ON INVARIANT SOLUTIONS TO THE SECOND-ORDER EQUATION OF INVISCID GAS FLOW

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Exact nonlinear wave equation seemingly first published by S.Goldstein in 1960 is used as expression adequately describing flow of inviscid gas. This is a second-order nonlinear partial differential equation in acoustic potential. Unfortunately, information about exact solutions to this equation is very poor, only different methods of approximate solution are proposed. Application of Lie group analysis to the equation under consideration is presented in the given communication. Some invariant solutions are found in this way.

Keywords: nonlinear acoustics of inviscid gas.

1. Introduction

Among methods of simplification of solution of complete system of hydrodynamic equations one proposed by S.Goldstein [1] stands out against others because for inviscid gas this approach leads to the single exact differential equation in acoustic potential. In the three-dimensional case it has the following form

(1) $$\frac{\partial^2\phi}{\partial t^2} - c_0^2\nabla^2\phi + \left[2\cdot\nabla\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla(\nabla\phi\cdot\nabla\phi)\right]\cdot\nabla\phi + (\gamma-1)\cdot\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi\cdot\nabla\phi)\right]\cdot\nabla^2\phi = 0$$

where ϕ is the acoustic potential, c_0^2 is the small-signal sound wave velocity, γ is the ratio of specific heats.

Equation (1) "is often used in aeroelasticity, and it frequently serves as a starting point for perturbation analyses in nonlinear acoustics" [2], but because of its nonlinearity it is not so easy to find its analytical solutions. Of course, numerical calculations are indispensable in modeling of real gas flow, nevertheless search of exact solutions remains important because they enable to make qualitative estimations as well as to develop and to debug modeling software.

The present report is devoted to application of Lie group approach to search of invariant solutions to the equation under consideration. Invariance means here preservation of a form of a solution at some changes of spatial coordinate and time. The standard technique allows reduction of partial differential equation (PDE) (1) to some ordinary differential equation (ODE) [3] and the latter one, as it was shown in the work, can be solved analytically in the case of a perfect inviscid gas. This reduction was successfully used for solution of nonlinear problems of acoustics [3], gas dynamics [4] as well as of biophysics and signal processing [5]. Further generalizations of Lie's method (by means of nonclassical reduction, for example), based on complementary demand of symmetry of boundary conditions, allows one to broaden number of invariant solutions of nonlinear equations [6-7].

2. Solution

A one-dimensional case is studied here, so Eq. (1) is considered in the next form

$$\frac{\partial^2\phi}{\partial t^2}-c_0^2\frac{\partial^2\phi}{\partial x^2}+2\frac{\partial^2\phi}{\partial t\partial x}\cdot\frac{\partial\phi}{\partial x}+(\gamma-1)\cdot\frac{\partial^2\phi}{\partial x^2}\cdot\frac{\partial\phi}{\partial t}+\frac{\gamma+1}{2}\cdot\frac{\partial^2\phi}{\partial t\partial x}\cdot\left(\frac{\partial\phi}{\partial x}\right)^2=0 \quad (2)$$

Lie group approach is based on analysis of behavior of an appropriate differential equation under small continuous changes of its variables [3]. S.Lie had stated that this behavior was completely determined with a gradient vector, which had components proportional to increments of all variables of equation under consideration. So as Eq. (2) includes three variables t,x,ϕ, the corresponding gradient vector has up to three components. Using the standard Lie's approach, the following four operators are obtained in this way. Each of them presents components of one of four independent gradient vectors along three possible axes t,x,ϕ

$$X_1=\frac{\partial}{\partial t},\qquad X_2=\frac{\partial}{\partial x},\qquad X_3=\frac{\partial}{\partial\phi},$$

$$X_4=Mt\frac{\partial}{\partial t}+Nx\frac{\partial}{\partial x}+\left[2c_0^2\frac{M-N}{\gamma-1}t+(2N-M)\phi\right]\frac{\partial}{\partial\phi}\equiv Mt\frac{\partial}{\partial t}+Nx\frac{\partial}{\partial x}+(rt+s\phi)\frac{\partial}{\partial\phi}, \quad (3)$$

where M and N are arbitrary constants. Components of differential operators (3) along any coordinate t,x,ϕ (i.e. coefficients before corresponding derivatives in the latter expressions) determine directions of 3D gradient vectors. In order to find invariant solutions to Eq. (2) one needs to solve the appropriate first-order partial differential equations as we do it now.

In the problem under consideration first three operators X_1, X_2 and X_3 correspond to translations of reference point along axes t, x and ϕ respectively and these operators allow to find only invariant solutions remaining constant along appropriate axes. The only nontrivial transformation operator X_4 leads to the next two differential equations

$$\frac{dt}{Mt}=\frac{dx}{Nx}=\frac{d\phi}{rt+s\phi} \quad (4)$$

Depending on relative values of M and N, two subsequent sets of invariants can be deduced from Eq. (4).

Case 1) $M=N\Rightarrow r=0, s=M$. The first family of invariant solutions to Eq. (2) looks like

$$\phi=t\cdot\Phi(\lambda)\ ,\quad \lambda=\left(\frac{x}{t}\right)^M \quad (5)$$

where $\Phi(\lambda)$ is an unknown function of the first invariant λ. This solution remains constant at simultaneous scaling of x and t. If we consider for simplicity $M=1$, then

$$\lambda=\frac{x}{t} \quad (5, a)$$

Substitution of the first of Eq. (5) subject to (5, a) into Eq. (2) reduces the latter one to the next second-order ODE in which λ is the only independent variable

(6) $$\Phi''(\lambda^2 - c_0^2) - 2\lambda\Phi''\Phi' + \Phi''\left[\frac{\gamma+1}{2}(\Phi')^2 + (\gamma-1)(\Phi - \lambda\Phi')\right] = 0$$

Evidently, Eq. (6) can be separated into two ODE. The first of them, $\Phi'' = 0$, leads to the trivial solution $\phi = A_1 x + B_1 t$, while from the second one the next couple of solutions follows

(7)
$$\phi_1 = \frac{c_0^2 t}{\gamma-1} + \frac{1}{\gamma+1}\cdot\frac{x^2}{t} + C \qquad \text{and}$$

$$\phi_2 = \frac{c_0^2 t}{\gamma-1} + \frac{x^2}{2t} + C,$$

where C is the integration constant. Both solutions (7) show similar dependences of acoustic velocity and pressure on space and time coordinates, namely

(8)
$$u_1 = \frac{2x}{(\gamma+1)t}, \quad p_1 = -\frac{c_0^2}{\gamma-1} + \frac{x^2}{t^2}$$

$$u_2 = \frac{x}{t}, \quad p_2 = -\frac{c_0^2}{\gamma-1} + \frac{2x^2}{(\gamma-1)t^2}$$

Expressions (8) describe acoustic pulses decaying at any point with a lapse of time.

Case 2) $M \neq N$. In this case the first equation of the system (4) leads to another first invariant of Eq. (2) in the form of a self-similar expression

(9) $$\lambda = \frac{x^M}{t^N}$$

When solving the second equation of (4) subject to the first invariant (9), the following expression for the second invariant of Eq. (2) can be found

(10) $$\phi = \frac{c_0^2 t}{(\gamma-1)} + t^{s/M}\cdot\Phi(\lambda)$$

We restrict consideration with the case $M = 2, N = 1 \Rightarrow s = 0.$ Thus

(11) $$\phi = \frac{c_0^2 t}{(\gamma-1)} + \Phi(\lambda) \qquad \text{where } \lambda = \frac{x^2}{t}$$

Substituting (11) into Eq. (2), one can get the next ODE in the function $\Phi(\lambda)$

(12) $$\lambda\Phi'' + 2\Phi' - 2(\gamma+3)(\Phi')^2 - 4\lambda(\gamma+1)\Phi''\Phi' + 4(\gamma+1)(\Phi')^3 + 8\lambda(\gamma+1)\Phi''(\Phi')^2 = 0$$

with λ as a new independent variable like previously. This equation can be easily integrated once, leading to the following first-order ODE

(13) $$\lambda^2 \cdot |\Phi'| \cdot \left|\Phi' - \frac{1}{\gamma+1}\right|^3 \cdot \left|\frac{\Phi' - \frac{1}{2}}{\Phi' - \frac{1}{\gamma+1}}\right|^{\frac{2}{\gamma-1}} = C_1$$

here C_1 is the constant of integration. Variable λ is real, therefore $C_1 \geq 0$ always. If $C_1 = 0$, solutions (7)-(8) follow from the latter equation. Otherwise, in general case, let we first consider the classical monoatomic perfect gas ($\gamma = 5/3$). Then Eq. (13) takes on a much simpler form

(14) $$\lambda^2 \cdot |\Phi'| \cdot \left|\Phi' - \frac{1}{2}\right|^3 = C_1 \ , \ C_1 \geq 0$$

Considering (14) as quartic algebraic equation in Φ' one can write its four solutions

(15, a) $$\Phi'_{1,2} = \frac{1}{2} \cdot \left(-\frac{1}{4} - R \mp \sqrt{\frac{1}{8} + \frac{4C_1}{\lambda^2} \cdot \sqrt[3]{\frac{1}{w}} - \frac{1}{3} \cdot \sqrt[3]{w} + \frac{1}{32R}} \right)$$

(15, b) $$\Phi'_{3,4} = \frac{1}{2} \cdot \left(-\frac{1}{4} + R \mp \sqrt{\frac{1}{8} + \frac{4C_1}{\lambda^2} \cdot \sqrt[3]{\frac{1}{w}} - \frac{1}{3} \cdot \sqrt[3]{w} - \frac{1}{32R}} \right)$$

where $R = \sqrt{\frac{1}{16} - \frac{4C_1}{\lambda^2} \cdot \sqrt[3]{\frac{1}{w}} + \frac{1}{3} \cdot \sqrt[3]{w}}$ and $w = -\frac{27}{8} \cdot \frac{C_1}{\lambda^2} + \sqrt{\left(\frac{12C_1}{\lambda^2}\right)^3 + \left(\frac{27C_1}{8\lambda^2}\right)^2}$.

Both acoustic velocity and acoustic pressure are expressed through partial derivatives of the acoustic potential so in one-dimensional geometry one obtains from Eqs. (11)

(16) $$u = \frac{\partial\phi}{\partial x} = 2\frac{x}{t}\Phi' \quad \text{and} \quad p = -\rho_0 \frac{\partial\phi}{\partial t} = -\rho_0 \frac{c_0^2}{\gamma - 1} + \frac{x^2}{t^2}\Phi'$$

Substituting solutions (15) into Eqs. (16), one obtains four exact analytical expressions for acoustic speed and pressure in the monoatomic perfect gas. It can be easily found that these solutions also describe monotonically spreading with time pulses. Indeed, absolute values of velocity given with Eqs. (15)-(16) asymptotically tend to zero as $t^{-1/2}$ when time tends to infinity, while values of pressure vanish even faster according to dependence $t^{-3/2}$. This behavior of invariant solutions seems to be owing to the fact that the first summands of Eqs. (10) or (11) after substitution into Eq. (2) exclude the second (”wave”) term of the latter equation. This fact shows the way of significant simplification of Eq. (2). Indeed, if one tries to find a solution in the form

$$\phi = \frac{c_0^2 t}{(\gamma - 1)} + \tilde{\phi}(x,t)$$

then initial Eq. (2) transforms into the next PDE in $\tilde{\phi}(x,t)$

$$\frac{\partial^2 \tilde{\phi}}{\partial t^2} + 2\frac{\partial^2 \tilde{\phi}}{\partial t \partial x} \cdot \frac{\partial \tilde{\phi}}{\partial x} + (\gamma - 1) \cdot \frac{\partial^2 \tilde{\phi}}{\partial x^2} \cdot \frac{\partial \tilde{\phi}}{\partial t} + \frac{\gamma + 1}{2} \cdot \frac{\partial^2 \tilde{\phi}}{\partial x^2} \cdot \left(\frac{\partial \tilde{\phi}}{\partial x}\right)^2 = 0$$

For a gas with polytropic index $\gamma = 3$, the latter equation looks like

$$\left[\frac{\partial}{\partial t} + 2\left(\frac{\partial^2 \tilde{\phi}}{\partial x^2}\right)\right] \cdot \left[\frac{\partial \tilde{\phi}}{\partial t} + \left(\frac{\partial \tilde{\phi}}{\partial x}\right)^2\right] = 0$$

It is interesting to note that the expression for acoustic speed in Eqs. (16) subject to solutions (15) does not contain any elastic parameters of the gas, only integration constant C_1 which has dimension of coefficient of diffusion.

For diatomic and polyatomic classical perfect gases (as well as for real gases) solutions (15) are valid only approximately.

3. CONCLUSION

Analytic expressions for acoustic velocity and pressure obeying the one-dimensional equation of flow of inviscid perfect gas are obtained using Lie group approach. One set of these expressions is invariant with respect to simultaneous uniform expansion along time and spatial coordinates while another one remains constant if ratio of quadratic expansion along the spatial coordinate to linear expansion along time axis is constant (self-similar solution).

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