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ACCURACY PREDICTION FOR DISTRIBUTED TRACKING SYSTEMS

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INTRODUCTION

The results to be presented in this note are the outcome of a study which was begun at Thorn EMI, Naval Systems Division, aimed at producing better methods of assessing the accuracy of tracking systems. The study was prompted by the need to analyse the error performance of different range configurations during the preparation of proposals. The use of existing methods involved extensive and time consuming numerical calculations.

The purpose of a tracking range is to locate an object which may be moving, within a volume of water. The type of system under consideration here comprises a number of receivers, transmitters or transponders located at fixed and known positions. The distances between the object and those positions are measured and then converted into X,Y,Z position or range and bearing, by a suitable numerical transformation carried out by a computer. Distance may be found by measuring the time taken for pulses to travel between the object and the known positions, and then multiplying by the speed of sound.

Two key aspects of tracking range performance are the time to produce a fix, and its accuracy. The calculation time becomes important if the object is moving quickly. In underwater applications, however, speeds are low, the update rate does not need to be high and finding a computer which can handle the data collection and transformation tasks in real time should not be difficult.

A number of factors determine accuracy, some largely under the control of the designer, others uncontrollable and perhaps unpredictable. In the first category are the minimum increment in time, and the numerical precision of the algorithm. In the second are the true value of the speed of sound, and its variation from day to day and from place to place on the same day. The distribution of the datum positions throughout the water volume, and the accuracy with which they can be specified also affect accuracy. Ideally the distribution is controllable, but in practice it may be restricted, for example by features on the sea floor.

Accuracy may be assessed by modelling the errors in observed variables and applying the numerical transformations which convert travel time into the desired co-ordinates. Particular values may be assumed for the errors, in which case exact knowledge about performance under one set of conditions is gained, but a number of carefully chosen cases must be studied to form an overall picture.

Alternatively, the errors can be assigned appropriate statistical distributions, and average values and the scatter of fixing error found over a number of trials. As the error distributions will vary with the environmental conditions, this process must be carried out a number of times also. With either method, the amount of calculation required is considerable.

The method outlined in this note is to determine upper limits on the accuracy which can be achieved for known levels of errors in the data. The use of the method is demonstrated for a two dimensional tracking range with three datum positions, using a specific transformation of the distances, but the principles can be applied more generally.

MATHEMATICAL DISCUSSION

If the distance between a fixed point and a variable one is known, the variable point lies on circle whose centre is the fixed point. When the distances to several points are known, the unknown position is located by finding the intersection of circles centred at the known points. More than two distances must be known, as in general, two circles intersect twice, but if there are no measurement errors, three circles will intersect at a single point. Errors will shorten or lengthen the circle radii, and instead of three double intersections and one triple intersection, there will be six double intersections (see fig 1). The problem is to make an estimate of where the true position of the object is.

Let $(x_i, y_i), i = 1 .. 3$ be the known co-ordinates,

(x, y) be the unknowns,

t_i $i = 1 .. 3$ be the measured travel times, and let

c_i $i = 1 .. 3$ be the speeds of sound along each path.

Then

$$(x - x_i)^2 + (y - y_i)^2 = (c_i t_i)^2 \quad i = 1 .. 3 \tag{1}$$

Solution

It is a fairly obvious first step in finding the unknown coordinates, to remove the squared terms in the unknowns by subtracting pairs of equations. Thus, taking the first and second equations, and the second and third equations,

$$(x_1 - x_2)x + (y_1 - y_2)y = 0.5(r_1^2 - r_2^2) + 0.5(d_2^2 - d_1^2) \tag{2a}$$

$$(x_2 - x_3)x + (y_2 - y_3)y = 0.5(r_2^3 - r_3^2) + 0.5(d_3^2 - d_2^2) \tag{2b}$$

where

$$r_i^2 = (c_i t_i)^2 \tag{3}$$

$$d_i^2 = x_i^2 + y_i^2 \tag{4}$$

For simplicity, choose the origin to be the circumcentre of the triangle. Then all the distances are equal and the second term on the right hand side of the equation disappears.

These equations are straight lines passing through intersections of pairs of circles and are the common chords of those circles. Note that any two pairs of equations from the set could be chosen, but that forming a third equation from the remaining pair gives no new information. The third line passes through the intersection of the other two.

When there are no measurement errors, the lines intersect at the triple circle intersection. Provided errors are small, the solution of the equations which gives the intersection of the chords is a good working estimate of the position of the object.

The chord equations can be represented in matrix form by

$$Ax = b$$

where A is the matrix of coefficients, x is the vector of unknowns and b is the vector containing differences of measured distances. The solution of these equations, which is the common chord estimate of position, is then given by

$$x = A^{-1}b$$

where A^{-1} is the inverse of A.

Allowing for errors in the data, the equations can be written

$$(A + E)y = b + f$$

where E is the matrix containing the errors in specification of datum positions, and f contains the errors in measurement of distance.

The error in the fix is the difference between the solutions of the ideal and actual equations

$$\begin{aligned} x - y &= A^{-1}b - (A + E)^{-1}(b + f) \\ &= (A + E)^{-1}(Ex - f) \qquad (5) \end{aligned}$$

ACCURACY ANALYSIS

The error performance of a tracking system can be specified in terms of the length of the error vector $x - y$ throughout the range coverage. Starting from equation 5, the upper limit for fixing error is found in terms of the norms of the input matrices and vectors expressing the measured values and their errors.

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A vector norm is a generalised measure of length, and with it is associated a matrix norm, as discussed in [1]. There are many norms, but the appropriate one for this analysis corresponds to the usual definition of vector length,

$$||x|| = (x_1^2 + x_2^2)^{0.5}$$

The associated norm for a matrix A, $||A||$, is shown in [1] to be the largest eigenvalue of the the matrix A^TA.

Following the analysis given in [1], it can be shown that if the data errors are small, the relative error in a fix satisfies

$$||x - y||/||x|| \leqslant ||A||\,||A^{-1}||(||E||/||A|| + ||f||/||b||) \qquad (6)$$

The quantity $||A||\,||A^{-1}||$, known as the condition number of A, and written K(A), determines the sensitivity of the range to measurement errors. It is not dependent on any of those errors, but scales their effects. Range datum geometries with large condition numbers will be more sensitive to measurement error.

This inequality relating input and output error can be used to determine the performance of a tracking range. The condition number, and the position uncertainty term need only be calculated once, leaving just the effects of varying the distance measurement error to be considered in detail.

PERFORMANCE OF A 2-D RANGE WITH THREE DATUM POSITIONS

To demonstrate the use (6), first the condition number for a 2-D range with a triangular arrangement of datum positions will be found, then the effects of errors in the speed of sound and travel times will be investigated.

Condition Number

In the particular case under consideration, it is possible to relate the sensitivity to the properties of the datum triangle in an explicit way.

First, after calculating A^TA, it can be shown that its characteristic equation is

$$s^2 - (l_1^2 + l_2^2)s + (l_1 l_2 \sin(th))^2 = 0$$

where l_1 and l_2 are the lengths of two sides of the triangle and 'th' is the included angle. The norm of A is the larger of the two solutions of this equation and the condition number is their ratio [1]. The roots can be expressed as the product of a length squared with a term depending only on the length ratio and the angle, and hence the condition number is independent of the size of the triangle, being affected only by its shape. Let the ratio between the sides be $r = l_1/l_2$, then

$$||A|| = (l^2/2)[1 + r^2 + [(1 + r^2)^2 - (2r.\sin(th))^2]^{0.5}]$$

$$K(A) = [1 + r^2 + [(1 + r^2)^2 - (2r.\sin(th))^2]^{0.5}]^2/(2r\sin(th))^2 \quad (7)$$

Equation 7 is plotted for different values of 'r' and 'th' in figs 2A and 2B, and it can be seen that the condition number has a minimum of 1 for r = 1, th = 90. In this case, the datum triangle is right-angled and isoceles. When that shape is distorted, the condition number increases and the fixes are less accurate for a given set of measurements. Long narrow triangles show greater sensitivity to data errors than more symmetrical arrangements.

There will be in general, three condition numbers associated with a triangle, but only two if it is isoceles, and one if it is equilateral. For an isoceles right angled triangle, the two condition numbers are 1 and 6.85. For an equilateral triangle, the condition number is 3. With an arbitrary triangle, since the choice of equations to solve is arbitrary, it is better to choose the two equations which give the smallest condition number to take advantage of the lower sensitivity to error.

Measurement Errors

Datum Position Error. From the limit on relative error (6), it can be seen that errors in specifying datum positions induce a relative error in the fix which is constant over the range, and hence the absolute error increases away from the origin.

Speed of Sound and Timing Error. Allowing for a bias of dc in the speed of sound, and dt in the time measurements, the relative error in distance measurement is (dc/c + dt/t). The relative error in a fix induced by speed of sound error is also constant, whereas travel time error induces a relative fix error which decreases away from the origin.

These predictions are compared with the behaviour of the actual errors for an equilateral triangle in fig 3. The fix errors are plotted at points equally spaced throughout a square with the origin of co-ordinates at the intersection of diagonals. The error magnitude is assigned to one of sixteen bands represented by the characters 0 - 9, A - F, with '0' being the lowest.

Fig 3A shows the relative fix error for a bias of 1 in 10^3 in timing and as predicted, the error decreases away from the origin.

Fig 3B shows the absolute fix error for a bias of 1 in 1.5×10^3 in assumed speed of sound. This plot exhibits an error increasing away from the origin in proportion to range, which is also in agreement with the prediction.

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DISCUSSION

The analysis has shown that the effects of the various sources of input error can be separated, and for small errors, the contributions from datum position uncertainty, speed of sound and timing, add to-gether. The range datum geometry has been shown to affect sensitivity, and this has been quantified.

These results allow a quick assessment of the importance of the different errors, and first choices of range parameters may be made without extensive numerical computations. If follow-on calculations are required, this type of analysis can be used to direct attention to the more important cases and so improve the efficiency of a more detailed study.

REFERENCES

[1] James M. Ortega, 'Numerical Analysis', Academic Press, 1972

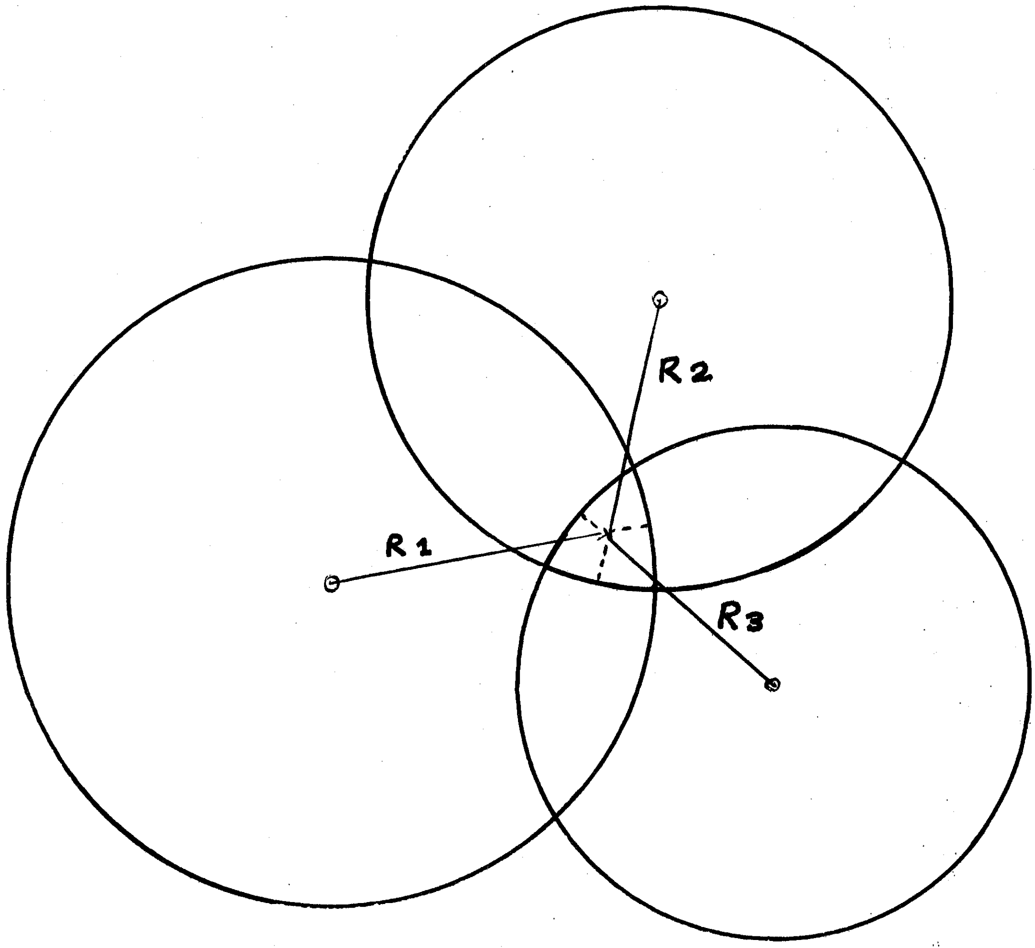

FIG 1 POSITION FIXING BY MEASURING DISTANCES

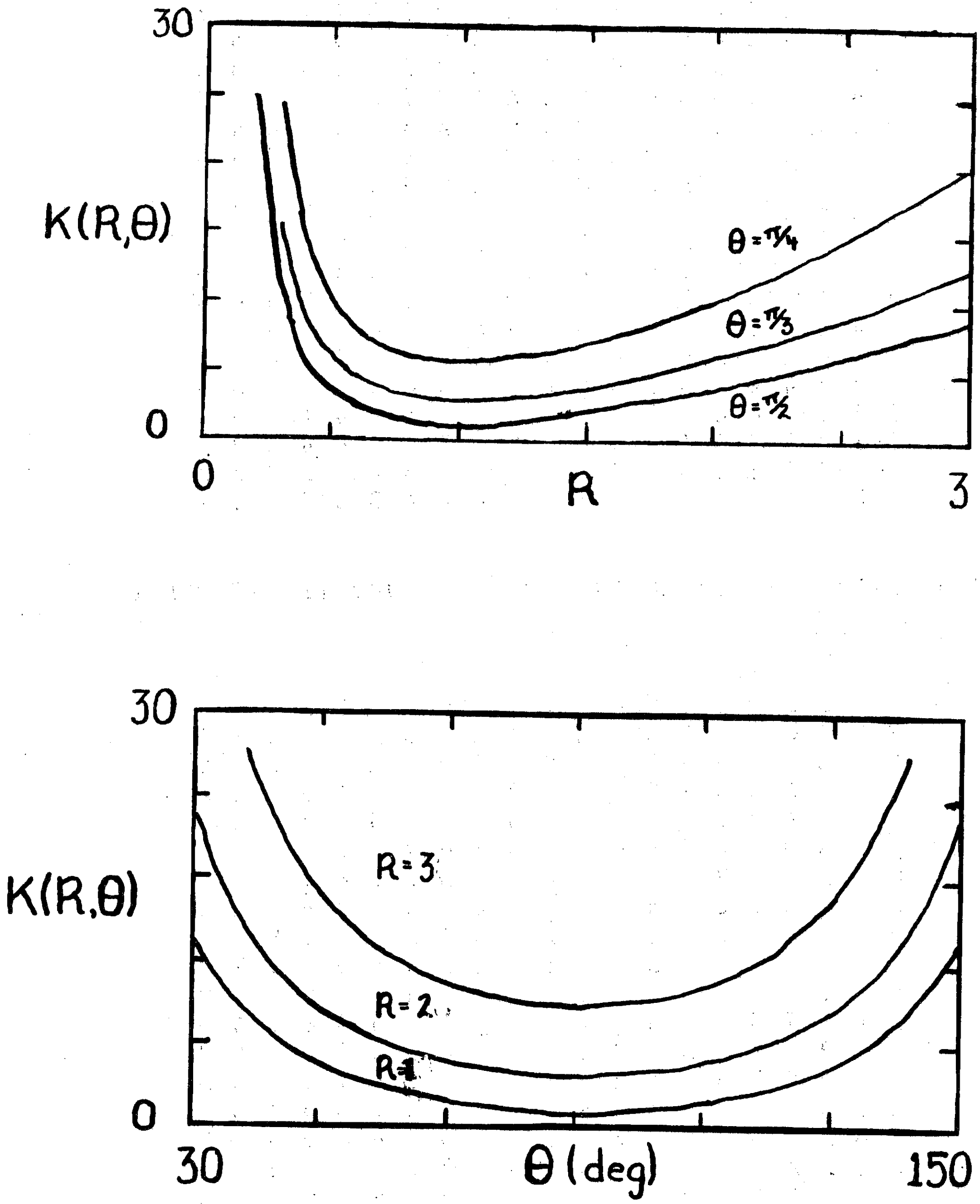

FIG 2 EFFECT OF TRIANGLE SHAPE ON THE CONDITION NUMBER.

2	2	3	3	3	3	3	3	3	3	4	4	4	4	4	3	3	3	3	3	2
2	3	3	3	3	3	3	4	4	4	4	4	4	4	4	4	4	3	3	3	3
3	3	3	3	3	3	4	4	4	4	5	5	5	5	4	4	4	4	3	3	3
3	3	3	3	4	4	4	4	5	5	5	5	5	5	5	5	4	4	3	3	3
3	3	3	4	4	4	5	5	5	6	6	6	6	6	6	5	5	4	4	3	3
3	3	4	4	4	5	5	5	6	7	7	8	8	7	6	6	5	4	4	3	3
3	4	4	5	5	5	6	6	7	8	9	A	A	8	7	6	5	5	4	4	3
4	4	4	5	6	6	7	7	8	9	B	D	C	A	8	6	5	5	4	4	3
4	4	5	5	6	7	8	9	A	A	D	F	E	A	8	6	5	5	4	4	3
4	4	5	6	7	8	A	C	D	C	E	F	C	9	8	6	5	5	4	4	3
4	4	5	6	7	9	B	F	F	F	0	C	B	9	7	6	5	5	4	4	3
4	4	5	6	7	8	A	C	D	C	E	F	C	9	8	6	5	5	4	4	3
4	4	5	5	6	7	8	9	A	A	D	F	E	A	8	6	5	5	4	4	3
4	4	4	5	6	6	7	7	8	9	B	D	C	A	8	6	5	5	4	4	3
3	4	4	5	5	5	6	6	7	8	9	A	A	8	7	6	5	5	4	4	3
3	3	4	4	4	5	5	5	6	7	7	8	8	7	6	6	5	4	4	3	3
3	3	3	4	4	4	5	5	5	6	6	6	6	6	6	5	5	4	4	3	3
3	3	3	3	4	4	4	4	5	5	5	5	5	5	5	5	4	4	3	3	3
3	3	3	3	3	3	4	4	4	4	5	5	5	5	4	4	4	4	3	3	3
2	3	3	3	3	3	3	4	4	4	4	4	4	4	4	4	4	3	3	3	3
2	2	3	3	3	3	3	3	3	3	4	4	4	4	4	3	3	3	3	3	2

FIG 3A RELATIVE FIX ERROR FOR 1 IN 1000 TIMING ERROR

F	F	E	D	D	C	C	B	B	B	B	B	B	B	C	C	D	D	E	F	F
F	E	D	C	C	B	B	A	A	A	A	A	A	A	B	B	C	C	D	E	F
E	D	C	C	B	A	A	9	9	9	9	9	9	9	A	A	B	C	C	D	E
D	C	C	B	A	9	9	8	8	7	7	7	8	8	9	9	A	B	C	C	D
D	C	B	A	9	8	8	7	7	6	6	6	7	7	8	8	9	A	B	C	D
C	B	A	9	8	8	7	6	6	5	5	5	6	6	7	8	8	9	A	B	C
C	B	A	9	8	7	6	5	5	4	4	4	5	5	6	7	8	9	A	B	C
B	A	9	8	7	6	5	4	4	3	3	3	4	4	5	6	7	8	9	A	B
B	A	9	8	7	6	5	4	3	2	2	2	3	4	5	6	7	8	9	A	B
B	A	9	7	6	5	4	3	2	1	1	1	2	3	4	5	6	8	9	A	B
B	A	9	7	6	5	4	3	2	1	0	1	2	3	4	5	6	7	9	A	B
B	A	9	7	6	5	4	3	2	1	1	1	2	3	4	5	6	8	9	A	B
B	A	9	8	7	6	5	4	3	2	2	2	3	4	5	6	7	8	9	A	B
B	A	9	8	7	6	5	4	4	3	3	3	4	4	5	6	7	8	9	A	B
C	B	A	9	8	7	6	5	5	4	4	4	5	5	6	7	8	9	A	B	C
C	B	A	9	8	8	7	6	6	5	5	5	6	6	7	8	8	9	A	B	C
D	C	B	A	9	8	8	7	7	6	6	6	7	7	8	8	9	A	B	C	D
D	C	C	B	A	9	9	8	8	7	7	7	8	8	9	9	A	B	C	C	D
E	D	C	C	B	A	A	9	9	9	9	9	9	9	A	A	B	C	C	D	E
F	E	D	C	C	B	B	A	A	A	A	A	A	A	B	B	C	C	D	E	F
F	F	E	D	D	C	C	B	B	B	B	B	B	B	C	C	D	D	E	F	F

FIG 3.B RELATIVE FIX ERROR FOR 1 IN 1500 SPEED OF SOUND ERROR