j$ or three modes i, j and k can be in internal resonance if $\omega_i = \omega_j + \omega_k$. Higher order nonlinearities such as cubics and above give more scope for more modes; the general internal resonance can be written $a\omega_i + b\omega_j + c\omega_k + \ldots = 0$ where a, b, c are smallish positive or negative integers.

Under these conditions it may be possible for energy to pass between modes and the resonant behaviour of the structure will be quite different from any linear prediction.

Various simple structural systems involving internal resonance have been examined both analytically and in the laboratory. One of these was basically a four mode model adjusted to meet the internal resonance condition $\omega_4 = \omega_3 + \omega_1$ associated with quadratic inertial nonlinearities. It was found that excitation of the model at frequency ω_4 results initially in a resonant growth of mode 4 but that this is shortly followed by a development of modes 1 and 3 together, with a reduction in the response of mode 4. In fact the interaction removes the peak of the usual steady-state resonant response of mode 4.

With the same model even more complex interaction is possible. If the external excitation frequency is taken equal to the sum $\omega_2 + \omega_4$ then modes 2 and 4 respond through the parametric combination resonance mechanism. However the growth of mode 4 then induces through the interactions just described the development of modes 1 and 3. Thus all four modes of the structure are excited by a single external frequency which is not simply related to any one of the natural frequencies. In the model this type of interaction led to a quasi-steady state in which the modal response developed a regular beat.

This more complex interaction between modes is an example of what we have chosen to refer to as 'cascading interaction'. In real structures there are very large numbers of modes and internal resonance conditions will generally be met or nearly met between many of them. External excitation may then excite mode

A say which through internal resonance excites modes B and C. Mode C may then in turn excite modes D and E and so on, so that the modal involvement from a single frequency excitation may be very extensive. In principle the frequency cascading effect can be both upward and downward though limited experiments seem to indicate a preference for modes to pass energy downwards to lower frequency modes rather than the reverse.

Similarly the modal response to a broad band form of excitation may be influenced by these forms of interaction and certain modes or modal groups may gain a predominance in the response. Certainly in the presence of modal interaction general concepts concerning modal response such as equi-partition of modal energies can be applied only with considerable reservation.

The analysis of the problem because of the assumed smallness of the parametric and nonlinear terms, can be approached using a perturbation method.

For instance the general form of the equation of motion of the rth mode of a structure retaining quadratic inertial nonlinearities and linear parametric terms can be written

$$\ddot{p}_r + \omega_r^2 p_r = F_r \cos\Omega_f t - \varepsilon[\kappa_{rj} p_j \cos\Omega_f t + \ell_{rij} p_i \ddot{p}_j + m_{rij} \dot{p}_i \dot{p}_j + \omega_r d_{rr} \dot{p}_r] \quad (3)$$

Summation on the repeated indices i, j only is implied. F_r is the direct forcing term, κ_{rj} the parametric term, both at frequency Ω_f, while ℓ_{rij}, m_{rij} are the constant coefficients of the quadratic nonlinearities. The single term $\dot{p}_r$ represents an assumed uncoupled modal viscous damping, ε is a small parameter.

The solution to equation (1) is taken in the form

$$p_r = A_{ro}(t) + A_r(t)\cos(\Omega_r t + \phi_r(t)) + f_r \cos(\Omega_f t + \psi_r) + \varepsilon a_r(t) + \varepsilon^2 b_r(t) \quad (4)$$

The first three terms are the principal part of the solution while the other terms in powers of ε allow for a perturbation approach. The term $A_{ro}(t)$ is regarded as varying only relatively slowly with time as are $A_r(t)$ and the phase $\phi_r(t)$. The first term is the DC or rectified component. The second term oscillates with frequency $\Omega_r(\approx\omega_r)$, the r^{th} mode response frequency, unknown at this stage. The third term is the component of forced response of the mode at the forcing frequency. The linear undamped response $f_r = F_r/(\omega_r^2 - \Omega_f^2)$ and $\psi_r = 0$ can often be assumed for this component as it is usually included in the solution only for excitation frequencies Ω_f reasonably well removed from ordinary resonance.

The assumed solution such as (4) is substituted in the equations of motion and various simplifications are made followed by separate consideration of the various powers of ε. The equations obtained considering only terms of order ε^0 are sometimes referred to as the variational equations. Terms which cause 'resonance' in the first order perturbation equation ε^1 are removed from it and put in with the variational equations. The DC terms and the cos and sin terms in $(\Omega_r t + \phi_r)$ are considered separately.

The resulting equations are generally simpler in form than the original equations of motion. Even if they are not analytically tractable computing directly from them is much faster and more economic. In most cases that have been tried they seem to model the response p_r obtained by straight numerical integration of the 'full' equations (3) tolerably well.