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A More Accurate Prediction of the Modal Frequencies of a Piano String

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1. Introduction

Improved FFT techniques at Edinburgh University have shown that the Fletcher formula $f_n = nf_0 (1 + Bn^2)^{\frac{1}{2}}$ is not completely adequate in predicting the modal frequencies of a piano string. Although a complete solution depends on the solution of a transcendental equation, the authors show that, by recourse to Rayleigh (1), a similar equation of the form $f_n = nf_0 (1 + Bn^2 - Cn^4)^{\frac{1}{2}}$ can be derived, which appears to be in agreement with the experimental evidence.

2. Prior Research

Dr Bell worked on the acoustics of the harp, prior to taking up his present post as lecturer in Physics and Musical Instrument Acoustics at Stevenson. Dr Parks has worked on a plethora of musical instrument problems at Edinburgh University. His other major research interest is the fauna and flora of Lapland.

3. Other Analyses

In 1894, Rayleigh (1) showed that the dependance of the nth modal frequency f_n on a piano string took the form:

$$f_n = nF(1 + An^2)$$

where F represented the fundamental frequency and A was a parameter determining the size of the stiffness correction.

In 1964, Fletcher (2) conducted a more accurate treatment of Rayleigh's method, considering both clamped and hinged boundary conditions. He found that the equation was of the form:

$$f_n = nF(1 + Bn^2)^{\frac{1}{2}}$$

Morse and Ingard (3) have also provided a treatment of the problem and they explain the consequences of various parts of Fletcher's method. In fact, they arrive at an equation that is a hybrid of Rayleigh and Fletcher's work.

4. This analysis

All the work is based on the solution of the general, differential equation:

$$T\frac{\delta^2 y}{\delta x^2} - EA K^2 \frac{\delta^4 y}{\delta x^4} = \rho A \frac{\delta^2 y}{\delta t^2}$$

Setting $y = C \exp(2\pi i(\mu x - vt)$ and applying a boundary condition, produces two equations of the form:

$$\tan(\pi l \mu_2) = -\surd(1 + 2\beta^2/\mu_2^2)\ \tanh(\pi l\ \surd\mu_2 + 2\beta^2)$$

(We are taking Morse and Ingard's notation for the equation, Fletcher uses different symbols)

Both Fletcher and Morse and Ingard proceed to tackle these two equations by making good assumptions about the value of the hyperbolic term and finding likely values for the trigonmetric term. They then find the formula as stated above. Morse and Ingard provide the caveat that the equation is only valid as long as the string tension is the predominant restoring force.

The analysis of the modal frequencies of the AO string on a Broadwood piano (4) show that fitting the experimental results against a plot of the form $nf_o(1 + Bn^2)^{\frac{1}{2}}$ reveals a residual discrepency. From the shape of the residue , it would appear to have a $-n^4$ form, which could be added to the Fletcher equation.

Our problem was to find a theoretical basis for this suspected extra term. We could of course tackle the transcedental equation head-on and solve it numerically, but we decided to start our work by returning to Rayleigh.

At one stage in his treatment of the differential equation underpinning all this work, Rayleigh writes that

$$f_n^2 = \frac{n^2\pi^2(a^2l^2 + n^2\pi^2k^2b^2)}{l^2\ (l^2 + n^2\pi^2k^2)}$$

where $a^2 = T/\sigma$ and $b^2 = (ES+T)/\sigma$. We have changed the symbols from the original in order to be consistent with the rest of this paper.
Rayleigh then makes certain simplifications to produce the equation of the form given above.

If we retain all the terms of this equation, gather terms and transpose, we can rewrite this as:

$$f_n^2 = \frac{n^2\pi^2a^2}{l^2}\left(1 + \frac{n^2\pi^2k^2b^2}{a^2l^2}\right)\left(1 + \frac{n^2\pi^2k^2}{l^2}\right)^{-1}$$

We can expand the $(1 + x)^{-1}$ term using a binomial method and multiply this with the adjacent term. We will discard all terms in n larger than n^4.

We gain:

$$f_n^2 = \frac{n^2\pi^2a^2}{l^2}\left[\left(1 + \frac{n^2\pi^2k^2}{l^2}\left(\frac{b^2}{a^2} - 1\right)\right) + \frac{n^4\pi^4k^4}{l^4}\left(1 - \frac{b^2}{a^2}\right)\right]$$

We now replace the terms a and b with T,E and A. We also replace k, a radius of gyration term, with a term in r (for radius). We obtain:

$$f_n^2 = \frac{n^2\pi^2 T}{l^2\sigma}\left[1 + \frac{n^2\pi^2 r^2}{4l^2}\left(\frac{EA}{T}\right) - \frac{n^4\pi^4 r^4}{16\, l^4}\left(\frac{EA}{T}\right)\right]$$

We now recall Fletcher's definition of $B = \frac{\pi^2 r^2 EA}{4l^2 T}$ and we also change the term before the square bracket into the frequency of the fundamental mode.

We gain:

$$f_n = nf_o(1 + Bn^2 - \frac{B^2 T}{E\,A}\ n^4)^{\frac{1}{2}}$$

We should repeat Morse and Ingard's warning that this formula is valid as long as n^2 is smaller than l^2T/π^2EAk^2.

5. Our Results

We found that we could fit the experimental results of the values of the modal frequencies of the A0 string of the Broadwood onto a function of the form:

$$f_n = nf_o(1 + 2.2 \times 10^{-4}\,n^2 - 2.0 \times 10^{-9}\ n^4)^{\frac{1}{2}}$$

The predicted values differed from the experimental values by-at most-1 Hz. This was a delightful surprise as the formula should only be valid for n less than n=17.
The value of B was calculated to be 1.7×10^{-4} from physical values of E,A, T,r etc. Given that we have completely ignored any effect of the restoring forces due to the double helical spring that constitutes the wrap of this string, we should be fairly pleased.
The value of the B^2T/EA term was calculated to be about 2×10^{-10}. This is wrong by a factor of 10, but it would appear that the line-fitting allows for quite large variation in the n^4 term, before there is a major change in fit.
In any case, we shouldn't be using this formula above n=17, and we are applying this on modes upto n= 60.
Our next step is to try to solve the transcendental equation numerically.

6. References

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