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THE AFFECT OF SENSOR POSITIONAL ERRORS ON THE ESPRIT ALGORITHM

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1. INTRODUCTION

An objective of a passive sonar system is to detect and estimate the bearings of multiple targets. These targets may have a small angular separation at the receiving array. Recently, several algorithms have been developed that are designed to resolve targets with an angular separation that is less than a beamwidth. Examples of these include the MUSIC [1], Minimum-Norm [2] and ESPRIT [3,4] algorithms.

A common sensor system used is a towed array. These consist of tens, or even hundreds of hydrophones towed in a line behind a vessel. It is impractical to make this array rigid and so it is allowed to flex in response to tow-ship motion and ocean currents [5]. This flexing may result in a difference between the assumed locations of the hydrophones and their actual location. These errors reduce the performance of bearing-estimation algorithms [6].

This paper considers the effect of sensor location errors on a particular high-resolution algorithm, ESPRIT. A first-order perturbation analysis is used to obtain expressions for the bias and variance of target-bearing estimates by the Least-Squares version of the algorithm.

2. ESPRIT AND LEAST-SQUARES ESPRIT

ESPRIT algorithms process observations from arrays with a specific structure - the so called ESPRIT array structure. This arrangement of sensors can be described in two ways. The array can be considered as a set of sensor doublets (see Figure 1). Each doublet consists of two hydrophones that have the same directivity pattern. All the doublets must have the same orientation and have the same separation between the hydrophones. This separation and direction form the doublet baseline. Alternatively the structure may be viewed as two identical subarrays that are displaced from each other. These two subarrays are normally called the **X** and **Y** subarrays.

Several algorithms have been proposed to process the observations from an ESPRIT array. These include Covariance ESPRIT [7], Least-Square ESPRIT [4], Total-Least-Squares ESPRIT [4] and Pro-ESPRIT [8].

ESPRIT has several advantages over other signal-subspace estimators. An exact knowledge of the subarray geometry is not required. It is computationally efficient as a search over the parameter space, as in MUSIC, is not necessary. It directly estimates target bearings and so a search for peaks in a spatial spectrum is not required.

In this paper, a simple version of the algorithm is considered, Least-Squares ESPRIT. This is both to make the algebra tractable, and because the assumptions made by the algorithm are in keeping with the physical situation. The algorithm has six main stages.

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1) The data matrices from the two subarrays, denoted X and Y, are combined to form a joint data matrix

$$Z = \begin{bmatrix} X \\ \cdots \\ Y \end{bmatrix}$$

2) The joint signal subspace $\mathcal{S}_z$ is estimated from this data matrix. This is done either directly, using a singular value decomposition [8], or by an eigen decomposition of the covariance matrix formed from Z. Estimates of the basis vectors of this subspace form the columns of the matrix E_z.

3) The joint subspace is partitioned into two halves ($\mathcal{S}_{zx}$ and $\mathcal{S}_{zy}$) corresponding to the X and Y subarrays. The basis vectors of these subspaces are the columns of the matrices E_{zx} and E_{zy} where

$$E_z = \begin{bmatrix} E_{zx} \\ \cdots \\ E_{zy} \end{bmatrix}$$

4) The matrix that rotates $\mathcal{S}_{zx}$ onto $\mathcal{S}_{zy}$, known as the subspace rotation matrix Ψ, is estimated - i.e. find Ψ such that

$$E_{zx}\, \Psi = E_{zy}$$

It is assumed (in Least-Squares ESPRIT) that there are errors only in the estimate of the $\mathcal{S}_{zy}$. Thus Ψ may be calculated using the equation

$$\Psi = (\, E_{zx}^{H}\, E_{zx}\,)^{-1}\, E_{zx}^{H}\, E_{zy} \qquad (1)$$

where ^{H} denotes the complex-conjugate transpose (Hermitian) operation.

5) The eigenvalues of the subspace rotation matrix, denoted $\underline{\Phi}$, are calculated.

6) The phases of the eigenvalues are calculated. These phases may then be directly translated into the target bearings.

3. PROBLEM FORMULATION

ESPRIT should theoretically be robust to sensor positional errors as it does not require an exact knowledge of the subarray shape. These sensor positional errors may, however, cause errors in the assumed ESPRIT array structure and this can be expected to degrade the algorithm's performance. This error mechanism is now considered in detail.

The analysis makes the following basic assumptions about the physical situation:

a) The array has a nominal ESPRIT structure but is otherwise arbitrary.

b) Two narrow-band far-field targets are present. They are not fully correlated.

c) There is no noise.

d) Enough snapshots are taken so that the data covariance matrix is ideal.

e) Two-dimensional errors are introduced into the positions of the Y subarray hydrophones.

f) The shape of the array does not change over the observation interval.

Some quantities used in the next section are now defined. ESPRIT uses the direction between the sensors in the doublet (the baseline) as a reference direction. It is convenient to define the target bearings and sensor positional errors relative to this direction.

The two targets, denoted P and Q are at bearings of θ_p and θ_q. They have wave numbers of

$$\underline{k}_p = \frac{2\pi}{\lambda_0}\begin{bmatrix}\cos\theta_p\\ \sin\theta_p\end{bmatrix} \qquad \text{and} \qquad \underline{k}_q = \frac{2\pi}{\lambda_0}\begin{bmatrix}\cos\theta_q\\ \sin\theta_q\end{bmatrix}$$

respectively. Here λ_0 is the operating wavelength.

Each subarray has N sensors: the baseline length is δ. The steering vectors from the X subarray to the targets are $\underline{a}_{px}$ and $\underline{a}_{qx}$, and from the Y subarray $\underline{a}_{py}$ and $\underline{a}_{qy}$. The lengths of these vectors are defined by

$$\underline{a}_{ij}^H\,\underline{a}_{ij} = N$$

where i is either p or q and j is x or y. It is also necessary to define the joint steering vectors

$$\underline{a}_{pz} = \begin{bmatrix}\underline{a}_{px}\\ \underline{a}_{py}\end{bmatrix} \qquad \text{and} \qquad \underline{a}_{qz} = \begin{bmatrix}\underline{a}_{qx}\\ \underline{a}_{qy}\end{bmatrix}$$

Positional errors in the Y subarray are assumed to exist both parallel to the baseline (x errors) and perpendicular to the baseline (y errors). These errors are grouped into the matrix

$$\Delta p = \begin{bmatrix}\underline{\Delta x} & \underline{\Delta y}\end{bmatrix}$$

where $\underline{\Delta x}$ is a column vector of x errors and similarly for $\underline{\Delta y}$.

If only source P were present, there would be a simple phase difference between the signal received at both hydrophones of a doublet. This phase delay is denoted ϕ_p. Similarly the phase delay associated with source Q is ϕ_q.

Finally, in the next section two distinct cases will be considered: ideal and perturbed. In the ideal case there are no errors in the positions of the Y subarray. In the perturbed case errors are introduced. Quantities relating to the perturbed case are denoted by $\tilde{}$ e.g. $\tilde{\underline{a}}_{qx}$.

4. OUTLINE SOLUTION

In this section an outline solution to the problem posed in the previous section is presented. It is not possible to present the whole solution and so only key results in the working are given. More detail will be available in [9].

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Because of the ESPRIT array structure, the steering vectors at the X and Y subarrays are related by

$$\underline{a}_{py} = e^{j\,\phi_p}\,\underline{a}_{px} \qquad \text{and} \qquad \underline{a}_{qy} = e^{j\,\phi_q}\,\underline{a}_{qx}$$

If the Y subarray is perturbed, then the Y subarray steering vectors will change. To first order these perturbed steering vectors may be written as

$$\tilde{\underline{a}}_{py} \simeq \underline{a}_{py} - j\;diag(\Delta p\;\underline{k}_p)\;\underline{a}_{py} \qquad \text{and} \qquad \tilde{\underline{a}}_{qy} \simeq \underline{a}_{qy} - j\;diag(\Delta p\;\underline{k}_q)\;\underline{a}_{qy}$$

The eigenvectors from the joint subspace estimation are given by

$$\tilde{E}_{zx} \simeq \left[\quad d_1\,(\,\underline{a}_{px} + e^{-j\,\alpha}\,\underline{a}_{qx}\,) \qquad d_2\,(\,\underline{a}_{px} - e^{-j\,\alpha}\,\underline{a}_{qx}\,)\quad\right]$$

$$\tilde{E}_{zy} \simeq \left[\quad d_1\,(\,\tilde{\underline{a}}_{py} + e^{-j\,\alpha}\,\tilde{\underline{a}}_{qy}\,) \qquad d_2\,(\,\tilde{\underline{a}}_{py} - e^{-j\,\alpha}\,\tilde{\underline{a}}_{qy}\,)\quad\right]$$

Here d_1 and d_2 are calculated so that

$$diag(\,\tilde{E}_{zx}^{\,H}\,\tilde{E}_{zx}) = diag(\,\tilde{E}_{zy}^{\,H}\,\tilde{E}_{zy}) = 0.5$$

i.e.

$$d_1 = \frac{1}{2\sqrt{N\left(1+\cos\gamma\,\cos\rho\right)}} \qquad \text{and} \qquad d_2 = \frac{1}{2\sqrt{N\left(1-\cos\gamma\,\cos\rho\,\right)}}$$

In these equations α is the phase angle between the joint steering vectors, $\cos\gamma$ is the steering vector correlation and ρ is half the difference between the phase delays for the two targets:

$$\alpha = \angle\{\,\underline{a}_{pz}^{\,H}\,\underline{a}_{qz}\}\,, \qquad \cos\gamma = \frac{1}{N}\,\underline{a}_{px}^{\,H}\,\underline{a}_{qx}\,, \qquad \rho = \tfrac{1}{2}(\phi_q - \phi_p)$$

The subspace rotation matrix $\tilde{\Psi}$ is calculated using Equation 1. An intermediate stage is to calculate the pseudoinverse of $\tilde{E}_{zx}$. This may be approximated as

$$(\,\tilde{E}_{xz}^{\,H}\,\tilde{E}_{xz})^{-1}\,\tilde{E}_{xz}^{\,H} \simeq \frac{1}{2\,N\,\sin^2\gamma}\begin{bmatrix} \frac{1}{d_1}\left(\left(1-e^{j\rho}\cos\gamma\right)\underline{a}_{px}^{\,H} + \left(1-e^{-j\rho}\cos\gamma\right)e^{j\alpha}\underline{a}_{qx}^{\,H}\right) \\ \frac{1}{d_2}\left(\left(1+e^{j\rho}\cos\gamma\right)\underline{a}_{px}^{\,H} - \left(1+e^{-j\rho}\cos\gamma\right)e^{j\alpha}\underline{a}_{qx}^{\,H}\right) \end{bmatrix}$$

The subspace rotation matrix is then given by

$$\tilde{\Psi} = \begin{bmatrix} \psi_1-\Delta\zeta_1-\Delta\zeta_2-\Delta\zeta_3 & \frac{d_2}{d_1}(\psi_2-\Delta\zeta_1+\Delta\zeta_2-\Delta\zeta_4) \\ \frac{d_1}{d_2}(\psi_2-\Delta\zeta_1+\Delta\zeta_2+\Delta\zeta_4) & \psi_1-\Delta\zeta_1-\Delta\zeta_2+\Delta\zeta_3 \end{bmatrix}$$

In the previous equation

$$\psi_1 = \tfrac{1}{2}\left(e^{j\phi_p}+e^{j\phi_q}\right) \qquad , \qquad \psi_2 = \tfrac{1}{2}\left(e^{j\phi_p}-e^{j\phi_q}\right)$$

$$\Delta\zeta_1 = \frac{j}{2\sin^2\gamma} e^{j\phi_p}(\alpha_p - \cos\gamma\,\beta_p) \quad , \quad \Delta\zeta_2 = \frac{j}{2\sin^2\gamma} e^{j\phi_q}(\alpha_q - \cos\gamma\,\beta_q)$$

$$\Delta\zeta_3 = \frac{j}{2\sin^2\gamma} e^{j\frac{\phi_2+\phi_1}{2}}\Big((\beta_p+\beta_q) - \cos\gamma(\alpha_p+\alpha_q)\Big)$$

$$\Delta\zeta_4 = \frac{j}{2\sin^2\gamma} e^{j\frac{\phi_2+\phi_1}{2}}\Big((\beta_p-\beta_q) - \cos\gamma(\alpha_p-\alpha_q)\Big)$$

where

$$N\,\alpha_p = \underline{a}_{px}{}^H diag(\Delta\underline{p}\,\underline{k}_p)\,\underline{a}_{px} \quad , \quad N\,\alpha_q = \underline{a}_{px}{}^H diag(\Delta\underline{p}\,\underline{k}_q)\,\underline{a}_{px}$$

$$N\,\beta_p = \underline{a}_{qx}{}^H diag(\Delta\underline{p}\,\underline{k}_p)\,\underline{a}_{px} \quad \text{and} \quad N\,\beta_q = \underline{a}_{px}{}^H diag(\Delta\underline{p}\,\underline{k}_q)\,\underline{a}_{qx}$$

The eigenvalues of $\tilde{\Psi}$ are the roots of its characteristic equation. These are given by

$$\tilde{\lambda} = \psi_1 - \Delta\zeta_1 - \Delta\zeta_2 \pm (\,\psi_2 - \Delta\zeta_1 + \Delta\zeta_2 + \Delta b\,)$$

where Δb is approximated by

$$\Delta b \simeq \frac{\Delta\zeta_3{}^2 - \Delta\zeta_4{}^2}{2\,(\,\psi_2 - \Delta\zeta_1 + \Delta\zeta_2\,)}$$

Hence the eigenvalues are

$$\tilde{\lambda}_p \simeq e^{j\,\phi_p}\left(1 - j\,\frac{\alpha_p - \cos\gamma\,\beta_p}{\sin^2\gamma} + \Delta b\, e^{-j\,\phi_p}\right)$$

$$\tilde{\lambda}_q \simeq e^{j\,\phi_q}\left(1 - j\,\frac{\alpha_q - \cos\gamma\,\beta_q}{\sin^2\gamma} - \Delta b\, e^{-j\,\phi_q}\right)$$

The bearing error is calculated from the phase of the eigenvalues

$$\Delta\theta_p = \theta_p - \tilde{\theta}_p \simeq \frac{\lambda_0}{2\,\pi\,\delta\sin\theta_p}\left(\frac{\alpha_p - \cos\gamma\,\beta_p}{\sin^2\gamma} + j\,\Delta b\, e^{-j\,\phi_p}\right) \tag{2}$$

$$\Delta\theta_q = \theta_q - \tilde{\theta}_q \simeq \frac{\lambda_0}{2\,\pi\,\delta\sin\theta_q}\left(\frac{\alpha_q - \cos\gamma\,\beta_q}{\sin^2\gamma} - j\,\Delta b\, e^{-j\,\phi_q}\right) \tag{3}$$

For targets that have an angular separation of over half a subarray beamwidth, these expressions may be simplified. Assuming $\sin\gamma \simeq 1$, $\cos\gamma \simeq 0$ and $\Delta b \simeq 0$ then

$$\Delta\theta_p \simeq \overline{\Delta x}\cot\theta_p + \overline{\Delta y} \quad \text{and} \quad \Delta\theta_q \simeq \overline{\Delta x}\cot\theta_q + \overline{\Delta y} \tag{4,5}$$

where $\overline{\Delta x}$ and $\overline{\Delta y}$ are the mean errors in the x and y directions relative to the nominal intersensor separation. They are given by

$$\overline{\Delta x} = \frac{1}{N\,\delta}\sum_{i=1}^{N}\Delta x_i \quad \text{and} \quad \overline{\Delta y} = \frac{1}{N\,\delta}\sum_{i=1}^{N}\Delta y_i$$

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If it is assumed that the sensor positional errors are random, independent, zero mean and have variances of σ_x^2 in the x direction and σ_y^2 in the y direction, then Equations 4 and 5 may be used to show that the expected values (E{}) of the errors are

$$E\{\Delta\theta_p\}=0 \qquad \text{and} \qquad E\{\Delta\theta_q\}=0$$

The variances of the errors are

$$E\{(\Delta\theta_p)^2\} = \frac{1}{N}\left(\sigma_x^2 \cot^2 \theta_p + \sigma_y^2 \right) \qquad \text{and} \qquad E\{(\Delta\theta_q)^2\} = \frac{1}{N}\left(\sigma_x^2 \cot^2 \theta_q + \sigma_y^2 \right)$$

Several observations may be made from these results. Errors in the x direction (baseline length) affect the bearing error as a function of cot θ. For broadside targets they have no effect. For targets that are near the end fire direction of the array, baseline length errors have a large effect. Errors in the y direction (baseline orientation) are not affected by the target bearing.

For targets with an angular separation of less than half a subarray beamwidth, the term Δb becomes significant. This term introduces an opposite, and approximately equal in magnitude, bias into the bearing estimates. The bias increases as approximately $\sin^{-4}\gamma$, i.e. becomes very large for small target angular separations (where $\sin\gamma \rightarrow 0$). The Δb term also increases the variance at small sensor separations.

5. SIMULATIONS

A set of simulations was carried out to verify the analysis in the previous section and to show the limits of its applicability. A nominal uniformly-spaced rectilinear array, containing eight sensors, was considered. The sensor spacing was half the operating wavelength.

Errors were introduced onto the positions of the sensors, both in the x and the y directions. These errors were independent (sensor to sensor and between x and y directions), Gaussian and zero mean. Three standard deviations of the errors were studied: 2%, 4.5% and 9.4% of the nominal intersensor spacing.

Two sources were modeled, both were narrowband and in the far-field. The first had a bearing of 90° (broadside to the array). The bearing of the second was swept between 91° and about 150° in 45 steps. The covariance matrix was calculated as if there were infinite snapshots available. No noise was introduced into the covariance matrix.

There are many ways of mapping an ESPRIT structure onto a uniform-spaced rectilinear array [8]. For this simulation the two subarrays were chosen so that they overlaped by one element (see Figure 2). Six of the sensors are thus used by both the **X** and **Y** subarrays.

One hundred simulations were performed for each target separation and standard deviation of the sensor-positional error. Additionally, for each simulation, the bearing error was estimated using Equations 2 and 3. From these analytic and simulation results, the mean and standard deviation of the bearing error were calculated. These results, after smoothing, are plotted in Figures 3 to 8. The x-axis is scaled in terms of subarray beamwidths, about 16°.

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There is good agreement between the simulation and theory, in particular for target separations of over half a subarray beamwidth. When the targets are a small fraction of a subarray beamwidth apart, the theory underestimated the bias and variance. This underestimation becomes more noticeable as the positional errors increase.

6. CONCLUSIONS

Equations have been derived that predict the performance of the Least-Squares ESPRIT algorithm in the presence of sensor positional errors. These equations show the factors that cause errors in the bearing estimates and give a quantitative estimate of the magnitude of these errors.

7. REFERENCES

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An array with ESPRIT structure

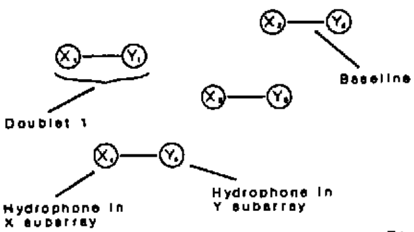

Figure 1

The array used in the simulations

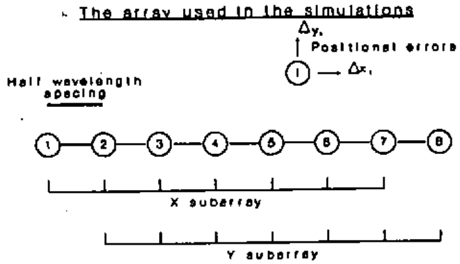

Figure 2

The bias produced by positional errors
Std. Deviation 2% of nominal separation

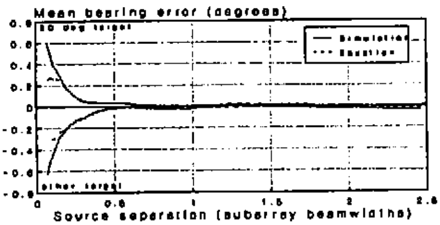

Figure 3

The standard deviation of the bearing
Std. Deviation 2% of nominal separation

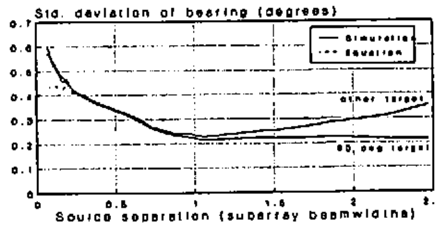

Figure 4

The bias produced by positional errors
Sd. Deviation 4.5% of nominal separation

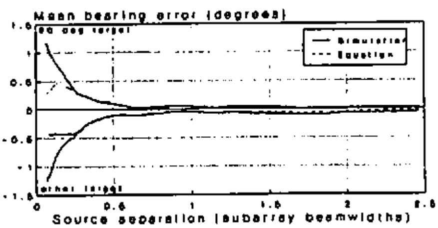

Figure 5

The standard deviation of the bearing
Sd. deviation 4.5% of nominal separation

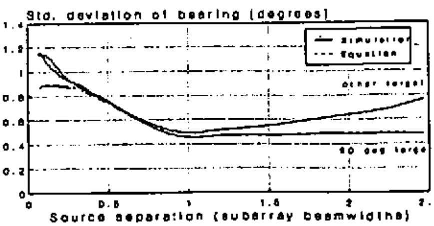

Figure 6

The bias produced by positional errors
Sd. Deviation 9.4% of nominal separation

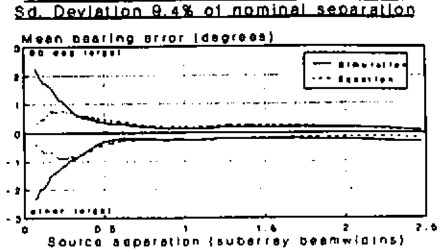

Figure 7

The standard deviation of the bearing
Sd. deviation 9.4% of nominal separation

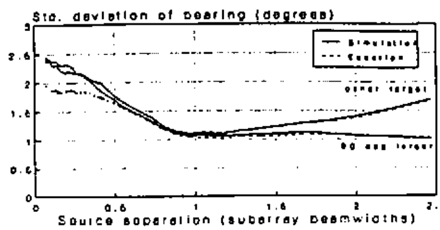

Figure 9